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Totally ⊕Generalized *Co finitely Supplemented Modules

Wasan Khalid Hassan, Raid Hasb Allah Dnan*
Department of Mathematics, College of Science, University of Baghdad, Baghdad. Iraq.

Abstract

Let R be an associative ring with identity, and let M be a unital left R-module, M is called totally ⊕generalized *cofinitely supplemented module for short (T⊕G*CS), if every submodule of M is a ⊕ Generalized *cofinitely supplemented (⊕G*CS). In this paper we prove among the results under certain condition the factor module of T ⊕G*CS is T⊕G*CS and the finite sum of T⊕G*CS is T ⊕G*CS.

Keywords: ⊕ generalized *cofinitely supplemented, totally ⊕ generalized *cofinitely supplemented modules.

المقاسات الكليه المكمله ضد المنتهيه ⊕ المعممه *

وسن خالد حسن, رائد حسب الله دنان*
قسم الرياضيات، كلية العلوم، جامعة بغداد، بغداد، العراق

الخلاصه

لتكن R حلقه تجميعيه ذات عنصر محايد وليكن M مقاس احادي ايسري على R.يقال بان M مقاس كلي مكمل ضد منتهي⊕ معمم * اذا كان كل مقاس جزئي من M هو مقاس مكمل ضد منتهي⊕ معمم *. في هذا البحث سوف نقوم باثبات وتحت شروط معينه اي عامل قسمه من M واي مجموع منتهي من مقاسات كليه مكمله ضد منتهيه ⊕ معممه * يبقى مقاس كلي مكمل ضد منتهي ⊕ معمم *.

1. Introduction:

Let R be an associative ring with identity, and let M be a unital left R- module, N≤M will mean submodule of M. E(M), Z*(M) will indicate the injective hull, co singular submodule of M, respectively. Where Z*(M) = {m∈M ; Rm is small in E(Rm) }. Let N and K be submodules of M. N is called a supplement of K in M if it is minimal with respect to M=N+K, equivalently M =N+K and N ∩K is small in N, for short (N ∩K ≪ N). Following [1], M is supplemented (⊕supplemented) if every submodule of M has a supplement (which is direct summand) in M, and M is called generalized * supplemented, for short (G*S), if for any submodule N of M , there is K ≤ M such that M =N+K and N ∩K ≤Z*(K), K is called a generalized * supplement of N in M,[2]. A submodule N of M is called cofinitely submodule if $\frac{M}{N}$ is finitely generated. A module M is called ⊕ generalized * cofinitely supplemented, for short (⊕G*CS), if for any cofinite submodule N of M, there exist submodules L,T of M such that M = N+L with N ∩L ≤Z*(L) and M =L⊕T,[2]. It is clear that every ⊕supplemented modules are ⊕G*CS modules.

*Email: read_master99@yahoo.com

Following [3], a module M is called totally cofinitely supplemented, if every submodule of M is cofinitely supplemented. In this paper we introduce a totally ⊕G*CS (T⊕G*CS), we called M is totally ⊕G*CS, if every submodule of M is ⊕G*CS, it is clear that, not every submodule of ⊕G*CS is ⊕G*CS, for example Q as Z- module is ⊕G*CS since the only cofinite submodule of Q is Q itself which is a direct summand, but Z not ⊕G*CS. In this work we prove some properties of T⊕G*CS modules.

2. The co singular sub modules:

Let M be an R- module, the radical Z*(M) were studied in [4], which is called the co singular submodule of M as a generalization of the Jacobson radical of M (Rad(M)), defined by

Z*(M) = { m ∈M ; Rm is small in E(M) }, where E(M) is the injective hull of M. equivalently:

Z*(M) = M ∩Rad (E(M))[5].

If M << E(M), then M is called small module,[6]. M is called co singular module if M = Z*(M), [5]. A ring R is called co singular if any R- module is co singular.

Every small module is co singular, but the converse is not true, for example Q as Z-module, Q is co singular but not small.

The following are some properties of Z*(M), which is appeared in [5], [7].

Proposition 2.1.[5] **:-** Let R be a ring and let M and L be two R –modules and let g:M →L be an R-homomorphism, then g(Z*(M)) ≤ Z*(L).

Proposition 2.2.[7]**:-**For any ring. If R = Z*(R), then M = Z*(M), for any R –module M.

Proposition 2.3.[5]**:-** Every Z –module is co singular.

Proposition 2.4. [7]:- Z*(N) = N ∩ Z*(M), for any submodule N of an R –module M.

Corollary 2.5. [5]:- Every submodule of co singular module is co singular.

Proposition 2.6.[5]**:-** Let $\{M_i\}$be a family of an R – modules, for any index set I; i∈I, if $M =\oplus_{i\in I}M_i$, then $Z^*(M) =\oplus_{i\in I} Z^*(M_i)$.

Proposition 2.7 [7]**:-** For any nonzero R –module M, Z*(M) =0 if and only if Rad (E(M)) =0.

The following definitions appeared in [2] as a generalization of generalized supplemented modules.

Definition 2.8 [2]**:-** Let N be a submodule of M, a submodule K of M is called generalized *supplement, for short (G*S) of N in M, if M= N+K and N ∩K≤ Z*(K). If every submodule of M has a generalized * supplement, then M is called generalized *supplemented module, clearly, every co singular is a G*S. A module M is called generalized *cofinitely supplemented module, for short (G*CS), if every cofinite submodule of M has generalized *supplement in M, [2].

As a generalization of ⊕ cofinitely supplemented modules, [2] introduce ⊕generalized cofinitely supplemented modules.

Recall that an R- module M is called ⊕generalized supplemented module if for M = N+ L with

N ∩L ≤ Rad (L), for N, L ≤ M, L is called generalized supplement of N in M [8]. And M is called ⊕ cofinitely supplemented module if every cofinite submodule of M has a supplement which is direct summand in M, [3] .

Definition 2.9 [2]:- An R –module M is called ⊕generalized *cofinitely supplemented module,(for short (⊕G*CS)), if every cofinite submodule of M has generalized *supplement in M that is a direct summand.

Clearly every ⊕ supplemented and ⊕generalized supplemented are ⊕G*CS. Notice that Q as Z-module is ⊕G*CS but not ⊕Supplemented,[2].

Proposition 2.10:-Let M be a G*CS module, then $\frac{M}{Z*(M)}$ is ⊕G*CS module.

Proof:- Let $\frac{N}{Z*(M)} \leq \frac{M}{Z*(M)}$ with $\frac{N}{Z*(M)}$ is cofinite submodule of $\frac{M}{Z*(M)}$, then N is cofinite in M, but since M is G*CS, then ∃ K ≤ M such that M = N+K and N ∩K ≤ Z*(K), then $\frac{M}{Z*(M)} = \frac{N}{Z*(M)} + \frac{K+Z*(M)}{Z*(M)}$ and $\frac{N}{Z*(M)} \cap \frac{K+Z*(M)}{Z*(M)} = \frac{(N\cap K)+Z*(M)}{Z*(M)} \leq \frac{Z*(K)+Z*(M)}{Z*(M)} = \frac{Z*(M)}{Z*(M)}$, then $\frac{M}{Z*(M)}$ is ⊕G*CS module.

Proposition 2.11:- Let M be any R –module such that every maximal submodule of M is a direct summand, then M is ⊕G*CS module.

Proof:- Let N be a cofinite submodule of M, then N is a direct summand, by [8.lemma 2.7].

i.e. M =N⊕K, for some K ≤ M. i.e. N ∩K = 0≤ Z*(K), then M is ⊕G*CS module.

Recall that a ring R is called a V- ring, if every ideal in R is an intersection of maximal ideals in R, equivalently, R is V-ring if and only if every simple R-module is injective if and only if Rad (M) =0, for every R – module M, [9].

Proposition 2.12:- Let R be a V- ring, then M is G*CS if and only if M is ⊕G*CS.

Proof :- Let N be a cofinite submodule of M, but since M is G*CS, then ∃ K ≤ M such that M = N+K and N ∩K ≤ Z*(K),but R is V – ring (Rad (E(M)) =0) hence by [prop. 2.7] Z*(M) =0, thus Z*(K) =0, then M is ⊕G*CS module. Conversely, trivial by definition.

3. Totally ⊕ Generalized * Cofinitely Supplemented Modules.

An R –module M is called totally cofinitely supplemented if every submodule is a cofinitely supplemented module, [3]. As a generalization of totally cofinitely supplemented module, we introduce the following definition.

Definition 3.1: An R –module M is called totally ⊕generalized * cofinitely supplemented module, for short (T⊕G*CS), if every submodule of M is ⊕G*CS.

Notice that Q as Z- module is ⊕G*CS, since the only cofinite submodule of Q is Q itself which is direct summand, but Z not ⊕G*CS module, hence Q as Z –module is not T⊕G*CS.

Clearly, every (semi simple, small, hollow, local) module is T⊕G*CS (⊕G*CS).

The following give some properties of T⊕G*CS.

Proposition 3.2:- Let M be a T⊕G*CS, then each finitely generated submodule N ≤ M be written as N = K ⊕L, where Z*(L) = L and K ≤ N.

Proof:- Let N be a finitely generated submodule of M, then $\frac{N}{Z*(N)}$ be a finitely generated module, hence Z*(N) is cofinite submodule of N , but M is T⊕G*CS (N ≤ M), then N is ⊕G*CS, i.e. ∃ K ,L ≤ N such that N = Z*(N) + K with Z*(N) ∩ K = Z*(K) and N = K⊕L, hence by [prop. 2.6] Z*(N) = Z*(K) ⊕ Z*(L), but L ≅ $\frac{N}{K}$ = $\frac{Z*(N)+K}{K}$ ≅ $\frac{Z*(N)}{K\cap Z*(K)}$ = $\frac{Z*(N)}{Z*(N)}$ ≅ Z*(L).

Recall that a submodule N of an R – module M is called fully invariant if for every f ∈End(M), f(N) ≤ N and M is called duo module if every submodule of M is fully invariant,[10].

Proposition 3.3:- Let M be a T⊕G*CS module, then for every fully invariant submodule N of M, $\frac{M}{N}$ is T⊕G*CS.

Proof:- Let $\frac{K}{N} \leq \frac{M}{N}$, and let $\frac{L}{N}$ be a cofinite submodule of $\frac{K}{N}$, then L is cofinite in K ≤ M, but M is T⊕G*CS, hence K is ⊕G*CS, then ∃ H ,T ≤ K such that K = L +H with L∩ H ≤ Z*(H) and K = H ⊕T . Now, $\frac{K}{N}=\frac{L}{N}+\frac{H+N}{N}$ with $\frac{L}{N}\cap\frac{(H+N)}{N}=\frac{(L\cap H)+N}{N} \leq \frac{Z*(H)+N}{N} \leq Z*(\frac{H+N}{N})$, hence $\frac{K}{N}=\frac{T+N}{N}\oplus\frac{H+N}{N}$, then $\frac{M}{N}$ is T⊕G*CS.

Recall that an R –module M is called distributive module if for N, H and L ≤ M, N + (L∩ H) = (N+L) ∩ (N+H) or N ∩(H+L) = (N∩ H) + (N∩ L) [1].

Proposition 3.4:-Let M be a distributive T⊕G*CS module, then $\frac{M}{N}$ is T⊕G*CS, for each N ≤ M.

Proof:- Let $\frac{K}{N} \leq \frac{M}{N}$, and let $\frac{L}{N}$ be a cofinite submodule of $\frac{K}{N}$, then L is cofinite in K ≤ M, but M is T⊕G*CS, hence K is ⊕G*CS, then ∃ H ,T ≤ K such that K = L +H with L∩ H ≤ Z*(H) and K = H ⊕T. Now , $\frac{K}{N}=\frac{L}{N}+\frac{H+N}{N}$ with $\frac{L}{N}\cap\frac{(H+N)}{N}=\frac{(L\cap H)+N}{N} \leq \frac{Z*(H)+N}{N} \leq Z*(\frac{H+N}{N})$, but $\frac{K}{N}=\frac{T+N}{N}+\frac{H+N}{N}$ and $\frac{T+N}{N}\cap\frac{H+N}{N}=\frac{(H+N)\cap(T+N)}{N}=\frac{T\cap(H+N)+H\cap(T+N)}{N}=\frac{(T\cap H)+N}{N}=\frac{N}{N}$, hence $\frac{K}{N}=\frac{T+N}{N}\oplus\frac{H+N}{N}$, then $\frac{M}{N}$is T⊕G*CS.

Proposition 3.5:-Let M be a T⊕G*CS module, let N ≤ M. If for any K ≤ M, N ≤ K such that K satisfies that for each direct summand L of K , $\frac{L+N}{N}$ is a direct summand of $\frac{K}{N}$, then $\frac{M}{N}$is T⊕G*CS.

Proof:- Let $\frac{P}{N}$ be a cofinite submodule of $\frac{K}{N}$, then P is cofinite in K ≤ M, but M is T⊕G*CS, hence K is ⊕G*CS, then ∃ L ,T ≤ K such that K = P +L with P ∩L ≤ Z*(L) and K = L ⊕T . Now, $\frac{K}{N}=\frac{P}{N}+\frac{L+N}{N}$ with $\frac{P}{N}\cap\frac{(L+N)}{N}=\frac{(P\cap L)+N}{N} \leq \frac{Z*(L)+N}{N} \leq Z*(\frac{L+N}{N})$, but by assumption $\frac{L+N}{N}$is direct summand of $\frac{K}{N}$, then ∃ $\frac{H}{N} \leq \frac{K}{N}$ such that$\frac{K}{N}=\frac{H}{N}\oplus\frac{L+N}{N}$, then $\frac{M}{N}$is T⊕G*CS.

Note that the following two propositions are hold if the ring R is commutative ring.

Proposition 3.6:- let R be noetherian ring and M is duo R- module such that $M = M_1 \oplus M_2$, where M_1 and M_2 are T⊕G*CS, then M is T⊕G*CS.

Proof:- Let $\frac{N}{L}$ be a finitely generated submodule ($L \leq N \leq M$), then $N = N \cap M = (N \cap M_1) \oplus (N \cap M_2)$ and $L = (L \cap M_1) \oplus (L \cap M_2)$, but since $\frac{N \cap M_1}{L \cap M_1} \oplus \frac{N \cap M_2}{L \cap M_2} \cong \frac{N}{L}$ (f.g), then both of $\frac{N \cap M_1}{L \cap M_1}$ and $\frac{N \cap M_2}{L \cap M_2}$ are finitely generated, so, $\exists$ H, $T \leq N \cap M_1$ such that $N \cap M_1 = L \cap M_1 + H = H \oplus T$ with $(L \cap M_1) \cap H \leq Z^*(H)$. Similarly $\exists$ A, $B \leq N \cap M_2$ such that $N \cap M_2 = (L \cap M_2) + A = A \oplus B$ with $(L \cap M_2) \cap A \leq Z^*(A)$. hence $N = (L \cap M_1) + (L \cap M_2) + (A+H)$, with $((L \cap M_1) + (L \cap M_2)) \cap (A+H) = (L \cap M_1) \cap H + (L \cap M_2) \cap A \leq Z^*(H) + Z^*(A) = Z^*(H+A)$, then $N = (N \cap M_1) + (N \cap M_2) = (H \oplus T) + (A \oplus B) = (H+A) \oplus (T+B)$, then M is T⊕G*CS.

Proposition 3.7:- let R be noetherian ring and M is duo R- module such that $M = \oplus_{i=1}^{n} M_i$, then M is T⊕G*CS if and only if M_i is T⊕G*CS for each i =1,...,n.

Before the following results we need the following definition.

Recall that an R- module M is said to be have summand sum property (SSP), if the sum of two direct summand submodules of M is direct summand in M [1].

Proposition 3.8:-Let M be a T⊕G*CS module such that, if every submodule of M has SSP, then every direct summand of M is a T⊕G*CS.

Proof:- Let $N \leq M$ such that N is a direct summand of M, then there is $K \leq M$ such that $M = N \oplus K$. It is enough to prove that $\frac{M}{K}$ is T⊕G*CS. Let $\frac{L}{K}$ be a cofinite submodule of $\frac{H}{K}$ in $\frac{M}{K}$, then L is a cofinite in H, but since M is T⊕G*CS, hence H is ⊕G*CS, then $\exists$ A, $B \leq H$, such that $H = A + L = A \oplus B$ with $A \cap L \leq Z^*(A)$. Since K is a direct summand in M, then K is a direct summand in H, and [H has SSP], hence A + K is a direct summand of H, [i.e. $H = (A+K) \oplus T$, for some $T \leq H$].

Now : $\frac{H}{K} = \frac{L}{K} + \frac{A+K}{K}$ and $\frac{L}{K} \cap \frac{A+K}{K} \leq Z^*(\frac{A+K}{K})$, then : $\frac{H}{K} = \frac{A+K}{K} + \frac{B+K}{K}$ with $\frac{(B+K) \cap (A+K)}{K} = \frac{K}{K}$, hence $\frac{H}{K} = \frac{A+K}{K} \oplus \frac{B+K}{K}$, then $\frac{M}{K}$ is T⊕G*CS.

Theorem 3.9:- Let R be a V- ring, and let M be an R – module, then the following are equivalent

1. M is T⊕G*CS.
2. M is G*CS.
3. Every finitely generated factor module is a direct summand.
4. M is semi simple module.

Proof:- (1 ⇒2) trivial by definition.

(2⇒ 3). Let N be a cofinite submodule of M, then by (2), $M = N + K$, for some $K \leq M$ with $N \cap K \leq Z^*(K) = 0$, (since R is V –ring, then $Z^*(K) = 0$), hence $M = N \oplus K$.

(3 ⇒ 4) by [3. Theorem. 10] .

(4 ⇒ 1). Let $K \leq M$, to show that K is ⊕G*CS. Let L be a cofinite submodule of $K \leq M$, then L is a direct summand of M, [i.e. $M = L \oplus T$, for some $T \leq M$]. Then $K = K \cap M = (K \cap L) \oplus (K \cap T) = L + (K \cap T)$ with $L \cap (K \cap T) = (L \cap T) \cap K = 0 \leq Z^*(T+K)$, then M is T⊕G*CS.

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