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A New Technique for Solving A Fractional Sharma-Tasso-Olever Equation

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Abstract

In this study, we present a modified analytical approximation method to find the time-fractional Sharma-Tasso-Olever issue solving. In order to tackle nonlinear fractional differential equations that arise in a variety of physical processes, we begin by providing an alternate foundation for the Laplace Residual Power Series Technique (LRPSM). Thus, the generalized Taylor series equation and residual functions serve as the foundation for this approach.

More precisely, our approach and the suggested solution produce good results. Moreover, the reliability, effectiveness, and simplicity of this approach are demonstrated for all classes of fractional nonlinear issues that arise in technological and scientific fields. Two examples are provided to exemplify how the considered scheme works in calculating various types of fractional ordinary differential equations. Finally, the obtained results in this article are compared with other methods such as Residual Power Series (RPS), Variational Iteration Method (VIM), and Homotopy Perpetration Method (HPM). The consequences of our method are good and effective.

Keywords: Fractional calculus**;** Formula of fractional Sharma-Tasso-Olever; Residual power series; Laplace residual power series; fractional derivative of Caputo.

تقنية جديدة لحل معادلة شارما-تاسو-أوليفر الكسرية

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الخلاصة

في هذه الدراسة، نقدم طريقة تقريب تحليلي معدلة لإيجاد حل لمعادلة شارما-تاسو-أوليفر الكسري الزمني. من أجل معالجة الصيغ التفاضلية الكسرية غير الخطية التي تنشأ في مجموعة متنوعة من العمليات الفيزيائية، نبدأ بتوفير أساس بديل لتقنية سلسلة الطاقة المتبقية لابلاس .(LRPSM) تعمل معادلة سلسلة تايلور المعممة والوظائف المتبقية كأساس لهذا النهج. نهجنا أو الحل المقترح يؤدي إلى نتائج جيدة. يتم توضيح موثوقية وفعالية وبساطة النهج المقترح لجميع فئات القضايا غير الخطية الجزئية التي تنشأ في المجالات التكنولوجية والعلمية. لتوضيح كيفية عمل المخطط المقترح في حساب أنواع مختلفة من المعادلات

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التفاضلية العادية الكسرية، تم تقديم مثالين تمت فيهما مقارنة نتائج طريقتنا المقترحة مع طرق أخرى مثل RPSو VIMو HPM وكانت نتائج طريقتنا جيدة وفعالة.

1. Introduction

Fractional calculus is a branch of applied mathematics that deals with derivation and integration like any real or complex ordering. Non-Newtonian calculus and extended calculus are other names for fractional calculus. In a famous letter, Leibniz asked what might occur if the order of derivatives is changed to $\frac{1}{2}$. His response in 1695 is recognized as that of the start of the field of fractional calculus [1,2]. In the disciplines of physics, biochemistry, biology, technology, viscoelasticity, operations research, optical fibers, communications, and finance, Fractional calculus is crucial [3,4,5]. While not all of these methods are regularly used, there are various techniques to define fractional differential equations.

There are various techniques to define fractional derivatives, however, not all of them are often used. The most frequently used fractional derivatives are those with fractional rank in terms of Conformable operators, Atangana-Baleanu, Riemann-Liouville and the Caputo fractional derivatives [6, 7, 8, 9, 10]. In some cases, fractional derivatives are preferable to integer-order derivatives when modelling because they can model and evaluate complicated systems with improved non-linear processes and higher-rank dynamics. This is caused by two main factors. Firstly, rather than being restricted to an integer order, we may select any order for derivative operators. Non-integer type derivatives depend on previous and local circumstances and are advantageous whenever the systems have such a long-term memory.

Differential equations are created when natural and biological processes are explained using mathematical methods in technology and research. The formula of movement, movement of simple harmonic, Beam deflecting, and other events are a few instances of phenomena that can be described by differential equations. Thus, the ideas of differential equations are important and helpful. Applications regularly come upon differential equations that are so complex. The close-form answers are sometimes not practical. The solution of the differential equations with given boundary conditions can be effectively replaced by numerical methods.

In recent years, the development of a number of techniques for dealing with fractional differential equations has been seen, this includes the Iterative Laplace transform technique [11], the adaptive approach of Shehu transform [12], the homotopy analytical technique [13], the variational iteration technique [14, 15], the technique of Elzaki transform decomposition [16, 17], the Laplace decomposition technique [18] , the technique of homotopy perturbation transform [19, 20], and the residual power series technique [21]. There are two main causes of this. Firstly, we are no longer restricted to an integer rank when choosing the rank for the derivative operator. Non-integer rank derivatives that are advantageous in systems with long-term memories are dependent on historical data and local circumstances.

The researcher Abu Arqub created the RPSM in 2013 [22]. The quasi technique is known as the RPSM is developed using Taylor's series and the residual error functions. All linear and non-linear differential equations convergence series are given. In 2013, RPSM was implemented to deal with fuzzy differential equations. Arqub et al. developed a novel collection of RPSM techniques to swiftly get series type solves for common differential equations [23]. Arqub et al. [24] also created a unique and interesting RPSM technique for solving of fractional nonlinear issues involving boundary values. In order to identify approximations of results to fractional rank KdV-burgers formulas, El-Ajou et al. created a new iteration RPSM approach [25]. Zhang et al. [26] presented an effective numerical method that incorporates the RPSM and least squares techniques. By merging two efficient methods, scientists have created a new method for solving fractional-order differential formulas (FODFs). Some of the methods that are mixed to define a few of these groups include the transform of Sumudu and the homotopy perturbation approach, the transform of natural the homotopy analysis method, the transformation of Shehu and the Adomian decomposition method and the Laplace transform with RPSM [27, 28, 29].

In this study, we employ the special combining method which is named the LRPSM to discover both approximation and precise results for the time-fractional Sharma-Tasso-Olver PDEs involving unknown parameters. The RPSM and Laplace transform technique are combined in this new technique. Moreover, graphical relevance is seen for various values of fractional-rank derivatives. As a consequence, the method is precise, quick, and impervious to computing iterations of errors. It also does not take up a lot of memory storage or processing time.

In order to explore the approximate solution of a nonlinear fractional Sharma-Tasso-Olever formula which is crucial in defining the non-linear phenomenon, we start to apply LRPSM in this study. Below is the shape of a time-fractional non- linear fractional Sharma-Tasso-Olever formula (FSTOF):

$$D_t^{\alpha}u + 3\lambda u_x^2 + 3\lambda u^2 u_x + 3\lambda u u_{xx} + \lambda u_{xxx} = 0\,, t > 0, 0 < \alpha \leq 1, \quad (1)$$

where λ is constant and $u \epsilon L^2(\Omega)$ is a function with respect x in bounded domain Ω and time t, and can be any random constant and alpha can be any factor defining the order of the fractional time-derivative.

Using the variation iteration approach, Adomian decomposition, and homotopy perturbation method, Song et al. [30] have solved the (1). Moreover, by employing the Residual Power Series Technique [31]. However, we solve equation (1) employing LRPSM.

The structure of the research is detailed below. Secondly, Part 2 uses the foundational ideas and findings of fractional calculus. The basis for the creative approach in Part 2 has some original outcomes that are provided. The results of time-fractional non-linear Sharma-Tasso-Olever are then discovered in Part 3 by using LRPSM. A few of the issues in Part 4 are overcome using LRPSM. A brief conclusion concludes Part 5.

2 Preliminaries

In this section, we go through some key terms, notions, and principles associated with the fractional derivative operations and the Laplace transform utilized in the present study:

Definition 2.1 [32]. The fractional derivative is the same in the Caputo as follows:

$$^{C}D^{\alpha}w(\mathrm{x},\eta) = \mathrm{J}^{\delta-\alpha}w^{\delta}(\mathrm{x},\eta),\ \delta - 1 < \alpha \leq \delta, \mathrm{x} > 0, \quad (2)$$

where the Riemann-Liouville integral operator is represented by $\mathbf{J}^{\alpha}$ as

$$\mathrm{J}^{\alpha}w(\mathrm{x},\eta) = \frac{1}{\Gamma(\alpha)}\int_0^{\kappa} (\kappa-\eta)^{\alpha-1}w(x,\eta)d\eta, \quad (3)$$

and $\delta \in \mathbb{N}$.

Definition 2.2. [32] The Laplace transform defined on function $w(\eta)$ is

$$\mathcal{L}\{w(\eta)\} = \int_0^{\infty} e^{-s\eta}w(\eta)d\eta,\ s > \alpha. \quad (4)$$

The inverse of the Laplace transform is given as follows:

$$\mathcal{L}^{-1}\{W(\mathrm{x},s)\} = \int_{c-i\infty}^{c+i\infty} e^{s\eta}W(\mathrm{x},s)ds,\ c = Re(\mathrm{s}) > c_0. \quad (5)$$

Lemma 2.3. [33] If we assume that $w(\mathrm{x},\eta)$ is a piece-wise continuous function with $(\mathrm{x},s)=\mathcal{L}\{w(\mathrm{x},\eta)\}$, then the following characteristics are genuine:

(i) $\mathcal{L}\{\mathrm{J}_*^{\alpha}w(\mathrm{x},\eta)\}=\frac{W(\mathrm{x},s)}{s^{\alpha}},\ \varrho>0$

(ii) $\mathcal{L}\{\mathrm{D}_*^{\alpha}w(\mathrm{x},\eta)\}=s^{\alpha}W(\mathrm{x},s)-\sum_{i=0}^{k-1}s^{\alpha-k-1}w^{k}(\mathrm{x},0),\ k-1<\alpha\leq k;$

(iii) $\mathcal{L}\{\mathrm{D}_*^{k\alpha}w(\mathrm{x},\eta)\}=s^{k\alpha}W(\mathrm{x},s)-\sum_{i=0}^{k-1}s^{(k-i)\alpha-1}\mathrm{D}_*^{i\alpha}w(\mathrm{x},0),\ 0<\alpha\leq 1.$

Proposition 2.4. [32] Take into account that $w(\mathrm{x},\eta)$ is piecewise continuous on $I\times[0,\infty)$ with an exponential order of $\mathfrak{I}$. Consider that the fractional expansions of $W(x,s)=\mathcal{L}\{w(\mathrm{x},\eta)\}$ is as follows:

$$W(\mathrm{x},s)=\sum_{m=0}^{\infty}\frac{\lambda_m(x)}{s^{1+m\alpha}},\ 0<\alpha\leq 1, s>\mathfrak{I}, \tag{6}$$

Hence, $\lambda_m(\mathrm{x})=\mathrm{D}_*^{i\alpha}w(\mathrm{x},0)$.

Remark 2.5.[34] Using the inverse of the Laplace transform to the provided (6), we get:

$$w(\mathrm{x},\eta)=\sum_{m=0}^{\infty}\frac{\mathbf{D}_*^{\alpha}w(\mathrm{x},0)}{\Gamma(1+m\alpha)}\eta^{m\alpha},\ 0<\alpha\leq 1, \eta\geq 0, \tag{7}$$

It is comparable to the fractional Taylor's equation presented in [35].

3. The Time -Fractional Sharma-Tasso-Olever Formula Solutions Using LRPS Technique

Take the following time -fractional Sharma-Tasso-Olever formula to demonstrate how the LRPS technique may be used to create a series solution to the FSTOF:

$$\begin{gathered}D_t^{\alpha}u(x,t)+3\lambda u_x^2(x,t)+3\lambda u^2(x,t)u_x(x,t)+3\lambda u(x,t)u_{xx}(x,t)\\ +\lambda u_{xxx}(x,t)=0, t>0, 0<\alpha\leq 1,\end{gathered} \tag{8}$$

where λ is constant and $u\epsilon L^2(\Omega)$ is a function with respect *x* in bounded domain Ω and time *t*. The initial condition is as follows:

$$u(x,0)=\zeta(x), \tag{9}$$

In the beginning, use the Laplace transform to Eq. (8), we obtain

$$\mathcal{L}[D_t^{\alpha}u(x,t)+3\lambda u_x^2(x,t)+3\lambda u^2(x,t)u_x(x,t)+3\lambda u(x,t)u_{xx}(x,t)+\lambda u_{xxx}(x,t)]=\mathcal{L}[0],$$

$$t\in\ I\times[0,\infty]. \tag{10}$$

with I is an open interval.

We may construct Eq. (10) as follows using Lemma 2.3:

$$s^{\alpha}U(x,s)-s^{\alpha-1}u(x,0)+3\lambda\mathcal{L}\left\{\left(\mathcal{L}^{-1}\left(U_x(x,s)\right)^2\right)\right\}+ 3\lambda\mathcal{L}\left\{\left(\mathcal{L}^{-1}\left(U(x,s)\right)^2\right)\mathcal{L}^{-1}\left(U_x(x,s)\right)\right\}+\lambda\mathcal{L}\left\{\left(\mathcal{L}^{-1}\left(U(x,s)\right)\right)\mathcal{L}^{-1}\left(U_{xx}(x,s)\right)\right\}+ \lambda U_{xxx}(x,s),\ s>0. \tag{11}$$

where $U(x,s)=\mathcal{L}[u(x,\eta)]$ and $U_{xxx}(x,s)=\mathcal{L}[u_{xxx}(x,\eta)]$.

The next form of Eq. (11) is produced by dividing it by s^{α} and applying the beginning circumstances from Eq. (11):

$$U(x,s)=\frac{\zeta(x)}{\mathrm{s}}-\frac{3\lambda}{s^{\alpha}}\left(\mathcal{L}\left\{\left(\mathcal{L}^{-1}\left(U_x(x,s)\right)^2\right)\right\}\right)-\frac{3\lambda}{s^{\alpha}}\left(\mathcal{L}\left\{\left(\mathcal{L}^{-1}\left(U(x,s)\right)^2\right)\mathcal{L}^{-1}\left(U_x(x,s)\right)\right\}\right)- \frac{3\lambda}{s^{\alpha}}\left(\mathcal{L}\left\{\left(\mathcal{L}^{-1}\left(U(x,s)\right)\right)\mathcal{L}^{-1}\left(U_{xx}(x,s)\right)\right\}\right)-\frac{\lambda}{s^{\alpha}}U_{xxx}(x,s),\ s>0. \tag{12}$$

Consider that extension of Eq. (12) result is as follows:

$$U(x,s)=\sum_{j=0}^{\infty}\frac{\zeta_j(x)}{s^{1+\alpha j}}, s>0. \tag{13}$$

According to (13), the kth-truncated series is

$$U_k(x,s) = \frac{\zeta(x)}{s} + \sum_{\boldsymbol{j=1}}^{\boldsymbol{k}} \frac{\zeta_j(x)}{s^{1+\alpha j}}\ , s > 0. \quad (14)$$

We can define the main LRPS techniques like the LRF of Eq. (12), in order to determine the unknown value of the parameter, $\zeta_j(x)$ is presented as follows:

$$\mathrm{LRes}(\mathrm{x,s}) = U(x,s) - \frac{\zeta(x)}{s} + \frac{3\lambda}{s^{\alpha}}\left(\mathcal{L}\left\{\left(\mathcal{L}^{-1}\left(U_x(x,s)\right)^2\right)\right\}\right) + \frac{3\lambda}{s^{\alpha}}\left(\mathcal{L}\left\{\left(\mathcal{L}^{-1}\left(U(x,s)\right)^2\right)\mathcal{L}^{-1}\left(U_x(x,s)\right)\right\}\right) + \frac{3\lambda}{s^{\alpha}}\left(\mathcal{L}\left\{\left(\mathcal{L}^{-1}\left(U(x,s)\right)\right)\mathcal{L}^{-1}\left(U_{xx}(x,s)\right)\right\}\right) + \frac{\lambda}{s^{\alpha}}U_{xxx}(x,s),\ s > 0. \quad (15)$$

thus, the kth-LRF is defined as:

$$\mathrm{LRes}_k(\mathrm{x,s}) = \frac{\zeta(x)}{s} + \frac{3\lambda}{s^{\alpha}}\left(\mathcal{L}\left\{\left(\mathcal{L}^{-1}\left(U_{(k)x}(x,s)\right)^2\right)\right\}\right) + \frac{3\lambda}{s^{\alpha}}\left(\mathcal{L}\left\{\left(\mathcal{L}^{-1}\left(U_k(x,s)\right)^2\right)\mathcal{L}^{-1}\left(U_{(k)x}(x,s)\right)\right\}\right) + \frac{3\lambda}{s^{\alpha}}\left(\mathcal{L}\left\{\left(\mathcal{L}^{-1}\left(U_k(x,s)\right)\right)\mathcal{L}^{-1}\left(U_{(k)xx}(x,s)\right)\right\}\right) + \frac{\lambda}{s^{\alpha}}U_{(k)xxx}(x,s),\ s > 0. \quad (16)$$

It is obvious that for $s > 0$ and $k = 0,1,2,3, \ldots.$ $\mathrm{Lim}_{k\to\infty}\mathrm{LRes}_k(x,s) = \mathrm{LRes}(x,s)$, $\mathrm{LRes}(x,s) = 0$. As a result, $\mathrm{Lim}_{s\to\infty}\left(s^k\mathrm{LRes}(x,s)\right) = 0$. Additionally, it was established [32, 35] and

$$\mathrm{Lim}_{s\to\infty}\left(s^{k+1}\mathrm{LRes}(x,s)\right) = \mathrm{Lim}_{s\to\infty}\left(s^{k+1}\mathrm{LRes}_k(x,s)\right) = 0, k = 1,2,3,.. \quad (17)$$

Given that $\mathrm{U}_1(x,s) = \frac{\zeta(x)}{s} + \frac{\boldsymbol{\zeta_1(x)}}{\boldsymbol{s^{1+\alpha}}}$, Eq. (16) signifies:

$$\mathrm{LRes}_1(\mathrm{x},s) = \frac{\boldsymbol{\zeta_1(x)}}{\boldsymbol{s^{1+\alpha}}} + \frac{3\lambda}{\mathrm{S}^{\alpha}}\left(\mathcal{L}\left\{\left(\mathcal{L}^{-1}\left(\frac{\zeta(x)}{s} + \frac{\boldsymbol{\zeta_1(x)}}{\boldsymbol{s^{1+\alpha}}}\right)\right)^2\right\}\right) + \frac{3\lambda}{\mathrm{S}^{\alpha}}\left(\mathcal{L}\left\{\left(\mathcal{L}^{-1}\left(\frac{\zeta(x)}{s} + \frac{\boldsymbol{\zeta_1(x)}}{\boldsymbol{s^{1+\alpha}}}\right)\right)^2\mathcal{L}^{-1}\left(\frac{\zeta'(\mathrm{x})}{\mathrm{S}} + \frac{\zeta_1'(\mathrm{x})}{\mathrm{S}^{\alpha+1}}\right)\right\}\right) + \frac{3\lambda}{\mathrm{S}^{\alpha}}\left(\mathcal{L}\left\{\mathcal{L}^{-1}\left(\frac{\zeta(x)}{s} + \frac{\boldsymbol{\zeta_1(x)}}{\boldsymbol{s^{1+\alpha}}}\right)\mathcal{L}^{-1}\left(\frac{\zeta''(\mathrm{x})}{\mathrm{S}} + \frac{\zeta_1''(\mathrm{x})}{\mathrm{S}^{\alpha+1}}\right)\right\}\right) + \frac{\lambda}{\mathrm{S}^{\alpha}}\left(\frac{\zeta'''(\mathrm{x})}{\mathrm{S}} + \frac{\zeta_1'''(\mathrm{x})}{\mathrm{S}^{\alpha+1}}\right), \mathrm{s} > 0. \quad (18)$$

By running the operator in Eq. (18), we can obtain the following simplified form:

$$\mathrm{LRes}_1(\mathrm{x},s) = \frac{\boldsymbol{\zeta_1(x)}}{\boldsymbol{s^{1+\alpha}}} + \frac{3\lambda\,\zeta'^2(x)}{S^{\alpha+1}} + \frac{6\lambda\,\zeta'(x)\zeta_1'(x)}{S^{2\alpha+1}} + \frac{3\lambda\,\zeta_1'^2(x)\Gamma(2\alpha+1)}{\Gamma(\alpha+1)^2S^{3\alpha+1}} + \frac{3\lambda\,\zeta'(x)\zeta^2(x)}{S^{\alpha+1}} + \frac{3\lambda\,\zeta_1'(x)\zeta^2(x)}{S^{2\alpha+1}} + \frac{6\lambda\,\zeta'(x)\zeta\,(x)\zeta_1(x)}{S^{2\alpha+1}} + \frac{6\lambda\,\zeta_1'(x)\zeta\,(x)\zeta_1(x)\Gamma(2\alpha+1)}{\Gamma(\alpha+1)^2S^{3\alpha+1}} + \frac{3\lambda\,\zeta'(x)\zeta_1^2(x)\Gamma(2\alpha+1)}{\Gamma(\alpha+1)^2S^{3\alpha+1}} + \frac{3\lambda\,\zeta_1'(x)\zeta_1^2(x)\Gamma(3\alpha+1)}{\Gamma(\alpha+1)^3S^{4\alpha+1}} + \frac{3\lambda\,\zeta(x)\zeta''(x)}{S^{\alpha+1}} + \frac{3\lambda\,\zeta(x)\zeta_1''(x)}{S^{2\alpha+1}} + \frac{3\lambda\,\zeta_1(x)\zeta''(x)\Gamma(\alpha+1)}{\Gamma(\alpha+1)S^{2\alpha+1}} + \frac{3\lambda\,\zeta_1(x)\zeta_1''(x)\Gamma(2\alpha+1)}{\Gamma(\alpha+1)^2S^{3\alpha+1}} + \frac{\lambda\,\zeta'''(x)}{S^{\alpha+1}} + \frac{\lambda\,\zeta_1'''(x)}{S^{2\alpha+1}}, \mathrm{s} > 0. \quad (19)$$

Next, multiplying $\boldsymbol{s^{1+\alpha}}$ by two parts of Eq. (19) yields

$$s^{1+\alpha}\mathrm{LRes}_1(\mathrm{x,s}) = \zeta_1(x) + 3\lambda\,\zeta'^2(x) + \frac{6\lambda\,\zeta'(x)\zeta_1'(x)}{S^{\alpha}} - \frac{3\lambda\,\zeta_1'^2(x)\Gamma(2\alpha+1)}{\Gamma(\alpha+1)^2S^{2\alpha}} + 3\lambda\,\zeta'(x)\zeta^2(x) + \frac{3\lambda\,\zeta_1'(x)\zeta^2(x)}{S^{\alpha}} + \frac{6\lambda\,\zeta'(x)\zeta\,(x)\zeta_1(x)}{S^{\alpha}} + \frac{6\lambda\,\zeta_1'(x)\zeta\,(x)\zeta_1(x)\Gamma(2\alpha+1)}{\Gamma(\alpha+1)^2S^{2\alpha}} + \frac{3\lambda\,\zeta'(x)\zeta_1^2(x)\Gamma(2\alpha+1)}{\Gamma(\alpha+1)^2S^{2\alpha}} + \frac{3\lambda\,\zeta_1'(x)\zeta_1^2(x)\Gamma(3\alpha+1)}{\Gamma(\alpha+1)^3S^{3\alpha}} + 3\lambda\,\zeta(x)\zeta''(x) + \frac{3\lambda\,\zeta(x)\zeta_1''(x)}{S^{\alpha}} + \frac{3\lambda\,\zeta_1(x)\zeta''(x)\Gamma(\alpha+1)}{\Gamma(\alpha+1)S^{\alpha}} + \frac{3\lambda\,\zeta_1(x)\zeta_1''(x)\Gamma(2\alpha+1)}{\Gamma(\alpha+1)^2S^{2\alpha}} + \lambda\,\zeta'''(x) + \frac{\lambda\,\zeta_1'''(x)}{S^{\alpha}}, \mathrm{s} > 0\,. \quad (20)$$

Next, utilizing the assumption in Eq. (17) and the limit as $s \to \infty$ from both parts of Eq. (20), we may quickly ascertain the value of $\zeta_1(x)$ via resolving the formula given of $\zeta_1(x)$:

$$0 = s^{1+\alpha}\mathrm{LRes}_1(\mathrm{x,s}) = \zeta_1(x) + 3\lambda\zeta(x)\zeta''(x) + \lambda\zeta'''(x), \mathrm{s} > 0\,. \quad (21)$$

It is simple to get the following by calculating $\zeta_1(x)$ in the ensuing algebraic formula (21).

$$\zeta_1(x) = -\left(3\lambda\zeta(x)\zeta''(x) + \lambda\zeta'''(x)\right),\ \ s > 0. \quad (22)$$

The 2nd-truncated series of Eq. (14), $U_2(x,s)=\frac{\zeta(x)}{s}+\frac{\zeta_1(x)}{s^{1+\alpha}}+\frac{\zeta_2(x)}{s^{1+2\alpha}}$, is substituted into in the 2nd -LRF to calculate the value of the next undetermined parameter $\zeta_2(x)$ as follows:

$$\mathrm{LRes}_2(s)=\frac{\zeta_1(x)}{s^{1+\alpha}}+\frac{3\lambda}{s^\alpha}\left(\mathcal{L}\left\{\left(\mathcal{L}^{-1}\left(\frac{\zeta(x)}{s}+\frac{\zeta_1(x)}{s^{1+\alpha}}+\frac{\zeta_2(x)}{s^{1+2\alpha}}\right)\right)^2\right\}\right)+\frac{3\lambda}{s^\alpha}\left(\mathcal{L}\left\{\left(\mathcal{L}^{-1}\left(\frac{\zeta(x)}{s}+\frac{\zeta_1(x)}{s^{1+\alpha}}+\frac{\zeta_2(x)}{s^{1+2\alpha}}\right)\right)^2\mathcal{L}^{-1}\left(\frac{\zeta'(x)}{s}+\frac{\zeta_1'(x)}{s^{\alpha+1}}+\frac{\zeta_2'(x)}{s^{1+2\alpha}}\right)\right\}\right)+\frac{3\lambda}{s^\alpha}\left(\mathcal{L}\left\{\mathcal{L}^{-1}\left(\frac{\zeta(x)}{s}+\frac{\zeta_1(x)}{s^{1+\alpha}}+\frac{\zeta_2(x)}{s^{1+2\alpha}}\right)\mathcal{L}^{-1}\left(\frac{\zeta''(x)}{s}+\frac{\zeta_1''(x)}{s^{\alpha+1}}+\frac{\zeta_2''(x)}{s^{1+2\alpha}}\right)\right\}\right)+-\frac{\lambda}{s^\alpha}\left(\frac{\zeta'''(x)}{s}+\frac{\zeta_1'''(x)}{s^{\alpha+1}}+\frac{\zeta_2'''(x)}{s^{1+2\alpha}}\right),\ s>0. \quad (23)$$

By running the operator in Eq (18) we can obtain the following simplified form:

$$\mathrm{LRes}_2(x,s)=\frac{\zeta_2(x)}{s^{2\alpha+1}}+\frac{3\lambda\,\zeta_1'^2(x)\Gamma(2\alpha+1)}{\Gamma(\alpha+1)^2s^{3\alpha+1}}+\frac{3\lambda\zeta_2'^2(x)\Gamma(4\alpha+1)}{\Gamma(2\alpha+1)^2s^{5\alpha+1}}+\frac{6\lambda\zeta'(x)\zeta_1'(x)}{s^{2\alpha+1}}+\frac{6\lambda\zeta'(x)\zeta_2'(x)}{s^{3\alpha+1}}+\frac{6\lambda\zeta_1'(x)\zeta_2'(x)\Gamma(3\alpha+1)}{\Gamma(\alpha+1)\Gamma(2\alpha+1)s^{4\alpha+1}}+\frac{3\lambda\,\zeta'(x)\zeta_1^2(x)\Gamma(2\alpha+1)}{\Gamma(\alpha+1)^2s^{3\alpha+1}}+\frac{3\lambda\,\zeta'(x)\zeta_2^2(x)\Gamma(4\alpha+1)}{\Gamma(2\alpha+1)^2s^{5\alpha+1}}+\frac{6\lambda\zeta'(x)\zeta(x)\zeta_1(x)}{s^{2\alpha+1}}+\frac{6\lambda\zeta'(x)\zeta(x)\zeta_2(x)\Gamma(3\alpha+1)}{\Gamma(2\alpha+1)s^{4\alpha+1}}+\frac{6\lambda\zeta'(x)\zeta_1(x)\zeta_2(x)\Gamma(3\alpha+1)}{\Gamma(\alpha+1)\Gamma(2\alpha+1)s^{4\alpha+1}}+\frac{3\lambda\zeta_1'(x)\zeta^2(x)}{s^{2\alpha+1}}+\frac{3\lambda\zeta_1'(x)\zeta_1^2(x)\Gamma(3\alpha+1)}{\Gamma(\alpha+1)^3s^{4\alpha+1}}+\frac{3\lambda\zeta_1'(x)\zeta_2^2(x)\Gamma(5\alpha+1)}{\Gamma(\alpha+1)\Gamma(2\alpha+1)^2s^{6\alpha+1}}+\frac{6\lambda\zeta_1'(x)\,\zeta(x)\zeta_1(x)\Gamma(2\alpha+1)}{\Gamma(\alpha+1)^2s^{3\alpha+1}}+\frac{6\lambda\zeta_1'(x)\zeta(x)\zeta_2(x)\Gamma(3\alpha+1)}{\Gamma(\alpha+1)\Gamma(2\alpha+1)s^{4\alpha+1}}+\frac{6\lambda\zeta_1'(x)\zeta_1(x)\zeta_2(x)\Gamma(4\alpha+1)}{\Gamma(\alpha+1)^2\Gamma(2\alpha+1)s^{5\alpha+1}}+\frac{3\lambda\zeta^2(x)\zeta_2'(x)}{s^{3\alpha+1}}+\frac{3\lambda\zeta_1^2(x)\zeta_2'(x)\Gamma(4\alpha+1)}{\Gamma(2\alpha+1)\Gamma(\alpha+1)^2s^{5\alpha+1}}+\frac{3\lambda\,\zeta_2^2(x)\zeta_2'(x)\Gamma(6\alpha+1)}{\Gamma(2\alpha+1)^3s^{7\alpha+1}}+\frac{6\lambda\,\zeta(x)\zeta_1(x)\zeta_2'(x)\Gamma(3\alpha+1)}{\Gamma(2\alpha+1)\Gamma(\alpha+1)s^{4\alpha+1}}+\frac{6\lambda\,\zeta(x)\zeta_2(x)\zeta_2'(x)\Gamma(4\alpha+1)}{\Gamma(2\alpha+1)^2s^{5\alpha+1}}+\frac{6\lambda\zeta_1(x)\zeta_2(x)\zeta_2'(x)\,\Gamma(5\alpha+1)}{\Gamma(\alpha+1)\Gamma(2\alpha+1)^2s^{6\alpha+1}}+\frac{3\lambda\zeta(x)\zeta_1''(x)}{s^{2\alpha+1}}+\frac{3\lambda\zeta(x)\zeta_2''(x)}{s^{3\alpha+1}}+\frac{3\lambda\zeta_1(x)\zeta''(x)}{s^{2\alpha+1}}+\frac{3\lambda\,\zeta_1(x)\zeta_1''(x)\Gamma(2\alpha+1)}{\Gamma(\alpha+1)^2s^{3\alpha+1}}+\frac{3\lambda\zeta_1(x)\zeta_2''(x)\Gamma(3\alpha+1)}{\Gamma(\alpha+1)\Gamma(2\alpha+1)s^{4\alpha+1}}+\frac{3\lambda\,\zeta_2(x)\zeta''(x)}{s^{3\alpha+1}}+\frac{3\lambda\,\zeta_2(x)\zeta_1''(x)\Gamma(3\alpha+1)}{\Gamma(2\alpha+1)\Gamma(\alpha+1)s^{4\alpha+1}}+\frac{3\lambda\,\zeta_2(x)\zeta''_2(x)\Gamma(4\alpha+1)}{\Gamma(2\alpha+1)^2s^{5\alpha+1}}+\frac{\lambda\,\zeta_1'''(x)}{s^{2\alpha+1}}+\frac{\lambda\,\zeta_2'''(x)}{s^{3\alpha+1}},\ s>0. \quad (24)$$

Next, multiplying $s^{1+2\alpha}$ by two parts of equation (24) yields

$$s^{1+2\alpha}\mathrm{LRes}_2(x,s)=\zeta_2(x)+\frac{3\lambda\,\zeta_1'^2(x)\Gamma(2\alpha+1)}{\Gamma(\alpha+1)^2s^{\alpha}}+\frac{3\lambda\zeta_2'^2(x)\Gamma(4\alpha+1)}{\Gamma(2\alpha+1)^2s^{3\alpha}}+6\lambda\zeta'(x)\zeta_1'(x)+\frac{6\lambda\zeta'(x)\zeta_2'(x)}{s^{\alpha}}+\frac{6\lambda\zeta_1'(x)\zeta_2'(x)\Gamma(3\alpha+1)}{\Gamma(\alpha+1)\Gamma(2\alpha+1)s^{2\alpha}}+\frac{3\lambda\,\zeta'(x)\zeta_1^2(x)\Gamma(2\alpha+1)}{\Gamma(\alpha+1)^2s^{\alpha}}+\frac{3\lambda\,\zeta'(x)\zeta_2^2(x)\Gamma(4\alpha+1)}{\Gamma(2\alpha+1)^2s^{3\alpha}}+6\lambda\zeta'(x)\zeta(x)\zeta_1(x)+\frac{6\lambda\zeta'(x)\zeta(x)\zeta_2(x)\Gamma(3\alpha+1)}{\Gamma(2\alpha+1)s^{2\alpha}}+\frac{6\lambda\zeta'(x)\zeta_1(x)\zeta_2(x)\Gamma(3\alpha+1)}{\Gamma(\alpha+1)\Gamma(2\alpha+1)s^{2\alpha}}+3\lambda\zeta_1'(x)\zeta^2(x)+\frac{3\lambda\zeta_1'(x)\zeta_1^2(x)\Gamma(3\alpha+1)}{\Gamma(\alpha+1)^3s^{2\alpha}}+\frac{3\lambda\zeta_1'(x)\zeta_2^2(x)\Gamma(5\alpha+1)}{\Gamma(\alpha+1)\Gamma(2\alpha+1)^2s^{4\alpha}}+\frac{6\lambda\zeta_1'(x)\,\zeta(x)\zeta_1(x)\Gamma(2\alpha+1)}{\Gamma(\alpha+1)^2s^{\alpha}}+\frac{6\lambda\zeta_1'(x)\zeta(x)\zeta_2(x)\Gamma(3\alpha+1)}{\Gamma(\alpha+1)\Gamma(2\alpha+1)s^{2\alpha}}+\frac{6\lambda\zeta_1'(x)\zeta_1(x)\zeta_2(x)\Gamma(4\alpha+1)}{\Gamma(\alpha+1)^2\Gamma(2\alpha+1)s^{3\alpha}}+\frac{3\lambda\zeta^2(x)\zeta_2'(x)}{s^{\alpha}}+\frac{3\lambda\zeta_1^2(x)\zeta_2'(x)\Gamma(4\alpha+1)}{\Gamma(2\alpha+1)\Gamma(\alpha+1)^2s^{3\alpha}}+\frac{3\lambda\,\zeta_2^2(x)\zeta_2'(x)\Gamma(6\alpha+1)}{\Gamma(2\alpha+1)^3s^{5\alpha}}+\frac{6\lambda\,\zeta(x)\zeta_1(x)\zeta_2'(x)\Gamma(3\alpha+1)}{\Gamma(2\alpha+1)\Gamma(\alpha+1)s^{2\alpha}}+\frac{6\lambda\,\zeta(x)\zeta_2(x)\zeta_2'(x)\Gamma(4\alpha+1)}{\Gamma(2\alpha+1)^2s^{3\alpha}}+\frac{6\lambda\zeta_1(x)\zeta_2(x)\zeta_2'(x)\,\Gamma(5\alpha+1)}{\Gamma(\alpha+1)\Gamma(2\alpha+1)^2s^{4\alpha}}+3\lambda\zeta(x)\zeta_1''(x)+\frac{3\lambda\zeta(x)\zeta_2''(x)}{s^{\alpha}}+3\lambda\zeta_1(x)\zeta''(x)+\frac{3\lambda\,\zeta_1(x)\zeta_1''(x)\Gamma(2\alpha+1)}{\Gamma(\alpha+1)^2s^{\alpha}}+\frac{3\lambda\zeta_1(x)\zeta_2''(x)\Gamma(3\alpha+1)}{\Gamma(\alpha+1)\Gamma(2\alpha+1)s^{2\alpha}}+\frac{3\lambda\,\zeta_2(x)\zeta''(x)}{s^{\alpha}}+\frac{3\lambda\,\zeta_2(x)\zeta_1''(x)\Gamma(3\alpha+1)}{\Gamma(2\alpha+1)\Gamma(\alpha+1)s^{2\alpha}}+\frac{3\lambda\,\zeta_2(x)\zeta''_2(x)\Gamma(4\alpha+1)}{\Gamma(2\alpha+1)^2s^{3\alpha}}+\lambda\,\zeta_1'''(x)+\frac{\lambda\,\zeta_2'''(x)}{s^{\alpha}},\ s>0. \quad (25)$$

To get the below formula, calculate the limit as $s\to\infty$ with both parts of Eq. (25) and then employ Eq. (17).

$$0=\zeta_2(x)+3\lambda\zeta_1(x)\zeta''(x)+3\lambda\zeta(x)\zeta_1''(x)+\lambda\,\zeta_1'''(x),\ s>0. \quad (26)$$

By resolving the algebraic equation that results for $\zeta_2(x)$, we obtain

$$\zeta_2(x)=-(3\lambda\zeta_1(x)\zeta''(x)+3\lambda\zeta(x)\zeta_1''(x)+\lambda\,\zeta_1'''(x)),\ s>0. \quad (27)$$

Similar to a previous stages, replace the 3rd -truncated series of Eq. (17), $U_3(x,s)=\frac{\zeta(x)}{s}+\frac{\zeta_1(x)}{s^{1+\alpha}}+\frac{\zeta_2(x)}{s^{1+2\alpha}}+\frac{\zeta_3(x)}{s^{1+3\alpha}}$ is substituted into in the 3rd -LRF , we get the value of the next undetermined parameter $\zeta_3(x)$ as follows:

$$\zeta_3(x)=-(3\lambda\zeta_2(x)\zeta''(x)+\frac{3\lambda\Gamma(1+2\alpha)\zeta_1(x)\zeta_1''(x)}{\Gamma(1+\alpha)^2}+3\lambda\zeta(x)\zeta_2''(x)+\lambda\zeta_2'''(x))\,,s>0. \quad (28)$$

As a result, we may write the results of Eq. (14) in an infinite series so they are described in the following:

$$U(x,s)=\frac{\zeta(x)}{s}-\frac{\left(3\lambda\zeta(x)\zeta''(x)+\lambda\zeta'''(x)\right)}{s^{1+\alpha}}-\frac{\left(3\lambda\zeta_1(x)\zeta''(x)+3\lambda\zeta(x)\zeta_1''(x)+\lambda\zeta_1'''(x)\right)}{s^{1+2\alpha}}-\frac{(3\lambda\zeta_2(x)\zeta''(x)+\frac{3\lambda\Gamma(1+2\alpha)\zeta_1(x)\zeta_1''(x)}{\Gamma(1+\alpha)^2}+3\lambda\zeta(x)\zeta_2''(x)+\lambda\zeta_2'''(x))}{s^{1+3\alpha}}-\ldots \quad (29)$$

The LRPS solution to Eqs. (8) and (9) is obtained by using the inverse of the Laplace transform of Eq. (29) in the given simple form:

$$u(x,t)=\zeta(x)-\frac{\left(3\lambda\zeta(x)\zeta''(x)+\lambda\zeta'''(x)\right)}{\Gamma(1+\alpha)}t^{\alpha}-\frac{\left(3\lambda\zeta_1(x)\zeta''(x)+3\lambda\zeta(x)\zeta_1''(x)+\lambda\zeta_1'''(x)\right)}{\Gamma(1+2\alpha)}t^{2\alpha}-\frac{(3\lambda\zeta_2(x)\zeta''(x)+\frac{3\lambda\Gamma(1+2\alpha)\zeta_1(x)\zeta_1''(x)}{\Gamma(1+\alpha)^2}+3\lambda\zeta(x)\zeta_2''(x)+\lambda\zeta_2'''(x))}{\Gamma(1+3\alpha)}t^{3\alpha}-\ldots \quad (30)$$

4. Numerical Issues

In this part, we look at the importance of the LRPSM in obtaining the solution to the CT.

Problem 4.1: Take into account the following fractional equation:

$$D_t^{\alpha}u(x,t)+3u_x^2(x,t)+3u^2(x,t)u_x(x,t)+3u(x,t)u_{xx}(x,t)+u_{xxx}(x,t)=0\ t>0,\quad 0<\alpha\leq 1, \quad (31)$$

with the initial condition

$$u(\mathrm{x},0)=\sin\mathrm{x} \quad (32)$$

Using Eq.(32), the Laplace transform is taken to Eq.(31) which gives

$$U(x,s)+\frac{\sin x}{s}+\frac{3}{s^{\alpha}}\left(\mathcal{L}\left\{\left(\mathcal{L}^{-1}\left(U_x(x,s)\right)^2\right)\right\}\right)+\frac{3}{s^{\alpha}}\left(\mathcal{L}\left\{\left(\mathcal{L}^{-1}\left(U(x,s)\right)^2\right)\mathcal{L}^{-1}\left(U_x(x,s)\right)\right\}\right)+\frac{3}{s^{\alpha}}\left(\mathcal{L}\left\{\left(\mathcal{L}^{-1}\left(U(x,s)\right)\right)\mathcal{L}^{-1}\left(U_{xx}(x,s)\right)\right\}\right)+\frac{1}{s^{\alpha}}U_{xxx}(x,s),\ s>0. \quad (33)$$

It is claimed that the kth-truncated series is

$$U_k(x,s)=\frac{\sin x}{s}+\sum_{j=1}^{k}\frac{\zeta_j(x)}{s^{1+\alpha j}}\ ,s>0. \quad (34)$$

Consequently, the kth LRFs are

$$\mathrm{LRes}_k(\mathrm{s})=\frac{\sin x}{s}-\frac{3}{s^{\alpha}}\left(\mathcal{L}\left\{\left(\mathcal{L}^{-1}\left(U_{(k)x}(x,s)\right)^2\right)\right\}\right)-\frac{3}{s^{\alpha}}\left(\mathcal{L}\left\{\left(\mathcal{L}^{-1}\left(U_k(x,s)\right)^2\right)\mathcal{L}^{-1}\left(U_{(k)x}(x,s)\right)\right\}\right)-\frac{3}{s^{\alpha}}\left(\mathcal{L}\left\{\left(\mathcal{L}^{-1}\left(U_k(x,s)\right)\right)\mathcal{L}^{-1}\left(U_{(k)xx}(x,s)\right)\right\}\right)-\frac{1}{s^{\alpha}}U_{(k)xxx}(x,s),\ s>0. \quad (35)$$

The kth-truncated series (34) is now placed into the kth LRF (35) to give $\zeta_j(x)$. After multiplying the resultant formula by $s^{1+\alpha j}$, we may calculate the relationship.

$$\mathrm{Lim}_{s\to\infty}\left(s^{k+1}\mathrm{LRes}_k(s)\right)=0, k=1,2,3,..$$

So, several values include:

$$\zeta_1(x)=\cos[x]-3\sin[x]^2(1+\sin[x]),$$

$$\zeta_2(x) = \frac{1}{64}(8504\cos[x] - \frac{12\Gamma[1+2\alpha](1+3\sin[x])(-6+4\cos[x]+6\cos[2x]-9\sin[x]+3\sin[3x])^2}{\Gamma[1+\alpha]^2} + 3(-4220+3718\cos[2x]-10356\cos[3x]-2436\cos[4x]+5580\cos[5x] + 378\cos[6x]-7707\sin[x]+6752\sin[2x]-8703\sin[3x]-9840\sin[4x]-1107\sin[5x] + 81\sin[7x])),$$

$$\zeta_3(x) = \frac{1}{64}(-8504\sin[x] - \frac{192\Gamma[1+3\alpha](\cos[x]-3\sin[x]^2(1+\sin[x]))^3}{\Gamma[1+\alpha]^3} - (384\Gamma[1+3\alpha](\cos[x] - 3\sin[x]^2(1+\sin[x]))(18\cos[x]^3+\sin[x](-1-3\sin[x]+15\sin[x]^2+45\sin[x]^3+ 27\sin[x]^4)-3(5+12\sin[x])\sin[2x])/\Gamma[1+\alpha]\Gamma[1+2\alpha] - (1152\Gamma[1+3\alpha]\sin[x](\cos[x]-3\sin[x]^2(1+\sin[x]))(18\cos[x]^3+\sin[x](-1- 3\sin[x]+15\sin[x]^2+45\sin[x]^3+27\sin[x]^4)-3(5+12\sin[x])\sin[2x])/\Gamma[1+\alpha]\Gamma[1+ 2\alpha] - (3\Gamma[1+2\alpha](2793\cos[x]-3072\cos[2x]-38313\cos[3x]+12288\cos[4x]+ 46125\cos[5x]-9261\cos[7x]+240\sin[x]+17168\sin[2x]-8424\sin[3x]- 52992\sin[4x]+9000\sin[5x]+27216\sin[6x])/\Gamma[1+\alpha]^2 - 3(7707\cos[x]- 54016\cos[2x]+234981\cos[3x]+629760\cos[4x]+138375\cos[5x]-27783\cos[7x]+ 29744\sin[2x]-279612\sin[3x]-155904\sin[4x]+697500\sin[5x]+81648\sin[6x]) - 6\sin[x](8504\cos[x] - \frac{\Gamma[1+2\alpha](1+3\sin[x])(-6+4\cos[x]+6\cos[2x]-9\sin[x]+3\sin[3x])^2}{\Gamma[1+\alpha]^2} + 3(-4220+ 3718\cos[2x]-10356\cos[3x]-2436\cos[4x]+5580\cos[5x]+378\cos[6x]- 7707\sin[x]+6752\sin[2x]-8703\sin[3x]-9840\sin[4x]-1107\sin[5x]+ 81\sin[7x])) - 9\sin[x]^2(8504\cos[x] - \frac{12\Gamma[1+2\alpha](1+3\sin[x])(-6+4\cos[x]+6\cos[2x]-9\sin[x]+3\sin[3x])^2}{\Gamma[1+\alpha]^2} + 3(-4220+3718\cos[2x]- 10356\cos[3x]-2436\cos[4x]+5580\cos[5x]+378\cos[6x]-7707\sin[x]+ 6752\sin[2x]-8703\sin[3x]-9840\sin[4x]-1107\sin[5x]+81\sin[7x]))), \quad (36)$$

and so on.

As a result, we may write the results of Eq. (34) in an infinite series so they are described in the following:

$$U(x,s) = \frac{sinx}{s} + \frac{(\cos[x]-3\sin[x]^2(1+\sin[x]))}{s^{1+\alpha}} + \left(\frac{1}{64}(8504\cos[x] - \frac{12\Gamma[1+2\alpha](1+3\sin[x])(-6+4\cos[x]+6\cos[2x]-9\sin[x]+3\sin[3x])^2}{\Gamma[1+\alpha]^2} + 3(-4220+ 3718\cos[2x]-10356\cos[3x]-2436\cos[4x]+5580\cos[5x]+378\cos[6x]- 7707\sin[x]+6752\sin[2x]-8703\sin[3x]-9840\sin[4x]-1107\sin[5x]+ 81\sin[7x]))\right)/s^{1+2\alpha} + \left(\frac{1}{64}(-8504\sin[x] - \frac{192\Gamma[1+3\alpha](\cos[x]-3\sin[x]^2(1+\sin[x]))^3}{\Gamma[1+\alpha]^3} - (384\Gamma[1+3\alpha](\cos[x]-3\sin[x]^2(1+\sin[x]))(18\cos[x]^3+\sin[x](-1-3\sin[x]+ 15\sin[x]^2+45\sin[x]^3+27\sin[x]^4)-3(5+12\sin[x])\sin[2x])/\Gamma[1+\alpha]\Gamma[1+2\alpha] - (1152\Gamma[1+3\alpha]\sin[x](\cos[x]-3\sin[x]^2(1+\sin[x]))(18\cos[x]^3+\sin[x](-1- 3\sin[x]+15\sin[x]^2+45\sin[x]^3+27\sin[x]^4)-3(5+12\sin[x])\sin[2x])/\Gamma[1+\alpha]\Gamma[1+ 2\alpha] - (3\Gamma[1+2\alpha](2793\cos[x]-3072\cos[2x]-38313\cos[3x]+12288\cos[4x]+ 46125\cos[5x]-9261\cos[7x]+240\sin[x]+17168\sin[2x]-8424\sin[3x]- 52992\sin[4x]+9000\sin[5x]+27216\sin[6x])/\Gamma[1+\alpha]^2 - 3(7707\cos[x]- 54016\cos[2x]+234981\cos[3x]+629760\cos[4x]+138375\cos[5x]-27783\cos[7x]+ 29744\sin[2x]-279612\sin[3x]-155904\sin[4x]+697500\sin[5x]+81648\sin[6x]) -$$

$6\sin[x](8504\cos[x]-\frac{12\text{gamma}[1+2\alpha](1+3\sin[x])(-6+4\cos[x]+6\cos[2x]-9\sin[x]+3\sin[3x])^2}{\Gamma[1+\alpha]^2}+3(-4220+3718\cos[2x]-10356\cos[3x]-2436\cos[4x]+5580\cos[5x]+378\cos[6x]-7707\sin[x]+6752\sin[2x]-8703\sin[3x]-9840\sin[4x]-1107\sin[5x]+81\sin[7x]))-9\sin[x]^2(8504\cos[x]-\frac{12\Gamma[1+2\alpha](1+3\sin[x])(-6+4\cos[x]+6\cos[2x]-9\sin[x]+3\sin[3x])^2}{\Gamma[1+\alpha]^2}+3(-4220+3718\cos[2x]-10356\cos[3x]-2436\cos[4x]+5580\cos[5x]+378\cos[6x]-7707\sin[x]+6752\sin[2x]-8703\sin[3x]-9840\sin[4x]-1107\sin[5x]+81\sin[7x])))\Big)/s^{1+3\alpha}+\ldots$ (37)

If we calculate LT's inverse, we obtain

$sinx+\frac{(\cos[x]-3\sin[x]^2(1+\sin[x]))}{\Gamma[1+\alpha]}t^{\alpha}+\Big(\frac{1}{64}(8504\cos[x]-\frac{12\Gamma[1+2\alpha](1+3\sin[x])(-6+4\cos[x]+6\cos[2x]-9\sin[x]+3\sin[3x])^2}{\Gamma[1+\alpha]^2}+3(-4220+3718\cos[2x]-10356\cos[3x]-2436\cos[4x]+5580\cos[5x]+378\cos[6x]-7707\sin[x]+6752\sin[2x]-8703\sin[3x]-9840\sin[4x]-1107\sin[5x]+81\sin[7x]))\Big)t^{2\alpha}/\Gamma[1+2\alpha]+\Big(\frac{1}{64}(-8504\sin[x]-\frac{192\Gamma[1+3\alpha](\cos[x]-3\sin[x]^2(1+\sin[x]))^3}{\Gamma[1+\alpha]^3}-(384\Gamma[1+3\alpha](\cos[x]-3\sin[x]^2(1+\sin[x]))(18\cos[x]^3+\sin[x](-1-3\sin[x]+15\sin[x]^2+45\sin[x]^3+27\sin[x]^4)-3(5+12\sin[x])\sin[2x])/\Gamma[1+\alpha]\Gamma[1+2\alpha]-(1152\Gamma[1+3\alpha]\sin[x](\cos[x]-3\sin[x]^2(1+\sin[x]))(18\cos[x]^3+\sin[x](-1-3\sin[x]+15\sin[x]^2+45\sin[x]^3+27\sin[x]^4)-3(5+12\sin[x])\sin[2x])/\Gamma[1+\alpha]\Gamma[1+2\alpha]-(3\Gamma[1+2\alpha](2793\cos[x]-3072\cos[2x]-38313\cos[3x]+12288\cos[4x]+46125\cos[5x]-9261\cos[7x]+240\sin[x]+17168\sin[2x]-8424\sin[3x]-52992\sin[4x]+9000\sin[5x]+27216\sin[6x])/\Gamma[1+\alpha]^2-3(7707\cos[x]-54016\cos[2x]+234981\cos[3x]+629760\cos[4x]+138375\cos[5x]-27783\cos[7x]+29744\sin[2x]-279612\sin[3x]-155904\sin[4x]+697500\sin[5x]+81648\sin[6x])-6\sin[x](8504\cos[x]-\frac{12\text{gamma}[1+2\alpha](1+3\sin[x])(-6+4\cos[x]+6\cos[2x]-9\sin[x]+3\sin[3x])^2}{\Gamma[1+\alpha]^2}+3(-4220+3718\cos[2x]-10356\cos[3x]-2436\cos[4x]+5580\cos[5x]+378\cos[6x]-7707\sin[x]+6752\sin[2x]-8703\sin[3x]-9840\sin[4x]-1107\sin[5x]+81\sin[7x]))-9\sin[x]^2(8504\cos[x]-\frac{12\Gamma[1+2\alpha](1+3\sin[x])(-6+4\cos[x]+6\cos[2x]-9\sin[x]+3\sin[3x])^2}{\Gamma[1+\alpha]^2}+3(-4220+3718\cos[2x]-10356\cos[3x]-2436\cos[4x]+5580\cos[5x]+378\cos[6x]-7707\sin[x]+6752\sin[2x]-8703\sin[3x]-9840\sin[4x]-1107\sin[5x]+81\sin[7x])))\Big)t^{3\alpha}/\Gamma[1+3\alpha]-\ldots$ (38)

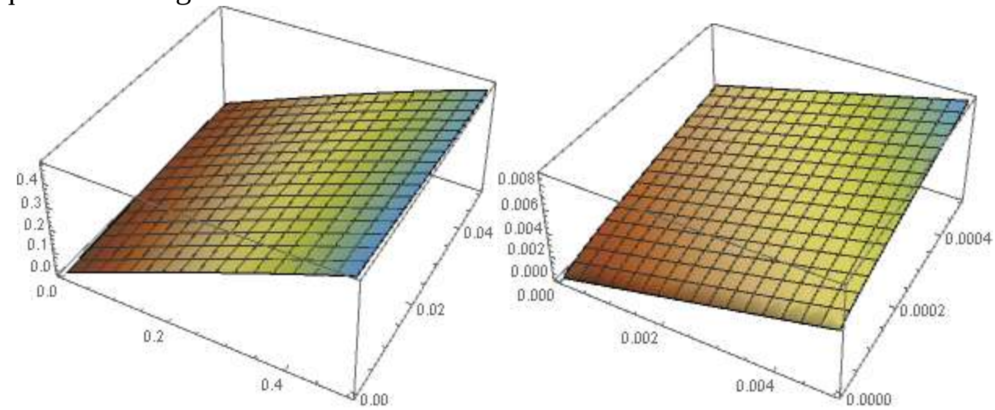
Since we cannot predict the pattern in the coefficients of the series solution in Eq. (38), we cannot reach the exact solution. Therefore, we test the results using the residual and relative errors which are defined as follows, respectively:

$$\text{Res. Err}(x,t)=|D_t^{\alpha}u(x,t)+3u_x^2(x,t)+3u^2(x,t)u_x(x,t)+3u(x,t)u_{xx}(x,t)+u_{xxx}(x,t)| \quad (39)$$

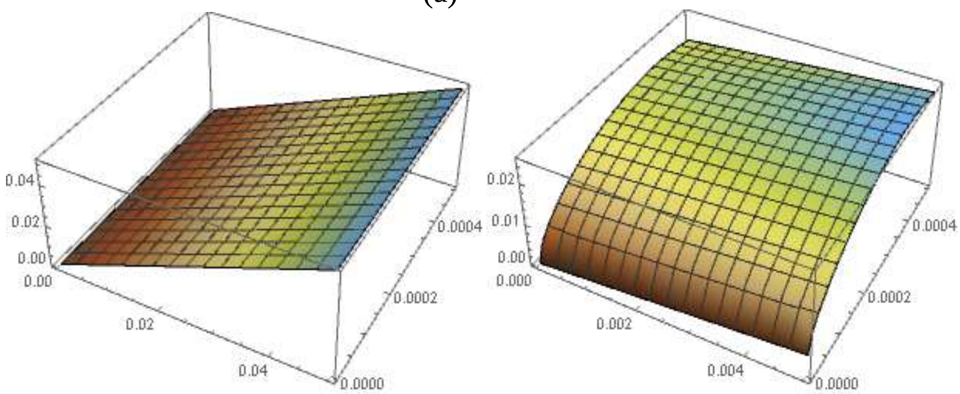
$$\text{Rel. Err}(x,t)=\left|\frac{u(x,t)-u_5(x,t)}{u(x,t)}\right|. \quad (40)$$

The graphs of the 5th approximation to Eqs. (31) and (32) in the range $(0,\infty)\times[0,1]$ is shown in Figure 4.1 a, b, c, d and e. The graph shows that the solutions to the initial value problems Eqs. (31) and (32) are strictly decreasing throughout the region.

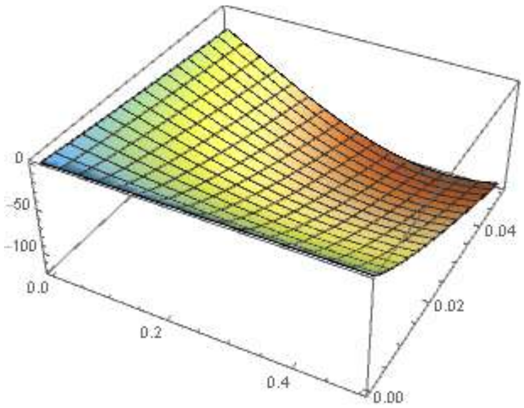
Tables 4.1, 4.2 and 4.3 present the numerical solutions to this issue. In additional to a residual and relative error at various values of α inside the range $(0,\infty)\times[0,1]$, it also shows the fifth approximate result. The outcomes show that the LRPS approach is a successful numerical technique for finding solutions to a non-linear FSTOF.



(a) (b)



(c) (d)



(e)

FIGURE 4.1. The graphs of Eqs. (31) at various values 0f α: (a) $\alpha = 1$, (b) $\alpha = 0.75$, (c) $\alpha = 0.90$, (d) $\alpha = 0.50$, (e) $\alpha = 0.25$.

Table 4.1 Numerical comparisons between the 5th-approximation of $u_5(x,t)$ and the residual error of $u(x,t)$ at $\alpha = 1$.

x	t	$u_5(x,t) -$ approximation	Res. Err. (x,t)	Rel. Err (x,t)
0.1	0.001	0.1008024100657677	0.10080241227202406	2.18869×10^{-8}
	0.002	0.10178477710131464	0.1017848127673913	3.50407×10^{-7}
	0.003	0.10277991314907793	0.10278009556134066	1.77478×10^{-6}
	0.004	0.10378715571384853	0.10378773808228293	5.61115×10^{-6}

0.3	0.001	0.29613668126581016	0.296136682585044	4.45481×10^{-9}
	0.002	0.2967536031980547	0.29675362378039	6.93583×10^{-8}
	0.003	0.2973699427316904	0.29737004426989455	3.41454×10^{-7}
	0.004	0.2979846455717039	0.29798495807607495	1.04873×10^{-6}
0.5	0.001	0.4792771112073483	0.47927711005871126	2.3966×10^{-9}
	0.002	0.479116299676806	0.4791162802561786	4.05343×10^{-8}
	0.003	0.478942185938138	0.4789420823438775	2.16298×10^{-7}
	0.004	0.47875389355708264	0.4787535494690805	7.18716×10^{-7}

Table 4.2: Numerical comparisons between the 5th-approximation of $u_5(x,t)$ and the residual error of $u(x,t)$ at $\alpha = 0.75$.

x	t	$u_5(x,t)$ − approximation	Res. Err. (x, t)	Rel. Err (x, t)
0.01	0.0001	0.010261580011503615	0.010261580025246643	1.33927×10^{-9}
	0.0002	0.010489340778599154	0.010489340946149237	1.59734×10^{-8}
	0.0003	0.01070637813888263	0.010706378863811426	6.771×10^{-8}
	0.0004	0.010917001167313953	0.010917003219477226	1.87979×10^{-7}
0.03	0.0001	0.03025646884584323	0.030256468860201054	4.74537×10^{-10}
	0.0002	0.030483512084002906	0.030483512258949064	5.73904×10^{-9}
	0.0003	0.030699830191552552	0.030699830948077888	2.46427×10^{-8}
	0.0004	0.030909721730315337	0.030909723870830325	6.92505×10^{-8}
0.05	0.0001	0.05023857058835623	0.05023857060322205	2.95905×10^{-10}
	0.0002	0.05046420794288065	0.05046420812391567	3.58739×10^{-9}
	0.0003	0.05067914951951529	0.0506791503019603	1.54392×10^{-8}
	0.0004	0.05088767047890543	0.0508876726916448	4.34828×10^{-8}

Table 4.3: Numerical comparisons between the 5th-approximation of $u_5(x,t)$ and the residual error of $u(x,t)$ at $\alpha = 0.90$.

x	t	$u_5(x,t)$ − approximation	Res. Err. (x, t)	Rel. Err (x, t)
0.01	0.0001	0.0021014848886265034	0.0021014926116914505	7.10807×10^{-7}
	0.0002	0.011866180914045356	0.011866245507276036	5.44344×10^{-6}
	0.0003	0.01254695107844321	0.012547173428538533	1.77211×10^{-5}
	0.0004	0.013179935736770773	0.01318047225373584	4.07054×10^{-5}
0.03	0.0001	0.0310926946658592	0.031092702886594357	2.64394×10^{-7}
	0.0002	0.03185497519483403	0.03185504233527371	5.44344×10^{-6}
	0.0003	0.03253244993321143	0.032532680599232776	7.09029×10^{-6}
	0.0004	0.033161968083111136	0.0331625237004097	1.67544×10^{-5}
0.05	0.0001	0.05106934683585255	0.05106935532468415	1.66222×10^{-7}
	0.0002	0.051826095485863886	0.0518261646564987	1.33467×10^{-6}
	0.0003	0.05249817780277047	0.052498414981692325	4.51783×10^{-6}
	0.0004	0.053122267642921946	0.053122837960231775	1.07358×10^{-5}

Problem 4.2: Take into account the fractional equation below:

$$D_t^{\alpha}u(x,t) + 3u_x^2(x,t) + 3u^2(x,t)u_x(x,t) + 3u(x,t)u_{xx}(x,t) + u_{xxx}(x,t) = 0$$

$$t > 0,\ 0 < \alpha \leq 1. \qquad (41)$$

The initial condition is as follows:

$$u(\mathrm{x},0)=\mathrm{e}^{x} \quad (42)$$

Using (42), the Laplace transform is taken to (41) which gives

$$U(x,s)+\frac{e^{x}}{s}+\frac{3}{s^{\alpha}}\left(\mathcal{L}\left\{\left(\mathcal{L}^{-1}\left(U_x(x,s)\right)^2\right)\right\}\right)+\frac{3}{s^{\alpha}}\left(\mathcal{L}\left\{\left(\mathcal{L}^{-1}\left(U(x,s)\right)^2\right)\mathcal{L}^{-1}\left(U_x(x,s)\right)\right\}\right)+\frac{3}{s^{\alpha}}\left(\mathcal{L}\left\{\left(\mathcal{L}^{-1}\left(U(x,s)\right)\right)\mathcal{L}^{-1}\left(U_{xx}(x,s)\right)\right\}\right)+\frac{1}{s^{\alpha}}U_{xxx}(x,s),\ s>0. \quad (43)$$

It is claimed that the kth-truncated series is

$$U_k(x,s)=\frac{e^{x}}{s}+\sum_{j=1}^{k}\frac{\zeta_j(x)}{s^{1+\alpha j}}\ ,s>0. \quad (44)$$

Consequently, the kth LRFs are

$$\mathrm{LRes}_k(s)=\frac{e^{x}}{s}-\frac{3}{s^{\alpha}}\left(\mathcal{L}\left\{\left(\mathcal{L}^{-1}\left(U_{(k)x}(x,s)\right)^2\right)\right\}\right)-\frac{3}{s^{\alpha}}\left(\mathcal{L}\left\{\left(\mathcal{L}^{-1}\left(U_k(x,s)\right)^2\right)\mathcal{L}^{-1}\left(U_{(k)x}(x,s)\right)\right\}\right)-\frac{3}{s^{\alpha}}\left(\mathcal{L}\left\{\left(\mathcal{L}^{-1}\left(U_k(x,s)\right)\right)\mathcal{L}^{-1}\left(U_{(k)xx}(x,s)\right)\right\}\right)-\frac{1}{s^{\alpha}}U_{(k)xxx}(x,s),\ s>0. \quad (45)$$

The kth-truncated series (44) is now placed into the kth LRF (45) to give $\zeta_j(x)$. After multiplying the resultant formula by $s^{1+\alpha j}$, we may calculate the relationship.

$$\mathrm{Lim}_{s\to\infty}\left(s^{k+1}\mathrm{LRes}_k(s)\right)=0,k=1,2,3,..$$

So, several values include:

$$\zeta_1(x)=e^{x}(1+3e^{x}+3e^{2x}),$$

$$\zeta_2(x)=e^{x}(1+27e^{x}+105e^{2x}+45e^{3x}+27e^{4x})$$

$$\zeta_3(x)=e^{x}(1+222e^{x}+3006e^{2x}+3753e^{3x}+4590e^{4x}+567e^{5x}+243e^{6x})+\frac{3e^{2x}(1+3e^{x})(1+3e^{x}+3e^{2x})^2\Gamma[1+2\alpha]}{\Gamma[1+\alpha]^2},$$

(46)

and so on.

As a result, we may write the results of Eq. (44) in an infinite series, so they are described in the following:

$$U(x,s)=\frac{e^{x}}{s}+\frac{\left(e^{x}(1+3e^{x}+3e^{2x})\right)}{s^{1+\alpha}}+\frac{\left(e^{x}(1+3e^{x}+3e^{2x})\right)}{s^{1+2\alpha}}+\frac{\left(e^{x}(1+222e^{x}+3006e^{2x}+3753e^{3x}+4590e^{4x}+567e^{5x}+243e^{6x})+\frac{3e^{2x}(1+3e^{x})(1+3e^{x}+3e^{2x})^2\mathrm{Gamma}[1+2\alpha]}{\mathrm{Gamma}[1+\alpha]^2}\right)}{s^{1+3\alpha}}+\ldots \quad (47)$$

If we calculate LT's inverse, we obtain

$$e^{x}+\frac{\left(e^{x}(1+3e^{x}+3e^{2x})\right)}{\Gamma[1+\alpha]}t^{\alpha}+\frac{\left(e^{x}(1+3e^{x}+3e^{2x})\right)}{\Gamma[1+2\alpha]}t^{2\alpha}+\frac{\left(e^{x}(1+222e^{x}+3006e^{2x}+3753e^{3x}+4590e^{4x}+567e^{5x}+243e^{6x})+\frac{3e^{2x}(1+3e^{x})(1+3e^{x}+3e^{2x})^2\mathrm{Gamma}[1+2\alpha]}{\mathrm{Gamma}[1+\alpha]^2}\right)}{\Gamma[1+3\alpha]}t^{3\alpha}+\ldots \quad (48)$$

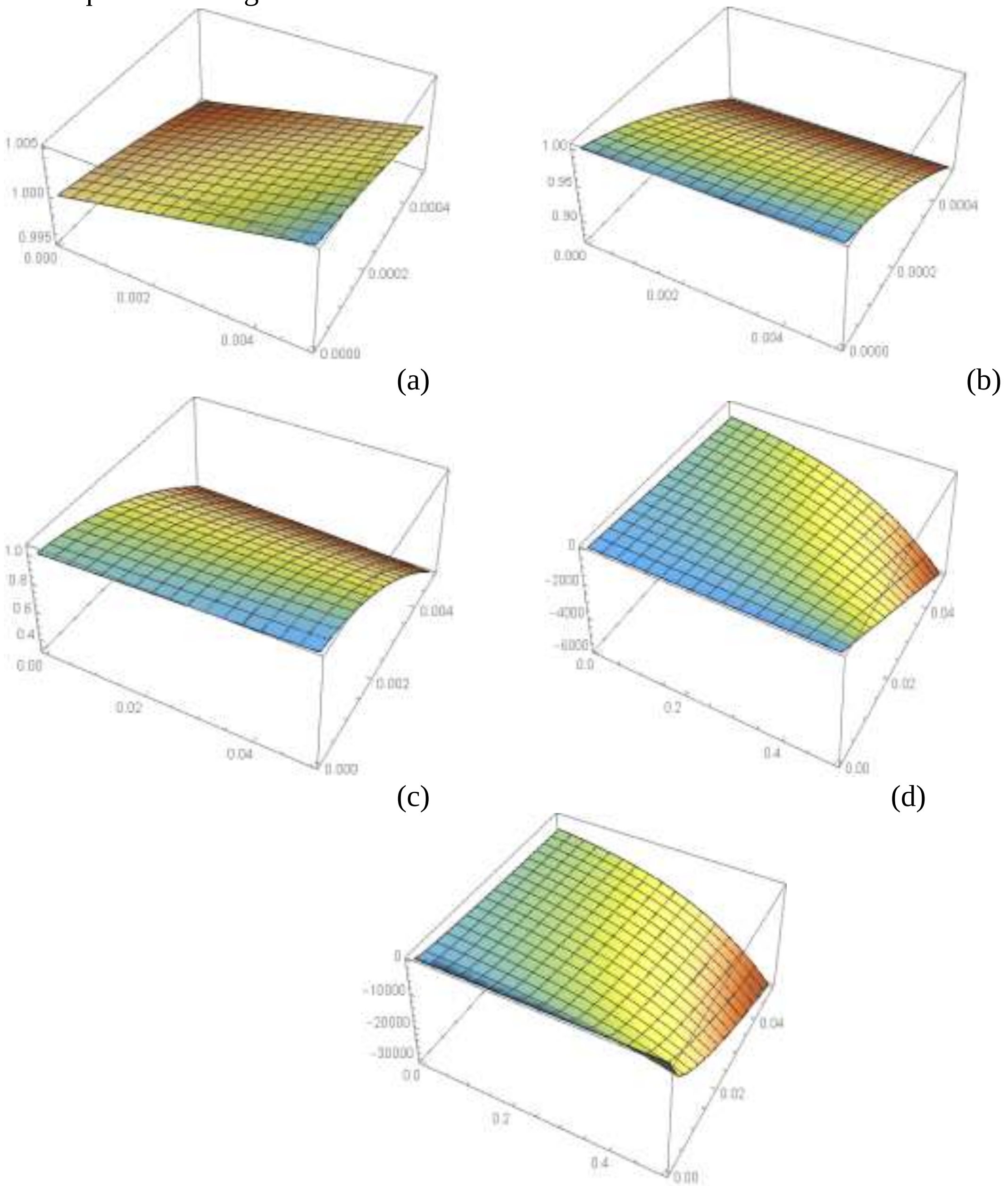
Since we cannot predict the pattern in the coefficients of the series solution in Eq. (48), we cannot reach the exact solution. Therefore, we test the results using the residual and relative errors which are defined as follows, respectively:

$$\mathrm{Res.Err}(x,t)=\left|\mathcal{L}^{-1}[LRes_5(x,s)]\right|=\left|D_t^{\alpha}u(x,t)+3u_x^2(x,t)+3u^2(x,t)u_x(x,t)+3u(x,t)u_{xx}(x,t)+u_{xxx}(x,t)\right| \quad (49)$$

$$\text{Rel.Err}(x,t) = \left|\frac{u_5(x,t)-u_3(x,t)}{u_5(x,t)}\right|. \quad (50)$$

The graphs of the 5th approximation to the (41) and (42) in the range $(0,\infty)\times[0,1]$ is shown in Figure 4.2 a, b, c, d and e. The graph shows that the IVP solutions (41) and (42) are strictly decreasing throughout the region.

Tables 4.4, 4.5 and 4.6 present the numerical solutions to this issue. In addition to a residual and relative error at various values of α inside the range $(0,\infty)\times[0,1]$, it also shows the 5th approximate result. The outcomes show that the LRPS approach is a successful numerical technique for finding solutions to a non-linear FSTOF.



(a) (b) (c) (d) (e)

FIGURE 4.2. The graphs of Eqs. (31) at various values 0f α: (a) $\alpha = 1$, (b) $\alpha = 0.75$, (c) $\alpha = 0.90$, (d) $\alpha = 0.50$, (e) $\alpha =\ 0.25$.

Table 4.4 Numerical comparisons between the 5th-approximation of $u_5(x,t)$ and the residual error of $u(x,t)$ at $\alpha = 1$.

x	t	$u_5(x,t) - $ approximation	Res. Err. (x,t)	Rel. Err (x,t)
0.1	0.001	1.086842304479863	1.0868421947511182	1.00961×10^{-7}
	0.002	1.0494942738181958	1.04949259346972	1.60111×10^{-6}
	0.003	0.993128225923869	0.9931201004238852	8.18179×10^{-6}
	0.004	0.917747177415517	0.9177227018060459	2.66699×10^{-5}
0.3	0.001	1.3125404394313007	1.312540111435795	2.49894×10^{-7}
	0.002	1.2289749781897146	1.2289700021668297	4.04894×10^{-6}
	0.003	1.099166189465696	1.099142374869939	2.16665×10^{-5}
	0.004	0.9231220400455659	0.9230511246459557	7.68272×10^{-5}
0.5	0.001	1.5668897614429629	1.5668887259810784`	6.60839×10^{-7}
	0.002	1.3678910212054012	1.3678755041761785`	1.13439×10^{-5}
	0.003	1.0517355594207647	1.0516623219125292`	6.96398×10^{-5}
	0.004	0.6184445567350575	0.6182298958172303`	3.47219×10^{-4}

Table 4.5: Numerical comparisons between the 5th-approximation of $u_5(x,t)$ and the residual error of $u(x,t)$ at $\alpha = 0.75$

x	t	$u_5(x,t) - $ approximation	Res. Err. (x,t)	Rel. Err (x,t)
0.001	0.0001	0.9840124910097807	0.9840122508549133	2.44057×10^{-7}
	0.0002	0.9617048780535125	0.9617030485695418	1.90234×10^{-6}
	0.0003	0.9349957869258534	0.9349898839016328	6.31346×10^{-6}
	0.0004	0.9045817749066819	0.9045683735376931	1.48152×10^{-5}
0.003	0.0001	0.985901761671138	0.9859015190316439	2.46153×10^{-7}
	0.0002	0.9634244945608885	0.9634226463122161	1.91842×10^{-6}
	0.0003	0.936506999907474	0.9364997369267588	6.3674×10^{-6}
	0.0004	0.905849351659414	0.9058353986447544	1.49437×10^{-5}
0.005	0.0001	0.9877941515233464	0.9877939063723574	2.4818×10^{-7}
	0.0002	0.965147990402824	0.9651439318259996	1.93465×10^{-6}
	0.0003	0.938014978240953	0.9380094740838009	6.42183×10^{-6}
	0.0004	0.90711872893738	0.9071001998019198	1.50734×10^{-5}

Table 4.6: Numerical comparisons between the 5th-approximation of $u_5(x,t)$ and the residual error of $u(x,t)$ at $\alpha = 0.90$.

x	T	$u_5(x,t) - $ approximation	Res. Err. (x,t)	Rel. Err (x,t)
0.001	0.0001	0.9987001061955921	0.9987001057291748	4.67024×10^{-10}
	0.0002	0.995951471022432	0.9959514654257643	5.61942×10^{-9}
	0.0003	0.992691270743497	0.9926941032240939	2.40258×10^{-8}
	0.0004	0.988949248482837	0.9889448583192284	6.72728×10^{-8}
	0.0001	1.0006917215912132	1.0006917211199005	4.70987×10^{-10}
	0.0002	0.9979259091911034	0.9979259035357918	5.66707×10^{-9}

0.003	0.0003	0.994647318273654	0.9946467077276158	2.42295×10^{-8}
	0.0004	0.990873047683787	0.990871237544524	6.78432×10^{-8}
0.005	0.0001	1.002672709957022	1.0026872705194398	4.74986×10^{-10}
	0.0002	0.999901558040727	0.9999041500894743	5.71515×10^{-9}
	0.0003	0.996609760930344	0.9966029517410601	2.4435×10^{-8}
	0.0004	0.9928011148451746	0.9928010469189296	6.84188×10^{-8}

5. CONCLUSION

In this paper, LRPSM has been effectively used to obtain the result of the fractional Sharma-Tasso-Olever equation. From the results obtained from the tables and graphs, we have discovered that LRPSM is very efficient and also more accurate in solving fractional-order differential equations, such as the Sharma-Tasso-Oliver equation. Thus, we can conclude that the LRPS approach is a very effective and sophisticated method for determining the approximate as well as analytical solution to many partial mathematical models that arise in various scientific fields [36-39].

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