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Lie Ideals with Multiplicative Homomultipliers on Near-Rings

Enaam Farhan
Department of Supervisory Specialization, Directorate General of Education in AL-Qadisiyah, AL-Qadisiyah, Iraq

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Abstract
In this study, the new types of mappings on the rings and near-rings are defined, which are named multiplicative homoleft multipliers, multiplicative homoright multipliers and multiplicative homo multipliers. We also present examples that show the existence of such mappings. Moreover, we study how these mappings effected on the construction of near-rings.

Keywords: 3-prime near ring, Lie ideal, multiplicative homoleft multipliers, multiplicative homoright multipliers multiplicative homo multipliers.

مثاليات لي مع المضاعفات التشاكلية الضربية على الحلقات المقتربة

انعام فرحان
قسم الاشراف الاختصاص. المديرية العامة لتربية القادسية. القادسية. العراق

الخلاصة
في هذه الدراسة، قدمنا أنواعا جديدة من الدوال على الحلقات والحلقات المقتربة، والتي تسمى المضاعفات التشاكلية اليسارية الضربية والمضاعفات التشاكلية اليسارية الضربية والمضاعفات التشاكلية الضربية ، ونقدم أيضًا أمثلة التي تظهر وجود مثل هذه الدوال. علاوة على ذلك، فإننا نبحث في كيفية تاثير هذه الدوال على البنية الجبرية للحلقات المقتربة.

1. Introduction

An algebraic structure known as a left near-ring can be defined as follows: Let $(K,+,\cdot)$ be a non-empty set with two binary operations, addition $(+)$ and multiplication $(\cdot)$, such that $(K,+)$ is a group (addition not necessarily abelian) and $(K,\cdot)$ is a semigroup. Addition achy, $a.(b+c) = a.b + a.c$ for each $a,b,c \in K$ (satisfying the left distributive law). If $(a+b).c = a.c + b.c$ is referred to a right near-ring. A left near-ring K is said to be zero symmetric if $0a = 0$ for every $a \in K$, and K is considered to be 3-prime near-ring if $aKb = \{0\}$ implies $a = 0$ *or* $b = 0$ for each $a, b \in K$. Moreover, K considered to be semi prime if $a \in K$ and $aKa = \{0\}$ implies $a = 0$. Furthermore, K is said to be 2-torsion free if $2a = 0$ implies $a = 0$ for each $a \in K$. The Jordan product and Lie product are denoted by $(a \circ b) = ab + ba$ and $[a,b] = ab - ba$, respectively, where $a, b \in K$. An additive subgroup U of K

*Email: enaamfa9@gmail.com

is called Lie ideal if $[w, \pi] \in U$ for each each $w \in U, \pi \in K$. For more information on near-rings, see [1].

In this article, K will refer to a zero symmetric left near-ring with the multiplicative center $\mathcal{Z}(K)$.

The commutativity theorem of 3-prime near-rings has been studied by various authors using different types of mappings, such as derivations, generalized derivations, right derivation, left derivation, multipliers, homoderivations, and generalized homoderivations, that satisfy certain conditions. For more see the references [2]- [11], where more references can be found. A new mapping on rings named by homoderivation was introduced in [8] by El Sofy, this concept includes both derivation and homomorphism. A homoderivation on a ring K is defined as an additive map G from K into itself such that $G(rq) = G(r)G(q) + G(r)q + rG(q)$ for each $r, q \in K$. The commutativity of prime rings with some algebraic conditions with a homoderivation has been studied. After that in [9]-[11], Boua et al studied the structure of near-rings with homoderivations and generalized homoderivations, which satisfy some algebraic identities. Motivated by these studies of El Sofy and Boua on homoderivations, we thought of merging homomorphism with multiplier (without additively condition) into one concept, we named it by *multiplicative homomultiplier*, and we study how these mappings effected the construction of rings and near-rings.

Definition 1. A mapping Γ from K into itself, is said to be a multiplicative homoright (or, homoleft) multiplier of K, if $\Gamma(rq) = \Gamma(r)\Gamma(q) + r\Gamma(q)$(or, $\Gamma(rq) = \Gamma(r)\Gamma(q) + \Gamma(r)q$) for each $r, q \in R$. If Γ is both multiplicative homoright multiplier of K and multiplicative homoleft multiplier of K, then Γ will be named by a multiplicative homomultiplier of K.
The example below demonstrates the existence of such ring mappings.

Example 1. Let B be a ring such that $Char\ B \neq 2$. Define
$R = \left\{ \begin{pmatrix} 0 & a & b \\ 0 & 0 & \mathrm{c} \\ 0 & 0 & 0 \end{pmatrix}, a, b, \mathrm{c}, 0 \in B \right\}$, and $\Gamma_1, \Gamma_2, \Gamma_3 \colon R \longrightarrow R$, such that
$\Gamma_1 \begin{pmatrix} 0 & a & b \\ 0 & 0 & \mathrm{c} \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & a & a \\ 0 & 0 & -\mathrm{c} \\ 0 & 0 & 0 \end{pmatrix}$, $\Gamma_2 \begin{pmatrix} 0 & a & b \\ 0 & 0 & \mathrm{c} \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -a & a \\ 0 & 0 & \mathrm{c} \\ 0 & 0 & 0 \end{pmatrix}$,
$\Gamma_3 \begin{pmatrix} 0 & a & b \\ 0 & 0 & \mathrm{c} \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -a & a \\ 0 & 0 & -\mathrm{c} \\ 0 & 0 & 0 \end{pmatrix}$.

We can easily show that K is a ring, Γ_1 is a multiplicative homoleft multiplier of R which is niether a homomorphism, nor a left multiplier. The map Γ_2 is a multiplicative homoright multiplier of R, which is neither a homomorphism nor right multiplier, while Γ_3 is a multiplicative homo multiplier of R (is both a multiplicative homoright multiplier and a multiplicative homoleft multiplier) which is neither a homomorphism nor a multiplier.

We started by studying these new concepts on rings and we had to an important conclusion in Lemma 4, that is a semi prime ring admits no non-zero multiplicative homoleft (nor, homoright) multiplier. This was a pivotal point in our research and a starting point for studying these mappings on the near-rings, in particular studying the commutativity of addition and we saw that Lie ideal was the most appropriate line of work in this field and Example 2 indicates the existence of such mappings on near-rings.

Example 2. Let T be a left near-ring, such that $(T,+)$ is not abelian, and $Char(T) \neq 2$. Define
$\mathrm{D} = \left\{ \begin{pmatrix} 0 & a & b \\ 0 & 0 & \mathrm{c} \\ 0 & 0 & 0 \end{pmatrix}, a, b, \mathrm{c}, 0 \in T \right\}$, and $\Gamma_1, \Gamma_2, \Gamma_3 \colon \mathrm{D} \longrightarrow \mathrm{D}$, such that
$\Gamma_1 \begin{pmatrix} 0 & a & b \\ 0 & 0 & \mathrm{c} \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & a \\ 0 & 0 & -\mathrm{c} \\ 0 & 0 & 0 \end{pmatrix}$, $\Gamma_2 \begin{pmatrix} 0 & a & b \\ 0 & 0 & \mathrm{c} \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -a & \mathrm{c} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$,
$\Gamma_3 \begin{pmatrix} 0 & a & b \\ 0 & 0 & \mathrm{c} \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & -\mathrm{c} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$.

We can easily show that D is a left near-ring and $(\mathrm{D},+)$ is not abelian, Γ_1 is a multiplicative homoleft multiplier of D which is neither a homomorphism nor a left multiplier, Γ_2 is a homoright multiplier of D, which is neither a homomorphism nor right multiplier, while Γ_3 is a multiplicative homo multiplier of D (both a multiplicative homoright multiplier and a multiplicative homoleft multiplier) which is neither a homomorphism, nor a multiplier. Moreover, each of Γ_1, Γ_2 and Γ_3 are not additive mappings on D.

In Lemma 7 we show that if K be a semi prime, then K has no nonzero homomultiplier and we reached to new and very interesting results in Theorem 1, including that: if Γ is a multiplicative homoleft (or, homoright) multiplier on 3-prime near-ring K, so the zero element is the only element whose image under Γ is zero. Furthermore, there is no non-zero multiplicative homoleft (or, homoright) multiplier satisfying $\Gamma(U) \subseteq \mathcal{Z}(K)$, where U is a non-zero Lie ideal of K.

2. Preliminaries

Lemma 1: [2] Let K be a 3-prime. If $\mathrm{y} \in \mathcal{Z}(K)$ and r is any element of K with $r\mathrm{y}$ or $\mathrm{y}r \in \mathcal{Z}(K)$, then $r \in \mathcal{Z}(K)$.

Lemma 2: [3] Let K be a 3-prime and U be a non-zero Lie ideal of K.
(i) If $rU = \{0\}$, then $r = 0$.
(ii) If $U \subseteq \mathcal{Z}(K)$, then $(K,+)$ is abelian.

Lemma 3: Let K be a 3-prime and U be a non-zero Lie ideal of K. If $w^2 = 0$ for each $w \in U$, then $U = 0$.

Proof. Since $w^2 = 0$ for each $w \in U$, then $0 = w(w+v)(w+v) = wvw$ for each $w, v \in U$. So, $0 = vw[-v, \mathfrak{n}]w = vw\mathfrak{n}vw$ for each $w, v \in U, \mathfrak{n} \in K$. 3-primeness of K implies $vw = 0$ for each $w, v \in U$. Thus $0 = v[w, \mathfrak{n}] = -v\mathfrak{n}w = v\mathfrak{n}(-w)$ for each $w, v \in U, \mathfrak{n} \in K$. Therefore, $vK(-w) = \{0\}$ for each $w, v \in U$, we conclude that $U = 0$ by 3-primeness of K.

Lemma 4: If R is a semi prime ring, then there is a non-zero multiplicative homoleft (or, homoright) multiplier.

Proof. If Γ is a multiplicative homoright (or, homoleft) multiplier on R, then by finding $\Gamma(r(qy))$ and $\Gamma\big((rq)y\big)$ for each $r,q,y \in R$, we can easily conclude that, $\Gamma(r)q\Gamma(y)=0$ for each $r,q,y \in R$, by the semi primeness of R, we obtain that $\Gamma=0$.
Note that Example 1 shows that the semi primeness condition in Lemma 4 is not superfluous.

Lemma 5:
(i) If Γ is a multiplicative homoleft multiplier on K, then $\Gamma(rq)(\Gamma(y)+y)=\Gamma(r)\Gamma(q)\Gamma(y)+\Gamma(r)\Gamma(q)y+\Gamma(r)qy$ for each $r,q,y \in K$.
(ii) If Γ is a multiplicative homoright multiplier on K, then $\Gamma(rq)\Gamma(y)=\Gamma(r)\Gamma(q)\Gamma(y)+\Gamma(r)q\Gamma(y)+r\Gamma(q)\Gamma(y)$ for each $r,q,y \in K$.

Proof. By using the definition of Γ, we can find both $\Gamma\big(r(qy)\big)$ and $\Gamma\big((rq)y\big)$ by simple calculations, and reach to the required result easily.

Lemma 7: If K be a semi prime, then K has no non-zero multiplicative homomultiplier.

Proof. Let Γ be a multiplicative homomultiplier of K, then $\Gamma(rq)=\Gamma(r)\Gamma(q)+\Gamma(r)q$ and also $\Gamma(rq)=\Gamma(r)\Gamma(q)+r\Gamma(q)$ for each $r,q \in K$. We find that $\Gamma(r)q=r\Gamma(q)$ for each $r,q \in K$. Putting qy in place of q in the previous relation and use it to get $r\Gamma(q)\Gamma(y)=0$ for each $r,q,y \in K$, we get $\Gamma(q)\Gamma(y)=0$ for each $q,y \in K$ because of semi primness of K. Therefore, $0=\Gamma(q)\Gamma(yr)=\Gamma(q)\Gamma(y)\Gamma(r)+\Gamma(q)y\Gamma(r)=\Gamma(q)y\Gamma(r)$ for each $r,q,y \in K$. Thus $\Gamma=0$ as K is a semi prime.

3. Main Results

Theorem 1. Let K be a 3-prime and $a \in K$, if Γ is a multiplicative homoleft or, (homoright) multiplier, such that $\Gamma(a)=0$, then either $\Gamma=0$ or $a=0$.

Proof. Suppose that Γ is a multiplicative homoleft multiplier, we can find both $\Gamma\big(r(aq)\big)$ and $\Gamma\big((ra)q\big)$ by using the fact $\Gamma(a)=0$, then for each $r,q \in K$, we have

$$\Gamma\big((ra)q\big)=\Gamma(ra)\Gamma(q)+\Gamma(ra)q=\Gamma(r)a\Gamma(q)+\Gamma(r)aq. \tag{1}$$

$$\Gamma\big(r(aq)\big)=\Gamma(r)\Gamma(aq)+\Gamma(r)aq=\Gamma(r)aq. \tag{2}$$

From (1) and (2), we can get

$$\Gamma(r)a\Gamma(q)=0 \text{ for each } r,q \in K. \tag{3}$$

Now, using (3) with the fact that $\Gamma(a)=0$, we get

$$\begin{aligned}\Gamma((ra)(yq)) &= \Gamma(ra)\Gamma(yq)+\Gamma(ra)yq\\ &= \Gamma(r)a\Gamma(yq)+\Gamma(r)ayq\\ &= \Gamma(r)ayq \text{ for each } r,q,y \in K,\end{aligned} \tag{4}$$

and, we have

$$\begin{aligned}\Gamma((r(ay))q) &= \Gamma(r(ay))\Gamma(q)+\Gamma(r(ay))q\\ &= \Gamma(r)ay\Gamma(q)+\Gamma(r)ayq \text{ for each } r,q,y \in K.\end{aligned} \tag{5}$$

Combining (4) and (5) to obtain $\Gamma(r)ay\Gamma(q)=\{0\}$ for each $r,q,y \in K$. Now by using 3-primeness of K, we obtain $\Gamma(r)a=0$ for each $r \in K$. Using the last result with the fact $\Gamma(a)=0$ and Lemma 5 (i) implies
$0=\Gamma(rq)a=\Gamma(rq)(\Gamma(a)+a)=\Gamma(r)\Gamma(q)\Gamma(a)+\Gamma(r)\Gamma(q)a+\Gamma(r)qa=\Gamma(r)qa$ for each $r,q \in K$, by 3-primeness of K, we obtain the required result.
Now, suppose that Γ is a multiplicative homoright multiplier, then for each $r,q \in K$, we have

$$\Gamma\big((ra)q\big)=\Gamma(ra)\Gamma(q)+ra\Gamma(q)=ra\Gamma(q). \tag{6}$$

$\Gamma\big(r(aq)\big) = \Gamma(r)\Gamma(aq) + r\Gamma(aq) = \Gamma(r)a\Gamma(q) + ra\Gamma(q).$ (7)
From (6) and (7), we can get $\Gamma(r)a\Gamma(q) = 0$ for each $r, q \in K$. thus $0 = \Gamma(r)a\Gamma(qy) = \Gamma(r)aq\Gamma(y)$ for each $r, q, y \in K$. Using 3-primeness of K, we get $\Gamma(r)a = 0$ for each $r \in K$. Now, using Lemma 5(ii) with the fact $\Gamma(a) = 0$, we get, for each $r, q, y \in K$

$$\begin{aligned}\Gamma\big(r(qa)\big)\Gamma(y) &= \Gamma(r)\Gamma(qa)\Gamma(y) + \Gamma(r)qa\Gamma(y) + r\Gamma(qa)\Gamma(y)\\ &= \Gamma(r)qa\Gamma(y). \qquad (8)\end{aligned}$$

And,

$$\Gamma(rqa)\Gamma(y) = \big(\Gamma(rq)\Gamma(a) + rq\Gamma(a)\big)\Gamma(y) = 0. \qquad (9)$$

Combining (8) and (9) to get $\Gamma(r)qa\Gamma(y) = 0$ for each $r, q, y \in K$, using 3-primeness of K, we obtain $a\Gamma(r) = 0$ for each $r \in K$, it follows that $0 = a\Gamma(rq) = ar\Gamma(q)$ for each $r, q \in K$. The 3-primeness of K this leads to the desired outcome.

Theorem 2. Let a be an any element of K and K is a 3-prime.
(i) If Γ is a multiplicative homoleft multiplier on K such that $\Gamma(a) + a \in \mathcal{Z}(K)$, then either $\Gamma = 0$ or $a = 0$.
(ii) If Γ is a multiplicative homoright multiplier on K such that $\Gamma(a) \in \mathcal{Z}(K)$, then either $\Gamma = 0$ or $a = 0$.

Proof. (i) By our assumption we have

$$\Gamma(rq)(\Gamma(a) + a) = (\Gamma(a) + a)\Gamma(rq) \text{ for each } r, q \in K. \qquad (10)$$

Using Lemma 5(i) to invoke the right-hand side of (10) which gives
$\Gamma(rq)(\Gamma(a) + a) = \Gamma(r)\Gamma(q)\Gamma(a) + \Gamma(r)\Gamma(q)a + \Gamma(r)qa$ for each $r, q \in K$.
While the right-hand side of (10) can be simplified as follows

$$\begin{aligned}(\Gamma(a) + a)\Gamma(rq) &= (\Gamma(a) + a)\Gamma(r)\Gamma(q) + (\Gamma(a) + a)\Gamma(r)q\\ &= \Gamma(r)\Gamma(q)\Gamma(a) + \Gamma(r)\Gamma(q)a + \Gamma(r)q\Gamma(a) + \Gamma(r)qa \text{ for each } r, q \in K.\end{aligned}$$

After simplifying, equate both sides of the equation (10) to get $\Gamma(r)q\Gamma(a) = 0$ for each $r, q \in K$, using the 3-primeness of K implies either $\Gamma = 0$ or $\Gamma(a) = 0$, by Theorem 1, we find that either $\Gamma = 0$ or $a = 0$.
(ii) By our assumption, we have

$$\Gamma(rq)\Gamma(a) = \Gamma(a)\Gamma(rq) \text{ for each } r, q \in K. \qquad (11)$$

Using Lemma 5(ii) to simplify the left-hand side of (11)
$\Gamma(rq)\Gamma(a) = \Gamma(r)\Gamma(q)\Gamma(a) + \Gamma(r)q\Gamma(a) + r\Gamma(q)\Gamma(a)$ for each $r, q \in K$.
While, the right-hand side can be finding as follows:
$\Gamma(a)\Gamma(rq) = \Gamma(a)\Gamma(r)\Gamma(q) + \Gamma(a)r\Gamma(q)$ for each $r, q \in K$.
After simplifying, equate both sides of the equation (11) to get $\Gamma(r)q\Gamma(a) = 0$ for each $r, q \in K$, using the 3-primeness of K and Theorem 1, we have either $\Gamma = 0$ or, $a = 0$.

Theorem 3. Let K be a 2-torsion free 3-prime and U be a Lie ideal of K,
(a) If Γ is a non-zero multiplicative homoright (or, homoleft) multiplier such that $\Gamma(w \diamond \mathfrak{n}) = 0$ or, for each $w \in U, \mathfrak{n} \in K$, then $U = 0$.
(b) If Γ is a non-zero multiplicative homoright multiplier of K satisfies any one of the following assertions:
(i) $\Gamma(w \diamond \mathfrak{n}) = (w \diamond \mathfrak{n})$ for each $w \in U, \mathfrak{n} \in K$,
(ii) $\Gamma([w, \mathfrak{n}]) = w \diamond \mathfrak{n}$ for each $w \in U, \mathfrak{n} \in K$,
(iii) $\Gamma(w \diamond \mathfrak{n}) = [w, \mathfrak{n}]$ for each $w \in U, \mathfrak{n} \in K$,
then $U = 0$.

Proof. (a) Suppose that Γ is a multiplicative homoright multiplier such that $\Gamma(w \diamond \mathfrak{n}) = 0$ for each $w \in U, \mathfrak{n} \in K$, by Theorem 1, we conclude $w \diamond \mathfrak{n} = 0$, $w\mathfrak{n} = -\mathfrak{n}w$, substituting $\mathfrak{n}r$ instead of $\mathfrak{n}$ we get $-\mathfrak{n}rw = w\mathfrak{n}r = \mathfrak{n}(-w)r$ for each $w \in U, \mathfrak{n}, r \in K$. Therefore, $\mathfrak{n}[r, -w] = 0$, hence by 3-primeness of K we conclude $U \subseteq \mathcal{Z}(K)$, thus $0 = \Gamma(w \diamond v\mathfrak{n}) = \Gamma\big((w \diamond \mathfrak{n})v\big) = \Gamma(w \diamond \mathfrak{n})\Gamma(v) + (w \diamond \mathfrak{n})\Gamma(v) = (w \diamond \mathfrak{n})\Gamma(v) = (2w)\mathfrak{n}\Gamma(v)$ for each $w, v \in U, \mathfrak{n} \in K$, using 2-torsion freeness and 3-primeness of K with Theorem 1, we obtain $U = 0$.
Now, if Γ is a multiplicative homoright multiplier such that $\Gamma(w \diamond \mathfrak{n}) = 0$ for each $w \in U, \mathfrak{n} \in K$, using the same a way as in the first case, with simple changing what is needed, we get the desired result.
(b) (i) From our hypothesis, we have

$$\Gamma(w \diamond \mathfrak{n}) = (w \diamond \mathfrak{n}) \quad \text{for each } w \in U, \mathfrak{n} \in K. \tag{12}$$

Replacing $\mathfrak{n}$ by $w\mathfrak{n}$ in our assumption, and using it again implies $w(w \diamond \mathfrak{n}) = \Gamma\big(w(w \diamond \mathfrak{n})\big) = \Gamma(w)\Gamma(w \diamond \mathfrak{n}) + w\Gamma(w \diamond \mathfrak{n}) = \Gamma(w)(w \diamond \mathfrak{n}) + w(w \diamond \mathfrak{n})$ for each $w \in U, \mathfrak{n} \in K$. Thus, $\Gamma(w)(w \diamond \mathfrak{n}) = 0$ for each $w \in U, \mathfrak{n} \in K$. That is $\Gamma(w)w\mathfrak{n} = -\Gamma(w)\mathfrak{n}w$ for each $w \in U, \mathfrak{n} \in K$, replace $\mathfrak{n}$ by $\mathfrak{n}t$ in last equation and use it to get $\Gamma(w)\mathfrak{n}[-w, t] = 0$ for each $w \in U, \mathfrak{n}, t \in K$, so either $\Gamma(w) = 0$ or $w \in Z(K)$ for each $w \in U$. Theorem 1 ensures that $U \subseteq \mathcal{Z}(K)$. Replacing $\mathfrak{n}$ by $v\mathfrak{n}$ in (12) and using it again implies $vv(w \diamond \mathfrak{n}) = \Gamma\big(v(w \diamond \mathfrak{n})\big) = \Gamma(v)\Gamma(w \diamond \mathfrak{n}) + v\Gamma(w \diamond \mathfrak{n}) = \Gamma(v)(w \diamond \mathfrak{n}) + v(w \diamond \mathfrak{n})$ for each $w, v \in U, \mathfrak{n} \in K$. Thus, $\Gamma(v)(w \diamond \mathfrak{n}) = 0$ for each $w, v \in U, \mathfrak{n} \in K$. That is $2\Gamma(v)w\mathfrak{n} = 0$ for each $w, v \in U, \mathfrak{n} \in K$, by 2-torsion freeness of K, we obtain $\Gamma(v)Kw = \{0\}$ for each $w, v \in U$, using 3-primeness of K with Theorem 1, we obtain the required result
(b)(ii) By assumption, we have

$$\Gamma([w, \mathfrak{n}]) = w \diamond \mathfrak{n} \text{ for each } w \in U, \mathfrak{n} \in K. \tag{13}$$

Replace $\mathfrak{n}$ by $w\mathfrak{n}$ in (13) and use it to get $w(w \diamond \mathfrak{n}) = \Gamma(w[w, \mathfrak{n}]) = \Gamma(w)\Gamma([w, \mathfrak{n}]) + w\Gamma([w, \mathfrak{n}]) = \Gamma(w)(w \diamond \mathfrak{n}) + w(w \diamond \mathfrak{n})$ for each $w \in U, \mathfrak{n} \in K$. Therefore, $\Gamma(w)(w \diamond \mathfrak{n}) = 0$ for each $w \in U, \mathfrak{n} \in K$. That is $\Gamma(w)w\mathfrak{n} = -\Gamma(w)\mathfrak{n}w$ for each $w \in U, \mathfrak{n} \in K$, replace $\mathfrak{n}$ by $\mathfrak{n}t$ in last equation and use it to get $\Gamma(w)\mathfrak{n}[-w, t] = 0$ for each $w \in U, \mathfrak{n}, t \in K$, so either $\Gamma(w) = 0$ or $w \in \mathcal{Z}(K)$ for each $w \in U$, since $\Gamma \neq 0$, Theorem 1 ensure that $U \subseteq \mathcal{Z}(K)$, return to (13) we find that $2w\mathfrak{n} = 0$ for each $w \in U, \mathfrak{n} \in K$, by 2-torsion free of K we obtain $w\mathfrak{n} = 0$ for each $w \in U, \mathfrak{n} \in K$, 3-primeness of K, we conclude that $U = 0$.
(b) (iii) By our hypothesis, we have

$$\Gamma(w \diamond \mathfrak{n}) = [w, \mathfrak{n}] \text{ for each } w \in U, \mathfrak{n} \in K. \tag{14}$$

Replace $\mathfrak{n}$ by $w\mathfrak{n}$ in (14) and use it to get $w[w, \mathfrak{n}] = \Gamma(w(w \diamond \mathfrak{n})) = \Gamma(w)\Gamma(w \diamond \mathfrak{n}) + w\Gamma(w \diamond \mathfrak{n}) = \Gamma(w)[w, \mathfrak{n}] + w[w, \mathfrak{n}]$ for each $w \in U, \mathfrak{n} \in K$. So, $\Gamma(w)[w, \mathfrak{n}] = 0$ for each $w \in U, \mathfrak{n} \in K$. That is $\Gamma(w)w\mathfrak{n} = \Gamma(w)\mathfrak{n}w$ for each $w \in U, \mathfrak{n} \in K$, replace $\mathfrak{n}$ by $\mathfrak{n}t$ in last equation and use it to get $\Gamma(w)\mathfrak{n}[w, t] = 0$ for each $w \in U, \mathfrak{n}, t \in K$, 3-primeness leads to either $\Gamma(w) = 0$ or $w \in \mathcal{Z}(K)$ for each $w \in U$, since $\Gamma \neq 0$, Theorem 1, implies that $U \subseteq \mathcal{Z}(K)$, return to (14), we get $\Gamma(w \diamond \mathfrak{n}) = 0$ for each $w \in U, \mathfrak{n} \in K$ and using (a) we find that $U = 0$.

Theorem 4. Let be K 3-prime and U be a non-zero Lie ideal of K.
(i) If Γ is a multiplicative homoright (or, homoleft) multiplier of R such that $\Gamma([w, \mathfrak{n}]) = 0$ for each $w \in U, \mathfrak{n} \in K$, then either $\Gamma = 0$ or $(K, +)$ is abelian.
(ii) If Γ is a nonzero multiplicative homoright multiplier of K, such that $\Gamma([w, \mathfrak{n}]) = [w, \mathfrak{n}]$ for each $w \in U, \mathfrak{n} \in K$, either $\Gamma = 0$ or $(K, +)$ is abelian.

Proof. (i) If Γ is a multiplicative homoright (or, homoleft) multiplier of R such that $\Gamma([w, \mathfrak{n}]) = 0$ for each $w \in U, \mathfrak{n} \in K$, then by Theorem 1, we get either $\Gamma = 0$ or $[w, \mathfrak{n}] = 0$

for each $w \in U, \mathfrak{n} \in K$ i.e., either $\Gamma = 0$ or, $U \subseteq \mathcal{Z}(K)$, by Lemma 2 (ii), last result can be written as $\Gamma = 0$ or $(K, +)$ is abelian.
(ii) By assumption, we have

$$\Gamma([w, \mathfrak{n}]) = [w, \mathfrak{n}] \text{ for each } w \in U, \mathfrak{n} \in K. \quad (15)$$

Replacing $\mathfrak{n}$ by $\mathfrak{n}w$ in (15) and using it again implies $w([w, \mathfrak{n}]) = \Gamma(w[w, \mathfrak{n}]) = \Gamma(w)\Gamma([w, \mathfrak{n}]) + w\Gamma([w, \mathfrak{n}]) = \Gamma(w)([w, \mathfrak{n}]) + w[w, \mathfrak{n}]$ for each $w \in U, \mathfrak{n} \in K$. Thus, $\Gamma(w)[w, \mathfrak{n}] = 0$ for each $w \in U, \mathfrak{n} \in K$. That is $\Gamma(w)w\mathfrak{n} = \Gamma(w)\mathfrak{n}w$ for each $w \in U, \mathfrak{n} \in K$, replace $\mathfrak{n}$ by $\mathfrak{n}t$ in last equation and use it to get $\Gamma(w)\mathfrak{n}[w, t] = 0$ for each $w \in U, \mathfrak{n}, t \in K$, 3-primeness leads to either $\Gamma(w) = 0$ or, $w \in \mathcal{Z}(K)$ for each $w \in U$, Theorem 1 ensures that either $\Gamma = 0$ or $U \subseteq \mathcal{Z}(K)$ it follows that either $\Gamma = 0$ or $(K, +)$ is abelian according to Lemma 2(ii).

Theorem 5. Let U be a Lie idea of a 3-prime near-ring K. If Γ is a multiplicative homoleft (or, homoright) multiplier on K such that $\Gamma(U) \subseteq \mathcal{Z}(K)$, then either $\Gamma = 0$ or, $U = 0$.

Proof. Suppose that Γ is a multiplicative homoleft multiplier of K and $\Gamma(U) \subseteq \mathcal{Z}(K)$, then $\Gamma([w, \mathfrak{n}]) \in \mathcal{Z}(K)$ for each $w \in U, \mathfrak{n} \in K$, thus
$\Gamma([w, w\mathfrak{n}]) = \Gamma(w[w, \mathfrak{n}]) = \Gamma(w)\Gamma([w, \mathfrak{n}]) + \Gamma(w)[w, \mathfrak{n}] = \Gamma(w)(\Gamma([w, \mathfrak{n}]) + [w, \mathfrak{n}]) \in \mathcal{Z}(K)$ for each $w \in U, \mathfrak{n} \in K$. Using Lemma 1, we get $\Gamma(w) = 0$ *or* $(\Gamma([w, \mathfrak{n}]) + [w, \mathfrak{n}]) \in \mathcal{Z}(K)$ for each $w \in U, \mathfrak{n} \in K$. By Theorem 1 and Theorem 2(i) then the last expression can be reduce to $\Gamma = 0$ or $w \in \mathcal{Z}(K)$ for each $w \in U$. If $w \in \mathcal{Z}(K)$, from hypothesis we also have $\Gamma(w) \in \mathcal{Z}(K)$, therefore, for each $w \in U, r, q \in K$, we have

$$\begin{aligned} \Gamma((rq)w) &= \Gamma(rq)\Gamma(w) + \Gamma(rq)w \\ &= \Gamma(w)\Gamma(rq) + w\Gamma(rq) \\ &= \Gamma(w)\Gamma(r)\Gamma(q) + \Gamma(w)\Gamma(r)q + w\Gamma(r)\Gamma(q) + w\Gamma(r)q. \quad (16) \end{aligned}$$

$$\begin{aligned} \Gamma(r(qw)) &= \Gamma(r)\Gamma(qw) + \Gamma(r)qw \\ &= \Gamma(r)\Gamma(q)\Gamma(w) + \Gamma(r)\Gamma(q)w + \Gamma(r)qw. \quad (17) \end{aligned}$$

Equalizing (16) and (17) forces, $\Gamma(w)\Gamma(r)q = 0$ for each $w \in U, r, q \in K$, that is $\Gamma(r)K\Gamma(w) = \{0\}$ for each $w \in U, r \in K$. By 3-primeness of K and Theorem 1 implies either $\Gamma = 0$ or $U = 0$.
Now, if Γ is a multiplicative homoright multiplier of K and $\Gamma(U) \subseteq \mathcal{Z}(K)$, by Theorem 2(ii) we obtain either $\Gamma = 0$ or, $U = 0$.

Theorem 6. Let U be a non-zero Lie idea of K, where K is a 3-prime. If Γ is a non-zero multiplicative homoright (or, homoleft) multiplier on K satisfying $\Gamma([w, \mathfrak{n}]) \in \mathcal{Z}(K)$ for each $w \in U, \mathfrak{n} \in K$, then $(K, +)$ is abelian .

Proof. If Γ is a multiplicative homoright multiplier and $\Gamma([w, \mathfrak{n}]) \in \mathcal{Z}(K)$, by assumption we have $\Gamma(w[w, \mathfrak{n}]) = \Gamma([w, w\mathfrak{n}]) \in \mathcal{Z}(K)$ for each $w \in U, \mathfrak{n} \in K$, so we obtain $\Gamma(w[w, \mathfrak{n}])\Gamma(y) = \Gamma(y)\Gamma(w[w, \mathfrak{n}])$ for each $w \in \mathrm{U}, \mathfrak{n}, y \in K$, using Lemma 5 (ii) lastly forces

$$\begin{aligned} &\Gamma(w)\Gamma[w, \mathfrak{n}]\Gamma(y) + \Gamma(w)[w, \mathfrak{n}]\Gamma(y) + w\Gamma[w, \mathfrak{n}]\Gamma(y) = \\ &\quad \Gamma(y)\Gamma(w)\Gamma([w, \mathfrak{n}]) + \Gamma(y)w\Gamma([w, \mathfrak{n}]) \text{ for each } w \in U, \mathfrak{n}, y \in K. \quad (18) \end{aligned}$$

Let $y = [v, m]$ where $v \in U, m \in K$, in (18) to get $\Gamma(w)[w, \mathfrak{n}]\Gamma([v, m]) = 0$ for each $w, v \in U, m, \mathfrak{n} \in K$, which can be written as $\Gamma([v, m])K\Gamma(w)[w, \mathfrak{n}] = \{0\}$ for each $w, v \in U, m, \mathfrak{n} \in K$, use 3-primeness of K conclude that either $\Gamma(w)[w, \mathfrak{n}] = 0$ for each $w \in U, \mathfrak{n} \in K$ or $\Gamma([v, m]) = 0$ for each $v \in U, m \in K$.

If $\Gamma(w)[w,\mathfrak{n}] = 0$ for each $w \in U, \mathfrak{n} \in K$, then $\Gamma(w)w\mathfrak{n} = \Gamma(w)\mathfrak{n}w$ for each $w \in U, \mathfrak{n} \in K$, put $\mathfrak{n} = \mathfrak{n}t$ in last equation and use it to implies $\Gamma(w)K[w,t] = \{0\}$ for each $w \in U, t \in K$, then we arrive at: either $\Gamma(w) = 0$ or $[w,t] = 0$ for each $w \in U, t \in K$, Since $\Gamma \neq 0$, Theorem 1 implies that $U \subseteq \mathcal{Z}(K)$, so we obtain the desired result according to Lemma 2(ii). Now, if $\Gamma([v,m]) = 0$ for each $v \in U, \mathrm{m} \in K$, since $\Gamma \neq 0$, by Theorem 4(i), we obtain $(K,+)$ is abelian.

Now, suppose that Γ is a multiplicative homoleft multiplier and $\Gamma([w,\mathfrak{n}]) \in \mathcal{Z}(K)$ for each $w \in U, \mathfrak{n} \in K$. If $\mathcal{Z}(K) = \{0\}$, then $\Gamma([w,\mathfrak{n}]) = 0$ for each $w \in U, \mathfrak{n} \in K$, since $\Gamma \neq 0$, we obtain $U \subseteq \mathcal{Z}(K) = \{0\}$ according to proof of Theorem 4(i), thus we conclude that $U = 0$ which contradicts our assumption.

Now, we suppose that $\mathcal{Z}(K) \neq 0$, then there exists a non-zero element $y \in \mathcal{Z}(K)$, from our assumption we find $\Gamma([w,y\mathfrak{n}]) = \Gamma([w,\mathfrak{n}]y) = \Gamma([w,\mathfrak{n}])(\Gamma(y)+y) \in \mathcal{Z}(K)$ for each $w \in U, \mathfrak{n} \in K$, using Lemma 1 lastly ensures that $\Gamma([w,\mathfrak{n}]) = 0$ or $(\Gamma(y)+y) \in Z(K)$ for each $w \in U, \mathfrak{n} \in K$. since $\Gamma \neq 0$ and $y \neq 0$, we find that $(K,+)$ is abelian according to Theorem 4(i) and Theorem 2(i).

Theorem 7. If U is a non-zero Lie idea of K, where K is a 3-prime, then there is no non-zero multiplicative homoright (or, homoleft) multiplier on K satisfying $\Gamma(w \diamond \mathfrak{n}) \in \mathcal{Z}(K)$ for each $w \in U, \mathfrak{n} \in K$.

Proof. If Γ is a non-zero multiplicative homoright multiplier and $\Gamma(w \diamond \mathfrak{n}) \in \mathcal{Z}(K)$for each $w \in U, \mathfrak{n} \in K$, by our assumption we have $\Gamma(w(w \diamond \mathfrak{n})) = \Gamma((w \diamond w\mathfrak{n})) \in \mathcal{Z}(K)$ for each $w \in U, \mathfrak{n} \in K$, therefore, we can say that $\Gamma(w(w \diamond \mathfrak{n}))\Gamma(y) = \Gamma(y)\Gamma(w(w \diamond \mathfrak{n}))$ for each $w \in U, \mathfrak{n}, y \in K$, using Lemma 5(ii) lastly forces

$$\Gamma(w)\Gamma(w \diamond \mathfrak{n})\Gamma(y) + \Gamma(w)(w \diamond \mathfrak{n})\Gamma(y) + w\Gamma(w \diamond \mathfrak{n})\Gamma(y) = \Gamma(y)\Gamma(w)\Gamma(w \diamond \mathfrak{n}) + \Gamma(y)w\Gamma(w \diamond \mathfrak{n}) \text{ for each } w \in U, \mathfrak{n} \in K. \quad (19)$$

If $y = (v \diamond m)$, where $v \in U, m \in K$, then (19) can be reduce to $\Gamma(w)(w \diamond \mathfrak{n})\Gamma(v \diamond m) = 0$ for each $w, v \in U, m \in K$, which can be written as $\Gamma(v \diamond m)K\Gamma(w)(w \diamond \mathfrak{n}) = \{0\}$ for each $w, v \in U, m \in K$, use 3-primeness of K to conclude that

$$\Gamma(w)(w \diamond \mathfrak{n}) = 0 \text{ for each } w \in \mathrm{U}, \mathfrak{n} \in K \text{ or } \Gamma(v \diamond m) = 0 \text{ for each } v \in U, m \in K. \quad (20)$$

If $\Gamma(w)(w \diamond \mathfrak{n}) = 0$ for each $w \in \mathrm{U}, \mathfrak{n} \in K$, that is $\Gamma(w)w\mathfrak{n} = -\Gamma(w)\mathfrak{n}w$ for each $w \in U, \mathfrak{n} \in K$, put $\mathfrak{n} = \mathfrak{n}t$ in last equation and use it to implies $\Gamma(w)K[-w,t] = \{0\}$ for each $w \in U, t \in K$, by 3-primeness, we arrive at either $\Gamma(w) = 0$ for each $w \in U$, or $U \subseteq \mathcal{Z}(K)$, the first cases leads to either $U = \{0\}$ according to Theorem 1 (which contradicts our assumption) or $\Gamma = 0$ (contradicts our assumption).
When $U \subseteq \mathcal{Z}(K)$, we obtain $\Gamma(w \diamond v\mathfrak{n}) = \Gamma\big(v(w \diamond \mathfrak{n})\big) = \Gamma(v)\Gamma(w \diamond \mathfrak{n}) + v\Gamma(w \diamond \mathfrak{n}) = \Gamma(w \diamond \mathfrak{n})(\Gamma(v)+v) \in \mathcal{Z}(K)$ for each $w, v \in U, \mathfrak{n} \in K$, using Lemma 1, we arrive at either $\Gamma(w \diamond \mathfrak{n}) = 0$ for each $w \in U, \mathfrak{n} \in K$ or $(\Gamma(v)+v) \in \mathcal{Z}(K)$ for each $v \in U$. According to Theorem 3(a) the first case leads to $U = \{0\}$, which contradicts our assumption. Now, suppose that $(\Gamma(v)+v) \in \mathcal{Z}(K)$ for each $v \in U$. It follows that
$\Gamma(rv)(\Gamma(v)+v) = (\Gamma(v)+v)\Gamma(rv)$ for each $v \in U$, $r \in K$, and we can simplify that last equation as follows:
$\Gamma(rv)\Gamma(v) + \Gamma(rv)v = (\Gamma(v)+v)\Gamma(r)\Gamma(v) + (\Gamma(v)+v)r\Gamma(v)$ for each $v \in U$, $r \in K$, using Lemma 5(ii) when we simplify the previous relation, we arrive at

$$\Gamma(r)\Gamma(v)\Gamma(v)+\Gamma(r)v\Gamma(v)+r\Gamma(v)\Gamma(v)+v\Gamma(r)\Gamma(v)+vr\Gamma(v)=$$
$$\Gamma(r)\Gamma(v)\Gamma(v)+\Gamma(r)\Gamma(v)v+r\Gamma(v)\Gamma(v)+r\Gamma(v)v \text{ for each } v\in U, r\in K.$$

Using the fact $v\in \mathcal{Z}(K)$ for each $v\in U$, the last relation can be reduce to $v\Gamma(r)\Gamma(v)=0$ for each $v\in U, r\in K$, it follows that $vK\Gamma(r)\Gamma(v)=\{0\}$ for each $v\in U, r\in K$, 3-primeness of K implies that $v=0$ or, $\Gamma(r)\Gamma(v)=0$ for each $v\in$ U, $r\in K$, we can say that $\Gamma(r)\Gamma(v)=0$ for each $v\in U$, $r\in K$, that is $0=\Gamma(rq)\Gamma(v)=\Gamma(r)\Gamma(q)\Gamma(v)+\Gamma(r)q\Gamma(v)+r\Gamma(q)\Gamma(v)=\Gamma(r)q\Gamma(v)$ for each $v\in U$, $r\in K$. i.e., $\Gamma(r)K\Gamma(v)=\{0\}$ for each $v\in U$, $r\in K$, 3-primeness of K implies that either $\Gamma(v)=0$ for each $v\in$ U, using Theorem 1 we conclude that $U=\{0\}$ or $\Gamma=0$, which is a contradiction.

Return to (20), when $\Gamma(v\diamond m)=0$ for each $v\in U, m\in K$, by Theorem 3(a), we conclude that $U=\{0\}$, which is a contradiction.

Now, Suppose that Γ is a non-zero multiplicative homoleft multiplier and $\Gamma(w\diamond \mathfrak{n})\in \mathcal{Z}(K)$ for each $w\in U$, $\mathfrak{n}\in K$. If $\mathcal{Z}(K)=0$, then $\Gamma(w\diamond \mathfrak{n})=0$ for each $w\in U, \mathfrak{n}\in K$, from Theorem 3(a), it follows that $U=0$ which contradicts our assumption. Now, if $\mathcal{Z}(K)\neq 0$, suppose that there exist $0\neq y\in \mathcal{Z}(K)$, from our assumption we find $\Gamma(w\diamond y\mathfrak{n})=\Gamma((w\diamond \mathfrak{n})y)=\Gamma(w\diamond \mathfrak{n})(\Gamma(y)+y)\in \mathcal{Z}(K)$ for each $w\in U, \mathfrak{n}, y\in K$, using Lemma 1 lastly ensures that $\Gamma(w\diamond \mathfrak{n})=0$ for each $w\in U, \mathfrak{n}\in K$ *or* $(\Gamma(y)+y)\in \mathcal{Z}(K)$ for each $y\in K$. If $\Gamma(w\diamond \mathfrak{n})=0$ for each $w\in U, \mathfrak{n}\in K$, using Theorem 3(a) implies $U=0$, if $(\Gamma(y)+y)\in \mathcal{Z}(K)$ for each $y\in K$, then Theorem 2(i) implies either $y=0$, hence both two last cases lead to a contradiction.

Theorem 8. Let U be a Lie ideal of K, where K is a 3-prime.
(i) If Γ is a non-zero multiplicative homoleft multiplier of K and acts as a homomorphism (or, left multiplier) on U, then $U=0$.
(ii) If Γ is a non-zero multiplicative homoright multiplier of K and acts as a homomorphism (or, right multiplier) on U, then $U=0$.

Proof. (i) Assume that Γ is a multiplicative homoleft multiplier of K and Γ acts as a homomorphism on U, then
$$\Gamma(wv)=\Gamma(w)\Gamma(v) \text{ for each } w,v\in U. \quad (21)$$
On the other hand
$$\Gamma(wv)=\Gamma(w)\Gamma(v)+\Gamma(w)v \text{ for each } w,v\in U. \quad (22)$$
From (21) and (22), we conclude that $\Gamma(w)v=0$, if we replace v by $[v,\mathfrak{n}]$ in last equation we obtain $0=\Gamma(w)[v,\mathfrak{n}]=\Gamma(w)v\mathfrak{n}-\Gamma(w)\mathfrak{n}v$ for each $w,v\in U,\mathfrak{n}\in K$, that is $\Gamma(w)\mathfrak{n}v=0$ for each $w,v\in U,\mathfrak{n}\in K$, which means that $\Gamma(w)Kv=\{0\}$ for each $w,v\in U$, 3-primeness of K implies that $\Gamma(U)=\{0\}$, since $\Gamma\neq\{0\}$. By Theorem 1 , we obtain $U=0$.
Now, suppose that Γ is a homoleft multiplier of K and Γ acts as a left multiplier on U, then
$$\Gamma(wv)=\Gamma(w)v \text{ for each } w,v\in U. \quad (23)$$
On another hand
$$\Gamma(wv)=\Gamma(w)\Gamma(v)+\Gamma(w)v \text{ for each } w,v\in U. \quad (24)$$
Comparing (23) with (24) forces
$$\Gamma(w)\Gamma(v)=0 \text{ for each } w,v\in U. \quad (25)$$
Therefore, using (25) in the following relation implies that
$\Gamma\big((wv)q\big)=\Gamma(wv)\Gamma(q)+\Gamma(wv)q=\Gamma(w)v\Gamma(q)+\Gamma(w)vq$ for each $w,v\in U, q\in K$.
Also, we have

$\Gamma\big(w(vq)\big) = \Gamma(w)\Gamma(vq) + \Gamma(w)vq = \Gamma(w)\Gamma(v)\Gamma(q) + \Gamma(w)\Gamma(v)q + \Gamma(w)vq = \Gamma(w)vq$ for each $w, v \in U, q \in K$.

From the two-above expression, we can get

$$\Gamma(w)v\Gamma(q) = 0 \text{ for each } w, v \in U, q \in K. \tag{26}$$

Now, we will find $\Gamma\big((wv)rq\big)$ and $\Gamma\big((wvr)q\big)$ for each $, v \in U, r, q \in K$, with using (25) and (26)

$$\begin{aligned}\Gamma\big((wv)rq\big) &= \Gamma(wv)\Gamma(rq) + \Gamma(wv)rq \\ &= \Gamma(w)v\Gamma(rq) + \Gamma(w)vrq = \Gamma(w)vrq.\end{aligned} \tag{27}$$

And also, we have

$$\begin{aligned}\Gamma\big((wvr)q\big) &= \Gamma\big((wv)r\big)\Gamma(q) + \Gamma((wv)r)q \\ &= (\Gamma(wv)\Gamma(r) + \Gamma(wv)r)\Gamma(q) + (\Gamma(wv)\Gamma(r) + \Gamma(wv)r)q \\ &= \Gamma(w)vr\Gamma(q) + \Gamma(w)vrq \text{ for each } w, v \in U.\end{aligned} \tag{28}$$

From the (27) and (28), we obtain $\Gamma(w)vr\Gamma(q) = 0$ for each $w, v \in U, r, q \in K$, thus $\Gamma(w)vK\Gamma(q) = \{0\}$ for each $w, v \in U, q \in K$ by 3-primeness of K, we conclude that $\Gamma = 0$ or $\Gamma(w)v = 0$ for each $w, v \in U$, therefore, $\Gamma(w)v = 0$ for each $w, v \in U$, using Lemma 2(i) forces $\Gamma(w) = 0$ for each $w \in U$, it follows that $U = 0$ according to Theorem 1.

(ii) Assume that Γ is a multiplicative homoright multiplier on K and Γ acts as a homomorphism on U. Then $\Gamma(wv) = \Gamma(w)\Gamma(v)$ for each $w, v \in U$ and on the other hand $\Gamma(wv) = \Gamma(w)\Gamma(v) + w\Gamma(v)$ for each $w, v \in U$, so, we can say that $w\Gamma(v) = 0$ for each $w, v \in U$, we can use the last result in the following equation,

$$\Gamma\big((wv)\mathfrak{n}\big) = \Gamma(w)\Gamma(v)\Gamma(\mathfrak{n}) + wv\Gamma(\mathfrak{n}) \text{ for each } w, v \in U, \mathfrak{n} \in K.$$

On the other hand

$$\begin{aligned}\Gamma\big(w(v\mathfrak{n})\big) &= \Gamma(w)\Gamma(v\mathfrak{n}) + w\Gamma(v\mathfrak{n}) \\ &= \Gamma(w)\Gamma(v)\Gamma(\mathfrak{n}) + \Gamma(w)v\Gamma(\mathfrak{n}) + w\Gamma(v)\Gamma(\mathfrak{n}) + wv\Gamma(\mathfrak{n}) \text{ for each } w, v \in U, \mathfrak{n} \in K.\end{aligned}$$

Combining the two last result, we obtain $\Gamma(w)v\Gamma(\mathfrak{n}) = 0$ for each $w, v \in U, \mathfrak{n} \in K$ and this result leads to $0 = \Gamma(w)v\Gamma(\mathfrak{n}t) = \Gamma(w)v\mathfrak{n}\Gamma(t)$ for each $w, v \in U, \mathfrak{n}, t \in K$, since $\Gamma \neq 0$, 3-primeness of K forces $\Gamma(w)v = 0$ for each $w, v \in U$, by Lemma 2(i), we obtain $\Gamma(U) = \{0\}$, thus $U = 0$, according to Theorem 1.

Now, If Γ is a multiplicative homoright multiplier on K and Γ acts as a right multiplier on U, then $\Gamma(wv) = w\Gamma(v)$ for each $w, v \in U$, also we have $\Gamma(wv) = \Gamma(w)\Gamma(v) + w\Gamma(v)$ for each $w, v \in U$, from the two above expression of $\Gamma(wv)$, we conclude, $\Gamma(w)\Gamma(v) = 0$ for each $w, v \in U$, then

$$\Gamma\big((wv)\mathfrak{n}\big) = \Gamma(wv)\Gamma(\mathfrak{n}) + wv\Gamma(\mathfrak{n}) = w\Gamma(v)\Gamma(\mathfrak{n}) + wv\Gamma(\mathfrak{n}) \text{for each } w, v \in U, \mathfrak{n} \in K. \tag{29}$$

On the other hand

$$\begin{aligned}\Gamma\big(w(v\mathfrak{n})\big) &= \Gamma(w)\Gamma(v\mathfrak{n}) + w\Gamma(v\mathfrak{n}) \\ &= \Gamma(w)\Gamma(v)\Gamma(\mathfrak{n}) + \Gamma(w)v\Gamma(\mathfrak{n}) + w\Gamma(v)\Gamma(\mathfrak{n}) + wv\Gamma(\mathfrak{n}) \\ &= \Gamma(w)v\Gamma(\mathfrak{n}) + w\Gamma(v)\Gamma(\mathfrak{n}) + wv\Gamma(\mathfrak{n}) \text{ for each } w, v \in U, \mathfrak{n} \in K.\end{aligned} \tag{30}$$

Commingling (29) and (30) involves: $\Gamma(w)v\Gamma(\mathfrak{n}) = 0$ for each $w, v \in U, \mathfrak{n} \in K$, i.e., $0 = \Gamma(w)v\Gamma(\mathfrak{n}t) = \Gamma(w)v\mathfrak{n}\Gamma(t)$ for each $w, v \in U, \mathfrak{n}, t \in K$, since $\Gamma \neq 0$, 3-primeness of K implies that $\Gamma(w)v = 0$, by Lemma 2(i), we obtain $\Gamma(U) = \{0\}$ and hence $U = 0$, according to Theorem 1.

The following example demonstrate that 3-primeness of K in each previous result which we find is not superfluous

Example 3. Let D, $\Gamma_1$and Γ_2, be defined as in Example 2. Define

$U = \left\{ \begin{pmatrix} 0 & 0 & b \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} : b, 0 \in T \right\}$, then U is a Le ideal of K, and let $\mathcal{A} = \begin{pmatrix} 0 & 0 & b \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

1. $\Gamma_1(\mathcal{A}) = 0$ and $\Gamma_2(\mathcal{A}) = 0$, but neither $\Gamma_1 = 0$ nor $\Gamma_2 = 0$ moreover, $\mathcal{A} \neq 0$.
2. $\Gamma_1(\mathcal{A}) + \mathcal{A} \in \mathcal{Z}(D)$ and $\Gamma_2(\mathcal{A}) \in \mathcal{Z}(D)$, but neither $\Gamma_1 = 0, \Gamma_2 = 0$ nor $\mathcal{A} = 0$.
3. $\Gamma_1(\mathcal{W} \diamond \mathcal{B}) = 0$, $\Gamma_2(\mathcal{W} \diamond \mathcal{B}) = 0$, $\Gamma_2(\mathcal{W} \diamond \mathcal{B}) = (\mathcal{W} \diamond \mathcal{B})$, $\Gamma_2([w, \pi]) = \mathcal{W} \diamond \mathcal{B}$ and $\Gamma_2(\mathcal{W} \diamond \mathcal{B}) = [\mathcal{W}, \mathcal{B}]$ for each $\mathcal{W} \in U, \mathcal{B} \in$ D, while $U \neq 0$.
4. $\Gamma_1([\mathcal{W}, \mathcal{B}]) = 0$, $\Gamma_2([\mathcal{W}, \mathcal{B}]) = 0$ an $\Gamma_2([\mathcal{W}, \mathcal{B}]) = [\mathcal{W}, \mathcal{B}]$ for each $\mathcal{W} \in U, \mathcal{B} \in K$, but neither $\Gamma_1 = 0$ nor $\Gamma_2 = 0$ moreover, $(\Gamma_1 = 0, +)$ is not abelian.
5. $\Gamma_1(U) \subseteq \mathcal{Z}(D)$ and $\Gamma_1(U) \subseteq \mathcal{Z}$(D) but neither $\Gamma_1 = 0$ nor $\Gamma_2 = 0$ moreover $U \neq 0$.
6. $\Gamma_1([\mathcal{W}, \mathcal{B}])$, $\Gamma_2([\mathcal{W}, \mathcal{B}]) \in \mathcal{Z}$(D) for each $\mathcal{W} \in U, \mathcal{B} \in K$, but (D, +) is not abelian .
7. $\Gamma_1(\mathcal{W} \diamond \mathcal{B})$, $\Gamma_2(\mathcal{W} \diamond \mathcal{B}) \in \mathcal{Z}$(D) for each $\mathcal{W} \in U, \mathcal{B} \in K, while\ \Gamma_1, \Gamma_2 \neq 0$.
8. Γ_1 acts as a homomorphism as well as a left multiplier on U and Γ_2 acts as a homomorphism as well as a right multiplier on U but $U \neq 0$.

4. Conclusion

This paper defines a new kinds of mappings in rings and near-rings, which are called multiplicative homoleft multipliers, multiplicative homoright multipliers and multiplicative homo multipliers, we reached important results and showed the importance of the 3-prime condition on these results. We suggest studying the generalization of these mappings in near-rings in the next works

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