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Some Geometric Properties of Close- to- Convex Functions

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Abstract
The objective of this paper is to present some geometric properties of the close to convex function f when f is an analytic, univalent self-conformal mapping defined in the open unit disk $\mathcal{D} = \{z \in \mathbb{C}: |z| < 1\}$, and on the boundary of $\mathcal{D}$. One of the goals of this work was determining the sharp bound for the function of form $f(z) = \frac{z}{(1-z)^{2\delta}}$, $0 < \delta < 1$, when $\Re\left(\frac{f'}{g'}(w_1)\right) \geq -2 + \frac{\delta}{w_1}$, for a some $w_1 \in \partial\mathcal{D}$ And another, if f has angular limit at $p \in \partial\mathcal{D}$. Then the inequality $\left|\frac{f'(z)}{g'(z)}\right| \geq \frac{-(1-2\delta)}{2(1+\delta)}$ is sharp with extremal functions $f(z) = \frac{z}{(1-z)^{2\delta}}$, and $g(z) = \frac{z}{1+z}$, where $0 < \delta < 1$. Finally, if f is extended continuously to the boundary of $\mathcal{D}$, then $\left|\left(\frac{f'}{g'}\right)(p)\right| \geq \frac{|1-2\delta|(|1-2\delta|-2)}{|p-\delta|}$; $0 < \delta < 1$.

Keywords: close - to- convex function, self- conformal mapping , angular derivative**,** univalent functions, analytic function.

بعض الخواص الهندسية للدوال القريبة من التحدب

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الخلاصة
الهدف من هذا البحث هو تقديم بعض الخصائص الهندسية للدالة القريبة من التحدب عندما تكون f دالة تحليلية أحادي التكافؤ ذاتي المطابقة معرّف في قرص الوحدة المفتوح $\mathcal{D} = \{z \in \mathbb{C}: |z| < 1\}$ ، و على حدود $\mathcal{D}$ (اي $\partial\mathcal{D}$) كما ان أهم موضوع تمت مناقشته هو قيد الدالة $f(z) = \frac{z}{(1-z)^{2\delta}}$> $0 < \delta$, عندما1 $\Re\left(\frac{f'}{g'}(w_1)\right) \geq -2 + \frac{\delta}{w_1}$, لبعض $w_1 \in \partial\mathcal{D}$ وفي حالة اخرى عندما تمتلك الدوال غاية الزاوية عند نقطة حدودية $p \in \partial\mathcal{D}$ فأن المتراجحة $\left|\frac{f'(z)}{g'(z)}\right| \geq \frac{-(1-2\delta)}{2(1+\delta)}$ تكون ذات قيمة عند الدالة القصوى $f(z)\frac{z}{(1-z)^{2\delta}}$ و الدالة $g(z) = \frac{z}{1+z}$ عندما $0 < \delta < 1$. اضافة الى نتائج حصلنا عليها عندما تكون الدالة f ذات استمرارية فأنها تتسع لتشمل حدود المنطقة $\mathcal{D}$ بحيث نحصل على
$\left|\left(\frac{f'}{g'}\right)(p)\right| \geq \frac{|1-2\delta|(|1-2\delta|-2)}{|p-\delta|}$; $0 < \delta < 1$.

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1. Introduction

Let $\mathcal{C}$ be the class of all analytic and univalent functions

$$f(z) = z + \sum_{n=2}^{\infty} e_n z^n \qquad \dots\dots\dots\dots (1)$$

defined in the unit disk $\mathcal{D} = \{z \in \mathbb{C}: |z| < 1\}$. Based on the preceding, let $\mathcal{C}_\delta$ be a subclass of $\mathcal{C}$ containing all of the functions that are a self-conformal mapping (analytic, univalent) function that is said to be a close - to - convex function of order δ, $0 < \delta < 1$; in $\mathcal{D}$, if and only if there exist a number $\Theta \in \mathbb{R}$ and a convex function in the form

$g(z) = \mathrm{k}_1 z + \sum_{n=2}^{\infty} k_n z^n$; ……….. (2)

which satisfies the condition $\Re\left(\frac{f'(z)}{g'(z)}\right) > \delta$ $\qquad \dots\dots\dots\dots (3)$

Furthermore, it appears that there is no loss of fundamental truth by making the assumption $f(0) = 0$; $f'(0) = 1$, with $g'(0) = \mathrm{k}_1 = e^{i\Theta}$ for each Θ belongs to $\frac{-\pi}{2} < \Theta < \frac{\pi}{2}$.

The set of all close- to -convex functions is denoted by $\mathcal{CV}$. (cf. [1], [2], [3]).

Obviously, in the form (3) the inverse function $g^{-1}(w)$ is analytic in the open unit disk $\mathcal{D}$ (convex domain $\mathcal{D} = g(\mathcal{D})$) , so that $\phi(w) = f\left(g^{-1}(w)\right): \mathcal{D} \longrightarrow \mathcal{D}$ satisfies

$$\Re(\phi'(w)) = \Re\left(\frac{f'(z)}{g'(z)}\right) > 0, \ \ w = g(z) \in \mathcal{D} \dots\dots\dots\dots (4)$$

where $\phi(w) = f(g^{-1}(w))$ with $f(z)$ is analytic function in $|z| < 1$, and $g^{-1}(w)$ is analytic function in the convex domain D.

A close-to-convex function is defined by a complex numerical factor $e^{i\varphi}$, which is sometimes incorrectly replaced by $e^{i\varphi}$. While experts in the field are aware that this replacement cannot be made without fundamentally changing the class, explicit reasons for this appear to be lacking in the literature. [4]

The main interests and central tools to study any geometric properties of analytic functions in the theory of analytic and geometric complex function theory are the Schwarz Lemma and Jack's Lemma. [5], [6].

Both lemmas are defined as follows:

Schwarz Lemma 1.1. [7]

Let $f: D \longrightarrow D$ be analytic function with $f(0) = 0$. Then $|f(z)| \leq |z|$ for all $z \in D$, and $|f'(0)| \leq 1$. In addition, if the equality $|f(z)| = |z|$ holds for any $z \neq 0$, or $|f'(0)| < 1$ then f is a rotation, that is, $f(z) = ze^{i\Theta}$, Θ real.

Jack's Lemma 1.2. [8]

Let $f(z)$ be a non- constant and analytic function in the unit disk D with $f(0) = 0$. If $|f(z)|$ attains its maximum value on the circle $|z| = r$ at the point z_0, then $\frac{z_0 f'(z_0)}{f(z_0)} = k$, where $k \geq 1$ is a real number.

As a result, we realized that these lemmas play a key role in describing the geometric concept for the maximum value of modulus of analytic function image on the circle $|z| = r$, since *Schwarz Lemma* is applied readily by the maximum modulus principle.

Also, the Schwarz Lemma is a key tool in complex function theory. This lemma is a significant result because it provides estimates for the values of analytic functions defined from the unit disk into itself. It is useful in many fields of analysis, particularly in the theory of geometric function hyperbolic geometry. The common formulation of the standard *Schwarz Lemma*, which is a direct application of the maximum modulus principle, [9], [10]. This topic's literature review can begin with the author [11] who is established the Jack's lemma, which is an essential lemma that serves as the foundation for many results in geometric function theory because the Jack's lemma has several applications and generalizations in the theory of differential subordinations, and then the author generalized the Nunokawa's lemma as an auxiliary tool in its work.

The references [12] and [13] provide historical context for the *Schwarz Lemma* and its applications on the unit disk boundary. Also, [14] and [12] so how a different application of *Jack's Lemma*.

Schwarz Lemma has such a broad range of applications, there have been numerous studies conducted on it (cf. [15], [16]). Therefore, *Schwarz Lemma* has the simple consequence that if f grows continuously to some boundary point p with $|p| = 1$, and if $|f'(p)|$ exists, and $|f'(p)| \geq 1$.

Hence, $|f'(p)| \geq \frac{2}{1+|f'(0)|}$, this is regarded to as the *Schwarz Lemma* on the boundary. As just a consequence, we will show the significance of *Jack's Lemma* in investigating the existence of a point $z_0 = g^{-1}(w_0) \in \mathcal{D}$, [17].

For historical background about the *Schwarz Lemma* and its applications on the boundary of the unit disk. Also, a different application of Jack's Lemma is shown in [9] .

Furthermore, we realized that when researchers study the geometric properties of the starlike function, convex function, and close to convex function in many methods such as convolution and subordination techniques, they focus on the bounded radius and bounded boundary rotations, respectively. Because the Schwarz Lemma has such a broad range of applications [18], there have been numerous studies conducted on it. Among these is the boundary version of the *Schwarz Lemma*, which is concerned with estimating the modulus of the derivative of the function at some boundary point of the unit disk [9].

That is why the properties of such close to convex depend on the given region, as discussed by [19], when the author shifted his attention from the unit disk to the unit disk, complicating matters further by the fact that composition operators do not have to be bounded on the Hardy or Bergman spaces of disk.

M. Jeong discovered a necessary and sufficient condition for analytic function with fixed points just at the unit disk's boundary, as well as certain associations with the function's derivatives at these fixed points [20] .

Ornek [21], investigated a boundary version of the Schwarz lemma for classes associated with boundary condition. In addition, at the boundary point $b, f(b) = b$, we estimate a modulus of the angular derivative of the $f(z)$ function. it has been reached more general results by taking into account the coefficients a_2 and a_3.

Mercer [22], establishes a variant of the *Schwarz Lemma* in which the images of two points are known. In addition, he considers some Schwarz and Carathoeodory inequalities at the boundary as a result of a Rogosinski Lemma [23], and demonstrates an application of Jack's Lemma for certain subclasses of analytic functions on the unit disk.It has also provided the *Schwarz Lemma* for this class. Furthermore, he provided the Schwarz Lemma for this class at the boundary.

In this paper, we clearly show how to apply Jack's Lemma to certain subclasses of close – to convex functions on the unit disk. We will also provide the *Schwarz Lemma* for this class. Furthermore, we will provide the *Schwarz Lemma* for this class at the boundary.

2. Main Results.

Regarding the previous section's introduction, which addressed the Schwarz lemma and jack's lemma in discovering the properties of convex function and starlike function. In this section and specifically in the theorem below, we engaged another criterion that played an important part in establishing geometric properties for the close to convex function is self - conformal mapping.

Subordination, Cauchy estimate, and Rogosinski Lemma all played essential parts in obtaining interesting results for close to convex functions, but in our paper, we chose the self conformal mapping to provide further geometric conditions.

Theorem 2.1. If $f(z) = z + e_2 z^2 + e_3 z^3 + \cdots$ be a self-conformal mapping and close to convex function (of order $\delta, 0 < \delta < 1$) in the open unit disk $\mathcal{D} = \{z \in \mathbb{C}: |z| < 1\}$. Then $\Re\left(\frac{f'}{g'}(w_1)\right) \geq -2 + \frac{\delta}{w_1}$, for a some $w_1 \in \partial\mathcal{D}$, and the sharp bound for the function in the form $f(z) = \frac{z}{(1-z)^{2\delta}}$, $0 < \delta < 1$, where $w = g(z) \in \mathcal{D}$.

Proof. Given $f(z) = z + e_2 z^2 + e_3 z^3 + \cdots$ be a self-conformal mapping defined in the open unit disk $\mathcal{D} = \{z \in \mathbb{C}: |z| < 1\}$.
Consider the function

$$\psi(w) = \frac{\phi(w) - 2w}{\phi(w) - 2\delta}, \qquad \psi(0) = 0. \qquad \dots\dots\dots\dots(5)$$

It seems from hypothesis that the inverse function $g^{-1}(w)$ is analytic in the open unit disk $\mathcal{D}$, so that $\phi(w) = f\left(g^{-1}(w)\right): \mathcal{D} \longrightarrow \mathcal{D}$ satisfies

$$\Re(\phi'(w)) = \Re\left(\frac{f'(z)}{g'(z)}\right) > 0, \quad w = g(z) \in \mathcal{D}.$$

From the *Schwarz Lemma*, we must clearly show that $|\psi(w)| < 1$, for $|z| < 1$ in $\mathcal{D}$, where $\psi: \mathcal{D} \longrightarrow \mathcal{D}$.
Suppose that there exist a point $w_1 \in \partial\mathcal{D}$ such that $\max_{|w| \leq |w_1|} |\psi(w)| = |\psi(w_1)| = 1$.
From *Jack's Lemma;* one can have

$$\psi(w_1) = e^{i\Theta}, \qquad \frac{w_1 \psi'(w_1)}{\psi(w_1)} = k \qquad \dots\dots\dots(6)$$

Therefore, from (5) and when $\psi(w_1) = \frac{\phi(w_1) - 2w_1}{\phi(w_1) - 2\delta}$ at a point $w_1 \in \mathcal{D}$, we obtain

$$\phi(w_1) = \frac{2\delta\, \psi(w_1) - 2w_1}{\psi(w_1) - 1}. \qquad \dots\dots\dots(7)$$

Next step in this technique we can find the following

$$\Re(\phi'(w_1)) = \Re\left(\frac{(\psi(w_1)-1)[2\delta\,\psi'(w_1)-2]-(2\delta\,\psi(w_1)-2w_1)[\psi'(w_1)]}{(\psi(w_1)-1)^2}\right)$$
$$= \Re\left(\frac{2\,\psi'(w_1)(w_1-\delta)+2\,[1-\psi(w_1)]}{(\psi(w_1)-1)^2}\right).$$

From (6) we obtain $\psi'(w_1) = \frac{2ke^{i\Theta}}{w_1}$, where $\psi(w_1) = e^{i\Theta}$ such that

$$\Re(\phi'(w_1)) = \Re\left(\frac{\frac{2ke^{i\Theta}}{w_1}(w_1-\delta)+2\left[1-e^{i\Theta}\right]}{(e^{i\Theta}-1)^2}\right)$$
$$= \Re\left(\frac{2ke^{i\Theta}(w_1-\delta)}{w_1(e^{i\Theta}-1)^2}+\frac{2}{e^{i\Theta}-1}\right).$$

As a result, we have to estimate each term in the real part of $\phi'(w_1)$ as follows

$$\frac{2ke^{i\Theta}(w_1-\delta)}{w_1(e^{i\Theta}-1)^2} = \frac{2k}{e^{-i\Theta}(e^{i\Theta}-1)^2}-\frac{2\delta\mathrm{k}}{w_1e^{-i\Theta}(e^{i\Theta}-1)^2}$$
$$= \frac{2k}{e^{-i\Theta}(e^{i\Theta}-1)^2}\left[1-\frac{\delta}{w_1}\right]$$
$$= \frac{2k}{-2+2\cos\Theta}\left[1-\frac{\delta}{w_1}\right]$$
$$= \frac{k}{\cos\Theta-1}-\frac{k\delta}{w_1(\cos\Theta-1)}.$$

Similarly, one can do the simplest for the second term as well

$$\frac{2}{e^{i\Theta}-1} = \frac{2}{(\cos\Theta-1)+i\sin\Theta}.$$

Therefore, we get

$$\Re(\phi'(w_1)) = \Re\left(\frac{2ke^{i\Theta}(w_1-\delta)}{w_1(e^{i\Theta}-1)^2}+\frac{2}{e^{i\Theta}-1}\right)$$
$$= \Re\left(\frac{k}{\cos\Theta-1}-\frac{k\delta}{w_1(\cos\Theta-1)}+\frac{2}{(\cos\Theta-1)+i\sin\Theta}\right).$$
$$\Re(\phi'(w_1)) = \frac{k}{-2\sin^2\frac{\Theta}{2}}+\frac{k\delta}{2w_1\sin^2\frac{\Theta}{2}}-\frac{-4\sin^2\frac{\Theta}{2}}{4\sin^4\frac{\Theta}{2}+\sin^2\Theta}.\ldots\ldots\ldots(8)$$

From (8) the maximum value for $\sin\frac{\Theta}{2} = \frac{1}{\sqrt{2}}$ when $\frac{-\pi}{2} < \Theta < \frac{\pi}{2}$.

$$\Re(\phi'(w_1)) = -k+\frac{k\delta}{w_1}-1$$
$$\Re(\phi'(w_1)) \geq -2+\frac{\delta}{w_1}. \qquad \ldots\ldots\ldots\ (9)$$

$$\Re(\phi'(w_1)) \geq -2 \qquad \ldots\ldots\ldots\ (10)$$

Since k is non negative and $|w_1| \leq 1$. This consequence indicate that there is no point in $\mathcal{D}$, where the modulus of $\psi(w_1) = 1$, for each $w_1 \in \mathcal{D}$, hence we obtain that $\psi(w) < 1$ for $w \in \mathcal{D}$.

Schwarz Lemma will make sense when $|\psi'(0)| \leq 1$, and

$$|\psi'(0)| = \frac{2\phi'(\delta+w)-2\phi}{(\phi-2\delta)^2}$$

$$|\psi'(0)| = \left|\frac{2\phi'(0)\delta - 2\phi(0)}{(\phi(0) - 2\delta)^2}\right| \leq 1. \quad \ldots\ldots\ldots\ldots(11)$$

At this point, we must derive the function $\phi(w)$ in order to obtain its value at $w = 0$, as shown below

$\phi'(g(z)) = \frac{f'(z)}{g'(z)}$; $g(z) = w$.

$$\phi'(w) = \frac{1 + 2e_1 z + 3e_3 z^2 + 4e_4\ z^3 \ldots}{k_1 + 2k_2 z + 3k_3 z^2 + 4k_4 z^3 + \cdots}$$

$$\phi'(w) = \frac{1}{k_1} + \left[\frac{2}{k_1}\left(e_2 - \frac{k_2}{k_1}\right)\right] z + \left[\frac{3}{k_1}\left(e_3 - \frac{k_2}{k_1}\right) - \frac{4k_2}{{k_1}^2}\left(e_2 - \frac{k_2}{k_1}\right)\right] z^2 + \cdots \ldots\ldots\ldots\ldots(12)$$

$$\phi'(g(z)) = \frac{f'(z)}{g'(z)} = c_0 + c_1 z + c_2 z^2 + c_3 z^3 + \cdots,$$

where $c_0 = \frac{1}{k_1}, c_1 = \frac{2}{k_1}\left(e_2 - \frac{k_2}{k_1}\right), c_2 = \frac{3}{k_1}\left(e_3 - \frac{k_2}{k_1}\right) - \frac{4k_2}{{k_1}^2}\left(e_2 - \frac{k_2}{k_1}\right). \ldots.$

As a result, we obtain $\phi(0) = 0$; with $\phi'(0) = c_1$.

By the form (11) for finding the modulus this

$$\left|\frac{2\phi'(0)\delta - 2\phi(0)}{(\phi(0) - 2\delta)^2}\right| = \left|\frac{2c_1\delta - 2c_0}{(c_0 - 2\delta)^2}\right|$$
$$= \frac{|2c_1\delta|}{4\delta^2}$$
$$= \frac{|c_1|}{2\delta} \leq 1.$$

Then, $|c_1| \leq 2\delta, 0 < \delta < 1.$

This result will be the sharp bound for the function in the form $f(z) = \frac{z}{(1-z)^{2\delta}}$, $0 < \delta < 1.$

Finally, we found that a geometric properties condition similar to the defining properties of convex and starlike function can characterize a close to convex function.■

Beyond Theorem 2.1, we will obtain another nice result related to the meaning of sharpness in complex analysis in particular, but this time when f has an angular limit at $\partial\mathcal{D}$. angular limit at $\partial\mathcal{D}$ as a concept will have an effect on the estimation for the close to convex condition when combined with another technique to achieve our goal.

Furthermore, it is known that if the function has an angular limit at the boundary domain, its derivative exists at any boundary point.

There is a main corollary for the finite angular derivative; the analytic function f has a finite angular derivative if and only if f' has the finite angular limit at the boundary point [9].

Theorem 2.2. If $f(z) = z + e_2 z^2 + e_3 z^3 + \cdots$ be a self-conformal mapping defined in the open unit disk $\mathcal{D} = \{z \in \mathbb{C}: |z| < 1\}$, and f has angular limit at $p \in \partial\mathcal{D}$. Then the inequality $\left|\frac{f'(z)}{g'(z)}\right| \geq \frac{-(1-2\delta)}{2(1+\delta)}$ is sharp with extremal functions $f(z) = \frac{z}{(1-z)^{2\delta}}$, and $g(z) = \frac{z}{1+z}$, where $0 < \delta < 1.$

Proof. Consider

$\psi(w) = \frac{\phi(w)-2w}{\phi(w)-2\delta}$, $\psi(0) = 0$, with $|\psi(w)| < 1.$

As known that $\phi(w) = f\left(g^{-1}(w)\right)$, $g^{-1}(w) = z.$

$$\phi(w_1) = \frac{2\delta\ \psi(w_1) - 2w_1}{\psi(w_1) - 1}.$$

Also, we have $|\psi(p)| = 1$ for $p \in \partial\mathcal{D}$.

$$\psi'(w) = \frac{2\phi'(w-\delta)}{(\phi(w)-2\delta)^2} - \frac{2}{(\phi(w)-2\delta)}.$$

From $|\psi'(p)| \geq 1$, and by *Schwarz Lemma*, one can obtain the follows

$$1 \leq |\psi'(p)| = \frac{2|\phi'(p)(p-\ \delta) - (\phi - 2\delta)|}{|\phi - 2\delta|^2}$$
$$\leq \frac{2|\phi'(p)||(p-\ \delta)| + |(\phi - 2\delta)|}{|\phi - 2\delta|^2}$$
$$\leq \frac{2|\phi'(p)|(1+\ \delta) + (1-2\delta)}{|1-2\delta|^2}$$
$$= \frac{2|\phi'(p)|(1+\ \delta)}{(1-2\delta)^2} + \frac{(1-2\delta)}{(1-2\delta)^2}.$$

Hence,

$\frac{-1}{1-2\delta} \leq 2|\phi'| \frac{1+\delta}{(1-2\delta)^2}$.

Finally, will have $|\phi'| \geq \frac{-(1-2\delta)}{2(1+\delta)}$. …………… (13)

Now it's time to show that the inequality (13) is sharp.

Consider $f(z) = \frac{z}{(1-z)^{2\delta}}$, and $g(z) = \frac{z}{1+z}$ such that $\varphi(z) = \frac{f(z)}{g(z)} = \frac{1+z}{1-z}$.

Differentiating both of $f(z)$ and $g(z)$ with respect to z as follows

$f'(z) = \frac{1+2\delta z(1-z)^{-1}}{(1-z)^{2\delta}}$, $g'(z) = \frac{1}{(1+z)^2}$.

Therefore,

$\varphi' = \frac{1+2z+z^2+2\delta z-2\delta z^2}{(1-z)^{2\delta}}$.

Since $\varphi' = \frac{f'}{g'}$.

Short calculation will yield

$$(1-z)^{2\delta}\varphi' = (1-2\delta)z^2 + 2z(1+\delta) + 1$$

$$\frac{(1-z)^{2\delta}}{2z^2(1+\delta)}\varphi' = \frac{1-2\delta}{2(1+\delta)} + \frac{1}{z} + \frac{1}{2z^2(1+\delta)}. \quad \text{……………} (14)$$

Estimate the second term and the last term in (14) when $z \to \infty$ then $\lim_{z\to\infty} \frac{1}{z} = 0$, and $\lim_{z\to\infty} \frac{1}{2z^2(1+\delta)} = 0$.

By L'Hospital's Rule, we have

$$\lim_{z\to\infty} \frac{(1-z)^{2\delta}}{2z^2(1+\delta)} = \lim_{z\to\infty} \frac{-2\delta(1-z)^{2\delta-1}}{4z(1+\delta)}$$

$= \lim_{z\to\infty} \frac{-\delta\, z^{2\delta}\left(\frac{1}{z}-1\right)^{2\delta-1}}{2z^2(1+\delta)}$.

Take $\lim \frac{-z^{2\delta}}{z^2}$ by setting $z = z^{\delta}$ then $\lim \frac{-z^{2\delta}}{z^2} = -\lim\left(\frac{z^{\delta}}{z}\right)^2 = -1$ as $z \to \infty$.

As a result, the equation takes the final form shown below

$$\varphi' = \frac{-(1-2\delta)}{2(1+\delta)}.$$

The inequality (13) can be expanded by letting a_1, the second coefficient in the function's expansion $(z) = 1 + a_1 z + a_2 z^2 + \cdots$.■

If f extends continuously to some boundary point and its value exists, then the modulus of deivative function f at the boundary point will be greater than one according to the classical

Schwarz Lemma. Furthermore, if we put a condition at the boundary domain that is fixed at zero, that is $f(0) = 0$, the modulus of the deivative function f at the boundary point will undoubtedly change [9].

Theorem 2.3. Let $f(z) = z + e_2 z^2 + e_3 z^3 + \cdots$ be a self-conformal mapping and close to convex function (of order $\delta, 0 < \delta < 1$) in the open unit disk $\mathcal{D} = \{z \in \mathbb{C}: |z| < 1\}$. Then, if f is extended continuously to the boundary of $\mathcal{D}$, then $\left| \left(\frac{f'}{g'}\right)(p) \right| \geq \frac{|1-2\delta|(|1-2\delta|-2)}{|p-\delta\,|}$ where $0 < \delta < 1$.

Proof.
Let $\psi(w) = \frac{\phi(w)-2w}{\phi(w)-2\delta}$, $\psi(0) = 0$ with $|\psi(w)| < 1$. If $p \in \partial\mathcal{D}$ that is f can be extended continuously to a point p, so that $|f(p)| = 1$ and $f'(p)$ exists. Then

$$|f'(p)| \geq \frac{2}{1 + |f'(0)|}$$

Thus $|f'(p)| \geq 1$.
From theorem (2.2), already we have $|\psi(p)| = 1$ for $p \in \partial\mathcal{D}$. Hence, it is worth to apply the form below with respect to point p.

$$\psi'(p) = \frac{2\phi'(p-\delta)}{(\phi(p) - 2\delta)^2} - \frac{2}{(\phi(p) - 2\delta)}.$$

Therefore,

$$\frac{2}{1 + |\psi'(0)|} \leq |\psi'(p)| = \frac{2\left|\phi'(p)\ \ [p-\delta]\ \ -[\phi(p)-2\delta]\,\right|}{|\phi(p)-2\delta|^2},$$

If $\phi(p) = 1$ then

$$\frac{2}{1 + |\psi'(0)|} \leq |\psi'(p)| = \frac{2|\phi'(p)\,[p-\delta] - [1-2\delta]\,|}{|1-2\delta|^2}$$
$$\leq 2|\phi'(p)| \frac{|p-\delta\,|}{|1-2\delta|^2} + \frac{2|1-2\delta|}{|1-2\delta|^2}.$$

Hence, we obtain

$$\frac{2}{1 + |\psi'(0)|} \leq |\psi'(p)| \leq 2|\phi'(p)| \frac{|p-\delta\,|}{|1-2\delta|^2} + \frac{2}{|1-2\delta|}.$$

Consider,
$\psi(w) = \frac{\phi(w)-2w}{\phi(w)-2\delta}$, $\psi(0) = 0$, with $|\psi(w)| < 1$.
Given $\phi(w) = f\big(g^{-1}(w)\big)$, $g^{-1}(w) = z$, such that

$$\psi(w) = \frac{f(z) - 2w}{f(z) - 2\delta} = \frac{z + e_2 z^2 + e_3 z^3 + \cdots\ 2w}{z + e_2 z^2 + e_3 z^3 + \cdots 2\delta}. \quad \ldots(15)$$

Drive (15) twice to get

$$\psi'(w) = \frac{2\,e_2 + 6e_3 z + \cdots}{2\,e_2 + 6e_3 z + \cdots} = 1.$$

Therefore, $|\psi'(0)| = 1$.
As a result, $1 \leq |\psi'(p)|$

$$1 \leq |\psi'(p)| \leq 2|\phi'(p)| \frac{|p-\delta\,|}{|1-2\delta|^2} + \frac{2}{|1-2\delta|}.$$

So that,

$$1 - \frac{2}{|1-2\delta|} \leq 2|\phi'(p)| \frac{|p-\delta\,|}{|1-2\delta|^2}$$

$$|\phi'(p)| \geq \frac{|1-2\delta|(|1-2\delta|-2)}{|p-\delta\ |}$$

$$\left|\left(\frac{f'}{g'}\right)(p)\right| \geq \frac{|1-2\delta|(|1-2\delta|-2)}{|p-\delta\ |} \quad \blacksquare$$

3. Conclusions

This paper considered the Self - conformal mapping with close to convex to show some interesting geometric properties related to the boundary of a given domain $\mathcal{D} = \{z \in \mathbb{C}: |z| < 1\}$, the angular derivative of f, and the extended continuously f, all of which are connected with sharp inequalities in this work.

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