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U- S Jordan Homomorphisim of Inverse Semirings

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Abstract
Let *S* be an inverse semiring, and *U* be an ideal of *S*. In this paper, we introduce the concept of *U-S Jordan homomorphism* of inverse semirings, and extend the result of Herstein on Jordan homomorphisms in inverse semirings.

Keywords: Inverse semiring, Jordan homomorphism, prime ring.

تشاكلات جوردان U–S على اشباه الحلقات المعكوسة

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الخلاصة
ليكن *S* شبه حلقة معكوسة و *U* مثالي في S. في هذا البحث قدمنا مفهوم تشاكلات جوردان *U–S* لاشباه الحلقات المعكوسة ووسعنا نتيجة Herstein لتشاكلات جوردان في اشباه الحلقات المعكوسة.

1. Introduction

A nonempty set *S* with a binary operation $*$ is said to be semigroup iff for all *x, y, z* ϵ *S* we have, $x*(y*z) = (x*y)*z$. The study of semiring dates back to H.S.Vandiver[1], a nonempty set *S* with two binary operations + and $*$ is said to be semiring iff (*S*, +) semigroup, (*S*, .) semigroup and, $x.(y+z) = x.y + x.z$ and $(y+z).x = y.x + z.x$ holds for all *x, y, z* ϵ *S.*

A semiring *S* is called inverse semiring, If for all $x \in S$ there exist $x' \in S$ such that $x + \acute{x} + x = x$ and $\acute{x} + x + \acute{x} = \acute{x}$ and this element is unique. Also *S* is called an additively inverse semiring if (*S*, +) is an inverse semigroup, that is for each $x \in S$ there exist $\acute{x} \in S$ such that $x + \acute{x} + x = x$ and $\acute{x} + x + \acute{x} = \acute{x}$, and this element is unique. We recall that an inverse semiring *S* is called semiprime if whenever $x S x = 0$ implies $x = 0$ for all $x \in S$, and *S* is called prime, if whenever, $x S y = 0$ implies either $x = 0$ or $y = 0$ for all $x, y \in S$.

A semiring *S* is said to be *n*-torsion free iff whenever $n.\ x = 0$ then $x = 0$ for all $x \in S$, where $n \neq 0$.
If *U* is nonempty subset of *S*, *U* is called left ideal of *S* if $x + y \in S$ for all $x, y \in I$, $r.\ x \in S$ for all $x \in U$, $r \in S$ and $U \neq S$ (Similarly right ideal).

Herstein in 1950 studied Jordan derivations [2] and Jordan homomorphisms [3, 4] in prime rings. Bresar [5- 7], Baxter and Martindale [8] generalized Herstein's results on semiprime rings.

In this paper, we introduce the definition of *U-S* Jordan homomorphism in inverse semirings and extend the result of Herstein on Jordan homomorphisms in the setting of inverse semirings.

2. Priliminaries

Definition 2.1[9]:

A non empty set *S* with a binary operation ($*$), is called semigroup iff

$$x*(y*z) = (x*y)*z \quad \text{for all } x, y, z \in S.$$

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Definition 2.2[9]:

A non empty set S with two binary operations (+) and (·), such that (S,+) and (S, .) are semigroups, where + is a commutative operation and $x\,.\,(y+z)= x\,.\,y + x\,.\,z$, $(y + z)\,.\,x = y\,.\,x + z\,.\,x$ holds for all $x, y, z \in S$, then, $(S,+,\cdot)$ is called semiring .

Definition 2.3[10]:

Let S be a semiring , S is called additively inverse if $(S,+)$ is an inverse semigroup (i.e) for each $x \in S$ there exists a unique element $x' \in S$ such that

$$x = x + x' + x \text{ and } x' = x' + x + x'.$$

Definition 2.4[10]:

Let S be a semiring, S is called an inverse semiring, if for each $x \in S$, there exists a unique element x' such that:

$$x = x + x' + x \text{ and } x' = x' + x + x'.$$

Definition 2.5[9]:

Let S be a semiring and let U be a subset of S , U is called a left ideal of S if :

i) $x, y \in U$ then $x + y \in U$.

ii) $x \in U$, $r \in S$ then $r.x \in U$.

The right ideal is defined in the similar way, and an ideal of S is a subset which is both a left ideal and a right ideal of S.

Examples 2.6 [11]:

1) If S is an inverse semiring, then clearly $M_n(S)$ under usual addition and multiplication is an inverse semiring for every positive integer n, for example

$$M_2(S) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in S \right\}$$

(M_2 (S) , + , .) is an inverse semiring .

2) Let S be any semiring, consider the set $S[x]$ of polynomials under usual addition and multiplication, for each $x \in S$ then $S[x]$ is an inverse semiring.

Definition 2.7 [10]:

Let S be a semiring then S is called

(i) Additively commutative iff for all x , y $\in S$,

$$x + y = y + x$$

(ii) Multiplicatively commutative if and only if for all $x, y \in S$,

$$x.\,y = y.\,x$$

(S , + , .) is called commutative semiring iff both (i) and (ii) hold.

Proposition 2.8 [10]:

Let S be an inverse semiring, then for each $x, y \in S$

(i) $x'' = x$

(ii) $(x + y)' = x' + y'$.

(iii) $(xy)' = x'y = xy'$.

(iv) $x'y' = xy$.

In [5] Jordan homomorphism in inverse semirings is defined as follows:

Let S and T be inverse semirings then, an additive mapping $\psi : S \to T$ is called a Jordan homomorphism if:

$\psi(xy + yx) + \psi(x)\,\psi(y)' + \psi(y)\,\psi(x)' = 0$ for all $x, y \in S$.

We need the following lemmas for our results.

Lemma 2.9[12]:

Let S be an inverse semiring and $x, y \in S$, if $x + y = 0$ then $x = y'$.

Lemma 2.10[12]:

Let S be a 2-torsion free semiprime inverse semiring if $x, y \in S$ such that $xry + yrx = 0$ then $xry = yrx' = 0$.

Lemma 2.11[13]:

Let S be a 2- torsion free prime inverse semiring, if $x, y \in S$ are such that $xry + yrx = 0$ for all $r \in S$.

Then either $x = 0$ or $y = 0$.

Throughtout this paper S and T are two inverse semirings and U be an ideal of S.

Now we introduce the definition of *U-S Jordan homomorphism* as follows:

Definition 2.12:

A map $\psi: S \to T$ is called *U -S Jordan homomorphism* if :

i) ψ is additive

ii) $\psi(xr + rx) + \psi(x)\,\psi(r)' + \psi(r)\psi(x)' = 0$ for all $r \in S$ and for all $x \in U$.

3. Results

Lemma 3.1:

Let a map $\psi: S \to T$ *U -S Jordan homomorphism*, and T is 2-torsion free, for all $a, c \in U$, $b \in S$, then the following statement are true.

(*i*) $\psi(a^2) = \psi(a)^2$

(*ii*) $\psi(aba) = \psi(a)\psi(b)\psi(a)$

(*iii*) $\psi(a\,b\,c + c\,b\,a) = \psi(a)\,\psi(b)\,\psi(c) + \psi(c)\,\psi(b)\,\psi(a)$.

Proof(i):

Let $a \in U$, $b \in S$

Since ψ is U-S Jordan homomorphism of inverse semiring S into 2- torsion free inverse semiring. Then

$$\psi(ab+ba) + \psi(a)\,\psi(b)' + \psi(b)\,\psi(a)' = 0 \quad (1)$$

We replace a by b in (1), we get:

$$\psi(a^2 + a^2) + \psi(a)\,\psi(a)' + \psi(a)\,\psi(a)' = 0$$

$$\psi(2a^2) + 2\,\psi(a)\,\psi(a)' = 0$$

$$2\psi(a^2) + 2\,\psi(a)\,\psi(a)' = 0$$

Since T is 2-torsion free, then

$$\psi(a^2) + \psi(a)\,\psi(a)' = 0.$$

And by Lemma (2.9), we get

$$\psi(a^2) = (\psi(a)\,\psi(a)')' = \psi(a)\,\psi(a)'' = \psi(a)^2.$$

Then, $\psi(a^2) = \psi(a)^2$ for all $a \in S$.

Proof (ii):

Let $a \in U, b \in S$

Since ψ is U-S Jordan homomorphism of an inverse semiring

Then,

$$\psi(ab + ba) + \psi(a)\,\psi(b)' + \psi(b)\,\psi(a)' = 0 \qquad \text{for all } a \in U, b \in S.$$

Put $b = ab + ba$

$\psi(a(ab + ba)) + (ab + ba)a) + \psi(a)\,\psi(ab + ba)' + \psi(ab + ba)\,\psi(a)' = 0$

$\psi(a^2b + aba + aba + ba^2) + \psi(a)\,\psi(ab + ba)' + \psi(ab + ba)\,\psi(a)' = 0$

$$\psi(a^2b + 2aba + ba^2) + \psi(a)\,\psi(ab + ba)' + \psi(ab + ba)\,\psi(a)' = 0 \quad (2)$$

In the view of Lemma 2.9 and by Definition(2.12)

$\psi(ab + ba) = (\psi(a)\,\psi(b)' + (\psi(b)\,\psi(a)')'$

$= (\psi(a)\,\psi(b)')' + (\psi(b)\,\psi(a)')'$

$= \psi(a)\,\psi(b) + \psi(b)\,\psi(a)$

$\psi(ab + ba)' = (\psi(a)\,\psi(b) + \psi(b)\,\psi(a))'$

$= (\psi(a)\,\psi(b))' + \psi(b)\,\psi(a))'$

$= \psi(a)\,\psi(b)' + \psi(b)\,\psi(a)'$

Then, we can replace

$\psi(ab + ba)$ by $\psi(a)\,\psi(b) + \psi(b)\,\psi(a)$ and $\psi(ab + ba)$ by $\psi(a)\,\psi(b) + \psi(b)\,\psi(a)$ in (2).

Thus,

$\psi(a^2b + ba^2 + 2aba) + \psi(a)\,(\psi(a)\,\psi(b)' + \psi(b)\,\psi(a)') +$
$(\psi(a)\,\psi(b) + \psi(b)\,\psi(a))\,\psi(a)' = 0$

$\psi(a^2b + ba^2) + \psi(2aba) + \psi(a)\,\psi(a)\,\psi(b)' + \psi(a)\,\psi(b)\,\psi(a)'$
$+ \psi(a)\,\psi(b)\,\psi(a)' + \psi(b)\,\psi(a)\,\psi(a)' = 0$

$\psi(a^2b + ba^2) + 2\psi(aba) + \psi(a)^2\psi(b)' + 2\psi(a)\,\psi(b)\,\psi(a)' + \psi(b)(\psi(a)\,\psi(a))' = 0$

$\psi(a^2b + ba^2) + 2\psi(aba) + \psi(a)^2\psi(b)' + 2\psi(a)\,\psi(b)\,\psi(a)' + \psi(b)\,(\psi(a)^2)' = 0$

Since $\psi(a^2b + ba^2) + \psi(a)^2\,\psi(b)' + \psi(b)(\psi(a)^2)' = 0$
$2\psi(aba) + 2\psi(a)\,\psi(b)\,\psi(a)' = 0$
Since T is 2- torsion free, then
$\psi(aba) + \psi(a)\,\psi(b)\,\psi(a)' = 0$
Then by Lemma (2.9) we get,
$\psi(aba) = (\psi(a)\,\psi(b)\,\psi(a)')'$
Since $\psi(a)\,\psi(b)\,\psi(a)'' = \psi(a)\,\psi(b)\,\psi(a)$
Therefore,

$$\psi(aba) = \psi(a)\,\psi(b)\,\psi(a)\,. \qquad \blacksquare$$

Proof (iii):
Let $a \in U$, $b \in S$
By linearizing the relation

$$\psi(aba) + \psi(a)\,\psi(b)\,\psi(a)' = 0$$

(i.e) Let $a \equiv a+c$ where $c \in U$
$\psi((a+c)\,b(a+c)) + \psi(a+c)\,\psi(b)\,\psi(a+c)' = 0$
$\psi(aba + abc + cba + cbc) + (\psi(a) + \psi(c)\,\psi(b)\,(\psi(a)' + \psi(c)') = 0$
$\psi(\text{aba}) + \psi(abc) + \psi(cba) + \psi(cbc) + \psi(a)\,\psi(b)\,\psi(a)' + \psi(a)\,\psi(b)\,\psi(c)' +$
$\psi(c)\,\psi(b)\,\psi(a)' + \psi(c)\,\psi(b)\,\psi(c)' = 0.$
By using (ii), we get,

$$\psi(abc) + \psi(cba) + \psi(a)\,\psi(b)\,\psi(c)' + \psi(c)\,\psi(b)\,\psi(a)' = 0$$

Therefore,

$$\psi(a\,bc + cba) = \psi(a)\,\psi(b)\,\psi(c)' + \psi(c)\,\psi(b)\,\psi(a)' = 0. \qquad \blacksquare$$

Now we put some notation
$a^b = \psi(ab) + \psi(a)\,\psi(b)'$ where $a \in U$, $b \in S$
$a_b = \psi(ab) + \psi(b)\,\psi(a)'$ where $a \in U$, $b \in S$
It's clear that by equation (1) we can get

$$a^b + b^a = 0.$$

Lemma 3.2:
Let a map $\psi: S \to T$ be *U -S Jordan homomorphism* such that T is 2- torsion free inverse semiring, then $a^b a_b = a_b a^b = 0$ for all $a \in U$, $b \in S$
<u>Proof</u> :
Let $a \in U$ $b \in S$
$a^b a_b = (\psi(ab) + \psi(a)\,\psi(b)')\,(\psi(ab) + \psi(b)\,\psi(a)')$
$= \psi(ab)\,\psi(ab) + \psi(ab)\,\psi(b)\,\psi(a)' + \psi(a)\,\psi(b)'\,\psi(ab) + \psi(a)(\psi(b)\,\psi(b)')\,\psi(a)'$
$= \psi(ab\,ab) + \psi(ab)\,\psi(b)\,\psi(a)' + \psi(a)\,\psi(b)'\psi(ab) + \psi(a)\,\psi(b^2)\,\psi(a)$
by Lemma 3.1 (ii) and(iii)
$\psi(aba) = \psi(a)\,\psi(b)\,\psi(a)$ and $\psi(cba + abc) + \psi(c)\,\psi(b)\,\psi(a)' + \psi(a)\,\psi(b)\,\psi(c)' = 0$
$\psi(a)\,\psi(b^2)\,\psi(a) = \psi(ab^2a)$
$a^b a_b = \psi(ab\,ab) + \psi(ab)\,\psi(b)\,\psi(a)' + \psi(a)\,\psi(b)\,\psi(ab)' + \psi(a)\,\psi(b^2)\,\psi(a)$
$= \psi(abab + abba) + \psi(ab)\,\psi(b)\,\psi(a)' + \psi(b)\,\psi(ab)' = 0$
Then,
$a^b a_b = 0$
To show $a_b\, a^b = 0$
$a_b a^b = (\psi(ab) + \psi(b)\,(\psi(a)')(\psi(ab) + \psi(a)\,\psi(b)')$
$= (\psi(ab)\,\psi(ab) + \psi(ab)\,\psi(a)\,\psi(b)' + \psi(b)\,(\psi(a)'\psi(ab) + \psi(b)\,\psi(a)'\psi(a)\,\psi(b)'$
$= \psi(ab)^2 + \psi(ab)\,(\psi(ab) + \psi(a)\,\psi(b)' + \psi(b)\,\psi(a)'\,\psi(ab) + \psi(b)\,\psi(a)'\psi(a)\,\psi(b)'$
By Lemma 3.1 $\psi(a)^2 = \psi(a^2)$
$= \psi(ab^2) + \psi(ab)\,\psi(a)\,\psi(b)' + \psi(b)\,\psi(a)'\psi(ab) + \psi(b)\,(\psi(a)\,\psi(a))'\,\psi(b)'$
$= \psi(ab\,ab) + \psi(ab)\,\psi(a)\,\psi(b)' + \psi(b)\,\psi(a)'\psi(ab) + \psi(b)(\psi(a)^2)'\psi(b)'$
$= \psi(ab\,ab) + \psi(ab) + \psi(a)\,\psi(b)' + \psi(b)\,\psi(a)'\psi(ab) + (\psi(b)\,\psi(a^2))'\psi(b)'$
$= \psi(ab\,ab) + \psi(ab)\,\psi(a)\,\psi(b)' + \psi(b)\,\psi(a)'\psi(ab) + \psi(b)\,\psi(a^2)\,\psi(b)$
By Lemma 3.1(ii)

$\psi(aba)= \psi(a)\ \psi(b)\ \psi(a)$
$\psi(cba+ abc)+ \psi(c)\ \psi(b)\ \psi(a)' + \psi(a)\ \psi(b)\ \psi(c)'=0$
$\psi(ab\,ab + ba\,ab) + \psi(ab)\ \psi(a)\ \psi(b)' + \psi(b)\ \psi(a)'\psi(ab) =0$
$\psi(ab\,ab + ba\,ab)+ \psi(ab)\ \psi(a)\ \psi(b)' +\psi(b)\ \psi(a)\ \psi(ab)'=0$
Then $a_b\, a^b=0$ ■

Lemma 3.3:
Let a map $\psi: S\to T$ be *U -S Jordan homomorphism* such that T is 2- torsion free inverse semiring , then for any $r,a\in U$ $b\in S$
$a^b\ \psi(r)\ a^b = a^b\ \psi([a, b]\ r)$
$a_b\ \psi(r)\ a_b = \psi([a, b]\ r)\ a_b$
Proof :
$\psi(r)\ a^b = \psi(r)\ (\psi(ab)+ \psi(a)\ \psi(b)')$
$\quad = \psi(r)\ \psi(ab) + \psi(r)\ \psi(a)\ \psi(b)'$
By Lemma 3.1 (iii)
$\psi(cba+ abc) + \psi(c)\varphi(b)\ \psi(a)' + \psi(a)\ \psi(b)\ \psi(c)'=0$
We get,
$\psi(rab + bar)+ \psi(r)\ \psi(a)\ \psi(b)'+ \psi(b)\ \psi(a)\ \psi(r)'=0$
$\psi(r)\ \psi(a)\ \psi(b)' = \psi(rab + bar)' +\psi(b)\ \psi(a)\ \psi(r)$
So,
$\psi(r)\ a^b = \psi(r)\ \psi(ab) +\psi(rab + bar)' + \psi(b)\ \psi(a)\ \psi(r)$
Since $r= r+r'+r$, we have,
$\psi(rab + bar)' = \psi((r+r'+r)\ ab + bar)'$
$\quad = \psi(rab + (r'+r)\ ab + bar)'$
$\quad = \psi(rab+ ab(r+r')+bar)'$
$\quad = \psi(rab + abr +abr'+bar)'$
$\quad = \psi(rab + abr)' + \varphi(abr'+bar)'$
Since ψ *is U-S* Jordan homomorphism , then
$\psi(rab + bar)' = \psi(r)\ \psi(ab)' + \psi(ab)\ \psi(r)'+\psi(abr'+bar)'$
Thus,
$\psi(r)\ a^b= \psi(r)\ \psi(ab) + \psi(r)\ \psi(ab)' + \psi(ab)\ \psi(r)'+ \psi(abr+ bar')+ \psi(b)\ \psi(a)\ \psi(r)$
$\quad = \psi(r)(\psi(ab) + \psi(ab)')+ \psi(b)\ \psi(a)\ \psi(r) + \psi(ab)\ \psi(r)' + \psi([a ,b]r)$
Note that
$\psi(abr + bar') = \psi(abr + ba'r)= \psi(ab + ba)r) =\psi([a,b]r)$
Thus $\psi(r)\ a^b = \psi(r)\ (\psi(ab)+ \psi(ab)')+ \psi(b)\ \psi(a)\ \psi(r)+ \psi(ab)\ \psi(r)'+ \psi([a,b]r)$
$\quad = (\psi(ab)+ \psi(ab)')\ \psi(r)+ \psi(ab)')\ \psi(r)+ \psi(b)\ \psi(a)\ \psi(r)+ \psi([a,b]r)$
$\quad = \psi(ab)'+ \psi(ab)+ \psi(ab)')\ \psi(r)+ \psi(b)\ \psi(a)\ \psi(r)+ \psi([a ,b]r)$
$\quad =(\psi(ab)'\psi(r) + \psi(b)\ \psi(a)\ \psi(r)+ \psi([a,b]r)$
$\quad = \psi(ab)\ \psi(r) ' +\psi(b)\ \psi(a)'\ \psi(r)'+ \psi([a,b]r)$
$\quad = (\psi(ab)+ \psi(b)\ \psi(a)')\ \psi(r)'+ \psi([a,b]r) =a_b\ \psi(r)' + \psi([a ,b]r)$
Thus,

$$\psi(r)\ a^b= a_b\ \psi(r)' + \psi([a ,b]r) \qquad (3)$$

$a^b\psi(r)\ a^b = a^b\ a_b\ \psi(r)' + a^b\psi([a,b]r) = a^b\ \psi([a, b]r)$
Now to prove that $a_b\ \psi(r)\ a_b = \psi([a, b]r)a_b$, multiply the equation (3) from the left by a_b , we get
$\psi(r)\ a^b a_b =a_b\ \psi(r)'\ a_b + \psi([a ,b]r).\ a_b$
Then $a_b\ \psi(r)'\ a_b + \psi([a ,b]r)\ a_b = 0$
So, by Lemma (2.9) $a_b\ \psi(r)a_b = \psi([a, b]r)\ a_b$. ■

Lemma 3.4:
Let a map $\psi: S\to T$ be *U -S Jordan homomorphism* such that T is 2- torsion free inverse semiring, if $a,r \in U$, $b\in S$.Then
$\psi([a, b]r) = \psi(r)\ a^b + a_b\ \psi(r)$

And,

$$\psi(r)\ [a,b] = a^b\,\psi(r) + \psi(r)\,a_b$$

Proof:

Let $a,r \in U$, $b \in S$

To prove that

$$\psi(r\,[a,b]) = a^b\,\psi(r) + \psi(r)\,a_b$$

Take rigtht hand

$a^b\,\psi(r) + \psi(r)\,a_b = \psi(ab)\,\psi(r) + \psi(a)\,\psi(b)'\,\psi(r) + \psi(r)\,\psi(ab) + \psi(r)\,\psi(b)\,\psi(a)'$

$$= \psi(abr + rab) + \psi(a)' + \psi(b)\,\psi(r) + \psi(r)\,\psi(b)\,\psi(a)'$$

$$= \psi(abr + rab) + \psi(a'br + rba')$$

By Lemma 3,1*(iii)*, we get,

$$= \psi(abr + rab + abr' + rba')$$

$$= \psi(ab(r + r') + rab + rba')$$

$$= \psi((r + r')\,ab + rab + rba')$$

$$= \psi((r + r' + r)\,ab + rba')$$

$$= \psi(rab + rba')$$

$$= \psi(r\,(ab + ba'))$$

$$= \psi(r[a,b])$$

Now,

$\psi(r)\,a^b + a_b\,\psi(r) = \psi(r)\,(\psi(ab) + \psi(a)\,\psi(b)') + \psi(ab)\,\psi(r) + \psi(b)\,\psi(a)'\,\psi(r)$

$$= \psi(r)\,\psi(ab) + \psi(r)\,\psi(a)\,\psi(b)' + \psi(ab)\,\psi(r) + \psi(b)\,\psi(a)'\,\psi(r)$$

$$= \psi(rab + abr) + \psi(r)\,\psi(a)\,\psi(b)' + \psi(b)'\psi(a)\,\psi(r)$$

$$= \psi(rab + abr) + \psi(rab' + b'ar)$$

$$= \psi(rab + abr + rab' + b'ar)$$

$$= \psi((r + r')\,ab + abr + b'ar)$$

$$= \psi(ab(r + r' + r) + b'ar)$$

$$= \psi(abr + b'ar)$$

$$= \psi((ab + b'a)r)$$

$$= \psi(ab + ba')r$$

$$= \psi([a,b]r) \qquad \blacksquare$$

Theorem 3.5 :

Let a map $\psi: S \rightarrow T$ be *U -S Jordan homomorphism* such that T is 2- torsion free inverse semiring, then for all $a,r \in U$, $b \in S$.

$$a_b\,\psi(r)\,a^b + a^b\,\psi(r)\,a_b = 0$$

Proof:

By Lemma 3.4 we have

$$\psi(r[a,b]) = a^b\,\psi(r) + \psi(r)\,a_b \qquad (4)$$

Replacing r by $[a,b]\,r$ in equation (4)

$$\psi([a,b]\,r\,[a,b]) = a^b\,\psi([a,b]r) + \psi([a,b])\,r)\,a_b$$

By Lemma (3.1) (ii) and Lemma (3.4) we get,

$$\psi([a,b]r) = \psi(r)\,a^b + a_b\,\psi(r)$$

Then $\psi[a,b]\ \psi(r)\ \psi[a,b] = a^b\,\psi(r)\,a^b + a^b\,a_b\,\psi(r) + (\psi(r)\,a^b + a_b\,\psi(r))\ a_b$

$$= a^b\,\psi(r)\,a^b + a^b\,a_b\,\psi(r) + \psi(r)\,a^b\,a_b + a_b\,\psi(r)\,a_b$$

$$\psi[a,b]\ \psi(r)\ \psi[a,b] = a^b\,\psi(r)\,a^b + a_b\,\psi(r)\,a_b \qquad (5)$$

Now

$$\psi[a,b] = \psi(ab + b'a) = \psi(ab + ab + ab' + b'a) = \psi(2ab) + \psi(ab' + b'a).$$

And since, ψ *is U-S* Joradan homomorphism ,then

$$\psi(ab' + b'a) = \psi(a)\,\psi(b)' + \psi(b)\,\psi(a)'$$

$$\psi(ab) + \psi(ab) = \psi(a)\,\psi(b)' + \psi(b)\,\psi(a)'$$

So

$$\psi[a,b] = a^b + a_b$$

Then the equation(5) will be

$$(a_b+a^b)\ \psi(r)(a_b+a^b) = a^b\ \psi(r)\ a^b + a_b\ \psi(r)\ a_b$$
$$a_b\ \psi(r)\ a_b + a_b\ \psi(r)\ a^b + a^b\ \psi(r)\ a_b + a^b\ \psi(r)\ a^b = a^b\psi(r)\ a^b+a_b\ \psi(r)a_b.$$

Adding $b_a\ \psi(r)\ a_b + b^a\ \psi(r)\ a^b$ on both sides of the equation above and take the left hand

$$b_a\ \psi(r)\ a_b + b^a\ \psi(r)\ a^b + a_b\ \psi(r)\ a_b + a_b\ \psi(r)\ a^b+ a^b\ \psi(r)\ a_b + a^b\ \psi(r)\ a^b$$
$$= (b_a + a_b)\ \psi(r)\ a_b + (b^a+a^b)\ \psi(r)\ a^b + a_b\ \psi(r)\ a^b + a^b\ \psi(r)\ a_b$$
$$= a_b\ \psi(r)\ a^b + a^b\ \psi(r)\ a_b$$

And when take the right hand

$$a^b\ \psi(r)\ a^b + a_b\ \psi(r)\ a_b + b_a\ \psi(r)\ a_b + b^a\ \psi(r)\ a^b$$
$$= (a^b+ b^a)\ \psi(r)\ a^b + (a_b+ b_a)\ \psi(r)\ a_b = 0.$$ ■

Lemma 3.6:

Let *S* be an inverse semiring , and *U* be an ideal in *S*. if *S* is 2-torsion free semiprime ,and a , $b \in U$, such that $axb + bxa=0$ for all $x \in U$. Then $axb=bxa=0$.

Proof:

Since, $axb + bxa = 0$ thus, $axb = (bxa)' = bxa'$

Thus,

$$axb = bxa \quad \text{….. (1)}$$

This satisfies for all $x \in U$

Then, $(bxa)\ y\ (bxa) = bx\ (ayb)\ x\ a = bx(bya')\ x\ a = a((xby)bxa \quad by\ (1)$

$$2(bxa)y\ (bxa) = (bxa)y\ (bxa) + (bxa)y(bxa) = a(xby)\ bxa + (bxa)y\ (bxa)$$
$$= (axb + bxa)y\ (bxa) = 0$$

Since *S* is 2- torsion free then, $bxa = 0$.

Since *U* is an ideal in *S*

$(b\ xa)\ U\ (bxa) = 0 \ \ (US \subseteq S)$

$US(bxa)\ US(b\ xa)=0$

$(bxa)US(bxa) \subseteq (b\ xa)U(bxa)= 0$

$US\ \ bxa\ SUS\ bxa =0$ and by semiprimness

$US\ bxa=0bxa\ S\ bxa = 0 \quad then\ \ bxa = 0$

by the same way will get $axb = 0$. ■

Lemma 3.7:

Let *S* be an inverse semiring , and *U* be an ideal in *S*. if *S* is 2-torsion free semiprime, and $a,b \in U$ are such that $axb + bxa =0$ for all $x \in U$, then either $a=0$ or $b=0$

Proof:

by Lemma (2.10)

$axb = bxa = 0$

Then either $a = 0 \ \ or\ b = 0$. ■

Theorem 3.8:

Let a map ψ: $S \to T$ be *U -S Jordan homomorphism* such that *T* is 2- torsion free semiprime inverse semiring, then either ψ is a homomorphism or an anti homomorphism on *U*.

Proof :

By Theorem (3.5) For all $a,b,r \in U$

$a_b\ \psi(r)\ a^b + a^b\ \psi(r)\ a_b= 0$

and by Lemma (2.11)

$$a_b\ \psi(r)\ a^b + a^b\ \psi(r)\ a_b= 0$$

then either $a_b=0$ or $a^b=0$

$$\psi(ab) + \psi(a)\ \psi(b)' = 0$$

and by Lemma (2.9)

$$\psi(ab) = \psi(a)\ \psi(b)$$

Or $a_b=0$

$$\psi(ab) + \psi(b)\ \psi(a)' = 0$$

and by Lemma (2.9)

$$\psi\ (ab) = \psi(b)\ \psi(a).$$

Then, ψ is either homomorphism or anti-hommorphism.. ■

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