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Mixed Galerkin- Implicit Differences Methods for Solving Coupled Parabolic Boundary Value Problems with Variable Coefficients

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Abstract
In this paper, an approximation technique is introduced to solve the coupled linear parabolic boundary value problems with variable coefficients by using mixed of the Galerkin finite element method in space variable with implicit finite difference method in the time variable. At any discrete time t_j this technique is transformed the coupled linear parabolic boundary value problems with variable coefficients into a linear algebraic system which is called a Galerkin a linear algebraic system, and then it is solved using the Cholesky Decomposition. Illustration examples are presented and the results are shown by figures and tables, and show the efficiency of the proposed method.

Keywords: Coupled parabolic boundary value problem, Galerkin finite element method, implicit difference method, Approximate solution.

مزج طريقتي كاليركن – الفروقات الضمنية لحل زوج من مسائل القيم الحدودية المكافئة ذات معاملات متغيرة

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الخلاصة
في هذا البحث، تم تقديم تقنية تقريبية لحل زوج من مسائل القيم الحدودية ذات معاملات متغيرة من النمط المكافئ باستخدام مزج طريقة كاليركن للعناصر المنتهية بالنسبة لمتغير الفضاء مع الطريقة الضمنية للفروقات المنتهية بالنسبة لمتغير الزمن . هذه التقنية حولت زوج من مسائل القيم الحدودية ذات معاملات متغيرة من النمط المكافئ إلى نظام جبري خطي وهو نظام كاليركن الجبري الخطي . والذي تم حله باستخدام طريقة جولسكي . تم إعطاء أمثلة توضيحية بالإشكال والجداول وتبين كفاءة الطريقة المقترحة.

1. Introduction

A wide range of applications in a natural science, engineering, and technology, are described in genera by mathematical modules and, in particular by parabolic boundary value

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problem (PBVP). Usually these problems are needed to be solved numerically; of course there are many numerical methods which are used to solve the PBVPs. Such as; in 2019 , the authors in [1] were used the mixed Galerkin finite element method (GFEM) with Crank-Nicolson method to solve nonlinear PBVPs(NLPBVPs), in 2020, the numerical solution of the PPDEs by using implicit method was introduced in [2], in 2021, the numerical solution of LBVPs by using the homotopy perturbation method were found [3], also the collocation method was presented in [4] for solving the PBVPVC, in 2022, [5] found the numerical solution of PPDEs in one and two space variable by using the central FDM (CFDM), while [6] were implemented the numerical solution of the PPDEs by using a novel collocation technique, and [7] was studied asymptotic behaviour on a PPDE and system modelling a production planning problem.

The numerical solution from solving couple elliptic BVP was studied by [8] in 2019, then the Ns for CPBVPs was introduced in 2021 by [9]. All the above methods motivated us to think about solving CLPBVPVC.

This paper deals with, the description of the continuous coupled linear parabolic boundary value problems with variable coefficients (CLPBVPVC), the weak form (WF)of the problem is found and the approximation problem obtained from the discretization of the continuous CLPBVPVC, by using the Galerkin finite element method (GFEM) in space variable with implicit finite difference method (IFDM) in the time variable, therefore the method is abbreviated by IGFEM. At any discrete time t_j this technique is transformed the CLPBVPVC into a linear algebraic system (LAS) which is called a Galerkin LAS (GLAS), and it is solved using the Cholesky Decomposition method (ChDeM). An algorithm to solve the problem is given. Finally, illustrations examples are presented to solve different problems using the proposed method with the help of the mathematical software- MATLAB, the results are given by tables and by figures which show the efficiency of this method.

2. Description of the Couple Linear Parabolic Boundary Value Problem (CLPBVPVC)

Let $\Omega = \{\vec{x} = (x_1, x_2) \in R^2:\ 0 < x_1, x_2 < 1\} \subset R^2$, be a region with boundary of Ω $(\partial\Omega)$, and let $I = [0, T]$, $Q = \Omega \times I$, $0 < T < \infty$, then the CLPBVPVC are given by:

$$U_{1t} - \sum_{r,s=1}^{2} \frac{\partial}{\partial x_s}\left[a_{rs}(\vec{x},t)\frac{\partial U_1}{\partial x_r}\right] + h_1(\vec{x},t)U_1 - g(\vec{x},t)U_2 = w_1(\vec{x},t), \text{ in } Q \quad (1)$$

$$U_{2t} - \sum_{r,s=1}^{2} \frac{\partial}{\partial x_s}\left[b_{rs}(\vec{x},t)\frac{\partial U_2}{\partial x_r}\right] + h_2(\vec{x},t)U_2 + g(\vec{x},t)U_1 = w_2(\vec{x},t), \text{ in } Q \quad (2)$$

$$U_1(\vec{x},0) = U_1^0(\vec{x}), \text{ in } \Omega \quad (3)$$

$$U_2(\vec{x},0) = U_2^0(\vec{x}), \text{ in } \Omega \quad (4)$$

$$U_1(\vec{x},t) = 0, \text{ on } \partial\Omega \times I \quad (5)$$

$$U_2(\vec{x},t) = 0, \text{ on } \partial\Omega \times I \quad (6)$$

where,$U_1 = U_1(\vec{x},t), U_2 = U_2(\vec{x},t) \in H^2(Q)$, $a_{rs}(\vec{x},t), b_{rs}(\vec{x},t)$ $(\forall r,s = 1,2), h_1(\vec{x},t)$, $h_1(\vec{x},t)\ and\ g(\vec{x},t)$ are positive non zero functions in $L^\infty(Q), w_1 = w_1(\vec{x},t), w_2 = w_2(\vec{x},t)$ are given functions in $L^2(Q)$ for all $\vec{x} \in \Omega$.

The "classical" of system (1-6) is $\vec{U} = \left(U_1(\vec{x},t), U_2(\vec{x},t)\right) \in \left(H^2(Q)\right)^2, s.t\ \vec{U} = 0\ on\ \partial\Omega$, for all $\vec{x}$ in Ω. Let $(.,.)$ and $\|.\|_0$, be denoted the inner product (IP) and norm (NO) in $L^2(\Omega)$, $\|.\|_1$ be denoted by SNO represent the norm in Sobolev space (SP), $V = H_0^1(\Omega)$, the duality bracket between V and its dual V^* will be denoted by $(.,.)$ and $\|.\|_q$ be the norm in $L^2(Q)$.

3. The Weak Formulation of the Couple Linear Parabolic Problem

Let $V = \{v: v = v(\vec{x}) \in H^1(\Omega), \forall \vec{x} \in \Omega, with\ v = 0\ on\ \partial\Omega\}$, then the WF of the CLPBVPVC (1-6) is given by

$(U_{1t}, v_1) + a_1(t, U_1, v_1) - (g(t)U_2, v_1) = (w_1, v_1), \forall\, v_1 \in H_0^1(\Omega), \vec{U} \in \left(H_0^1(Q)\right)^2$ (7)
$(U_1(0), v_1) = (U_1^0, v_1)$, in $H_0^1(\Omega)$ (8)
$(U_{2t}, v_2) + a_2(t, U_2, v_2) + (g(t)U_1, v_2) = (w_2, v_2), \forall\, v_2 \in H_0^1(\Omega), \vec{U} \in \left(H_0^1(Q)\right)^2$ (9)
$(U_2(0), v_2) = (U_2^0, v_2)$, in $H_0^1(\Omega)$ (10)
Where the following bilinear form are defined:
$a_1(t, U_1, v_1) = \sum_{r,s=1}^{2} a_{rs}(\vec{x}, t)\left(\frac{\partial U_1}{\partial x_s}, \frac{\partial v_1}{\partial x_r}\right) + h_1(\vec{x}, t)(U_1, v_1)$
$a_2(t, U_2, v_2) = \sum_{r,s=1}^{2} b_{rs}(\vec{x}, t)\left(\frac{\partial U_2}{\partial x_s}, \frac{\partial v_2}{\partial x_r}\right) + h_2(\vec{x}, t)(U_2, v_2)$

3.1 Assumptions: For $\gamma_i, \overline{\gamma_i}$ are positive constants $\forall\, i = 1,2$:
i. $|a_i(t, U_i, v_i)| \le \gamma_i \|U_i\|_1 \|v_i\|_1, \ \forall\, i = 1,2$
ii. $a_i(t, U_i, v_i) \ge \overline{\gamma_i} \|U_i\|_1^2, \ \forall\, i = 1,2.$

4. Discretization of the Weak Form

The WF of (7-10) is discretized by using the GFEM as follows: It stars by splitting the region Ω into subregions , i.e. let $M_1 > 0$ be an integer and let $O = O_i^n, i = 1,2, \dots, N, N = M^2$ where $M = M_1 - 1$ be an "admissible regular triangulation" of $\overline{\Omega}$, i.e. $\overline{\Omega} = \bigcup_{i=1}^{N} O_i$, let x_{i1}, x_{i2} be points a polyhedron domain $\overline{\Omega} = [0,1] \times [0,1]$, s.t.
$0 = x_{01} < x_{11} < \cdots < x_{i1} < \cdots < x_{M_1 1} = 1$ & $0 = x_{02} < x_{12} < \cdots < x_{i2} < \cdots < x_{M_1 2} = 1$.
Let $h = {}^{1}/_{M_1}$ i.e. $x_{i1} = ih$ and $x_{i2} = ih$, $i = 0,1,2, \dots, M_1$ and for every integer $NT > 0$, the interval $I = [0,1]$, can be divided into subintervals as $I_j = I_j^n \coloneqq [t_j^n, t_{j+1}^n]$ of equal length $\Delta t = \frac{T}{NT}$ with $t_j = j\Delta t, j = 0,1, \dots, NT - 1$.

4.1 The Approximation solution (APPS) of the CLPBVPVC

To find the APPS of $\vec{U}^n = (U_1^n, U_2^n)$ of (7-10), using the GFEM, let V_N be a subspace of (continuous piecewise linear function (PAF)) of a dimension N of $H_0^1(\Omega)$, and the following proceedings can be applied:
Step1: let $\vec{V}_N = V_N \times V_N$, when $V_N = \{v_i, i = 1,2, \dots, N, with\ v_i(\vec{x}) = 0\ on\ \partial\Omega\}$ be a PAF finite basis of V_N in Ω, then (7-10) for any $i = 1,2, \dots, N$ can be written as
$(U_{1t}^n, v_1) + a_1(t, U_1^n, v_1) - (g(t)U_2^n, v_1) = (w_1, v_1), \forall\, v_1 \in V_N, \vec{U}^n \in \vec{V}_N$ (11)
$(U_1(0), v_1) = (U_1^0, v_1)$, in V_N (12)
$(U_{2t}^n, v_2) + a_2(t, U_2^n, v_2) + (g(t)U_1^n, v_2) = (w_2, v_2), \forall\, v_2 \in V_N, \vec{U}^n \in \vec{V}_N$ (13)
$(U_2(0), v_2) = (U_2^0, v_2)$, in V_N (14)
Step2: Applying the GFEM [10] and the IFDM in [11], the APPS U_i^n is approximated by using the basis $(v_1, v_2, \dots, v_N)$ of V_N, i.e.
$U_1^n(\vec{x}, t_j) = \sum_{k=1}^{N} c_k(t_j) v_k(\vec{x})$, $U_2^n(\vec{x}, t_j) = \sum_{k=1}^{N} c_{k+N}(t_j) v_k(\vec{x})$
$U_1^n(\vec{x}, 0) = \sum_{k=1}^{N} c_k(0) v_k(\vec{x})$, $U_2^n(\vec{x}, 0) = \sum_{k=1}^{N} c_{k+N}(0) v_k(\vec{x})$
Where $c_k(t)$ and $c_{k+N}(t)$ are unknown coefficients to be found
Step3: Substitute $\vec{U}^n$ in ((11)-(14)) with $v_1 = v_2 = v_m$, to get the following GLAS with its ICs, i.e.
$(A + \Delta t B) C_k^{j+1} - \Delta t D C_{k+N}^{j+1} = A C_k^{j} + \Delta t \vec{b}_1(t_{j+1})$ (15)
$A C_k(0) = \vec{b}_1^0$ (16)
$(A + \Delta t E) C_{k+N}^{j+1} + \Delta t D C_k^{j+1} = A C_{k+N}^{j} + \Delta t \vec{b}_2(t_{j+1})$ (17)
$A C_{k+N}(0) = \vec{b}_2^0$ (18)

where $A = (a_{mk})_{N\times N}$, $a_{mk} = (v_k, v_m)$, $B = (b_{mk})_{N\times N}$, $b_{mk} = a_1(v_k, v_m)$, $a_1(v_k, v_m) = \sum_{r,s=1}^{2} a_{rs}(\vec{x})\left(\frac{\partial v_k}{\partial x_s}, \frac{\partial v_m}{\partial x_r}\right) + (h_1(\vec{x})v_k, v_m)$, $D = (d_{mk})_{N\times N}$, $d_{mk} = (g(\vec{x})v_k, v_m)$,
$E = (e_{mk})_{N\times N}$, $e_{mk} = a_2(v_k, v_m)$, $a_2(v_k, v_m) = \sum_{r,s=1}^{2} b_{rs}(\vec{x})\left(\frac{\partial v_k}{\partial x_s}, \frac{\partial v_m}{\partial x_r}\right) + (h_2(\vec{x})v_k, v_m)$
$C_k(t_j) = \left(C_k(t_j)\right)_{N\times 1}$, $C_{k+N}(t_j) = \left(C_{k+N}(t_j)\right)_{N\times 1}$, $\vec{b}_1 = (b_{1i})_{N\times 1}$, $b_{1i} = \left(w_1(\vec{x}, t_{j+1}), v_m\right)$, $\vec{b}_2 = (b_{2i})_{N\times 1}$, $b_{2i} = \left(w_2(\vec{x}, t_{j+1}), v_m\right)$, $\vec{b}_1^0 = (b_{1i}^0)_{N\times 1}$, $b_{1i}^0 = (U_1^0, v_i)$, $\vec{b}_2^0 = (b_{2i}^0)_{N\times 1}$, $b_{2i}^0 = (U_2^0, v_i)$, $\forall\, m, k = 1,2, \dots, N$.
Step4: Solve the GLAS (15-18) using the ChDeM, to find the APPS for the problem.

4.2Remark: The matrices A, B, D, and E are positive definite, hence the GLAS has a unique solution.

4.3Remark: The space of the basis V_N was choice as a space of continuous piecewise linear function, because the graph of such basis on a compact interval is a polygonal chain, and this play an important role in the GFEM.

4.4 The Cholesky Decomposition Method

The ChDeM is used to solve the GLAS with two conditions, $A = L.L^T$,(T is the transpose operator) every symmetric positive definite matrix A can be decomposed into a product of a unique lower triangular matrix L and its transpose [11] . The ChDeM can be represented in the following steps:

Step1: $L_{pp} = \left(a_{pp} - \sum_{z=1}^{p-1} L_{pz}^2\right)^{1/2}$ $for\ p = 1,2, \dots, N$

Step2: $L_{pq} = \frac{a_{pq} - \sum_{z=1}^{q-1} L_{qz}.L_{pz}}{L_{qq}}$ $for\ q = p + 1, \dots, N$

5. The Algorithm for Solving the Couple Linear Parabolic Boundary Value Problem

Step1: Solve the ICs (16) and (18) respectively to get C_k^0 and C_{k+N}^0
Step2: Set $j = 0$
Step3: Solve the GLAS (15) and (17), to get C_k^1 and C_{k+N}^1 respectively
Step4: Set $j = j + 1$
Step5: Repeated step 3-4, until $NT - 1$.

6. Stability and the Convergent

6.1 Lemma (Stability): For Δt is sufficiently small, and $\forall\, j = 0,1, \dots, NT - 1$, then
$\|\vec{U}_j^n\|_0^2 \le d_1^2$, $\Delta t \sum_{j=0}^{NT-1} \|\vec{U}_j^n\|_0^2 \le d_2^2$, $\sum_{j=0}^{NT-1} \|\vec{U}_{j+1}^n - \vec{U}_j^n\|_0^2 \le d_3^2$,& $\Delta t \sum_{j=0}^{NT-1} \|\vec{U}_{j+1}^n\|_1^2 \le d_4^2$

Proof: Using the IFDM in the WF ((11)&(13)), then substituting $\vec{v} = 2\Delta t \vec{U}_{j+1}^n$, to get

$$2[\left(U_{1j+1}^n, U_{1j+1}^n\right) - \left(U_{1j}^n, U_{1j+1}^n\right) + \Delta t a_1\left(U_{1j+1}^n, U_{1j+1}^n\right) - \left(g(t_{j+1})U_{2j+1}^n, U_{1j+1}^n\right)] = 2\Delta t\left(w_1(t_{j+1}), U_{1j+1}^n\right) \quad (19)$$

$$2[\left(U_{2j+1}^n, U_{2j+1}^n\right) - \left(U_{2j}^n, U_{2j+1}^n\right) + \Delta t a_2\left(U_{2j+1}^n, U_{2j+1}^n\right) + \left(g(t_{j+1})U_{1j+1}^n, U_{2j+1}^n\right)] = 2\Delta t\left(w_2(t_{j+1}), U_{2j+1}^n\right) \quad (20)$$

Using the assumptions on w_1 and w_2 in R.H.S. of (19) and (20), rewritten the 1st two terms in the L.H.S. of (19) and (20) in the norm form, then adding the obtained two inequalities, and finally taking then summing both sides for $j = 0\ to\ j = l - 1$, to get

$$\|\vec{U}_l^n\|_0^2 + \sum_{j=0}^{l-1} \|\vec{U}_{j+1}^n - \vec{U}_j^n\|_0^2 + 2\Delta t \sum_{j=0}^{l-1} [a_1\left(U_{1j+1}^n, U_{1j+1}^n\right) + a_2\left(U_{2j+1}^n, U_{2j+1}^n\right) \le$$

$c_4 + \Delta t \sum_{j=0}^{l-1} \| \vec{U}_{j+1}^n \|_0^2$, where $c_4 = l\Delta t c_3$, and $c_3 = c_1^2 + c_{12}^2$ (21)
Since $\|\vec{U}_{j+1}^n\|_0^2 \le 2\|\vec{U}_j^n\|_0^2 + 2\|\vec{U}_{j+1}^n - \vec{U}_j^n\|_0^2$, $\|\vec{U}_0^n\|_0^2 \le c_4$ (from the Projection theorem)&
$\sum_{j=0}^{l} a_i(U_{ij}^n, U_{ij}^n) - a_i(U_{i0}^n, U_{i0}^n) = \sum_{j=1}^{l-1} a_i(U_{ij+1}^n, U_{ij+1}^n)$, for $i = 1,2$.
Then, substituting the above inequality in the R.H.S of (21), the equality in its L.H.S., and using assumption A(i), to get
$\|\vec{U}_l^n\|_0^2 + (1-2\Delta t) \sum_{j=0}^{l-1} \|\vec{U}_{j+1}^n - \vec{U}_j^n\|_0^2 + \Delta t \sum_{j=0}^{l-1} [a_1(U_{1j+1}^n, U_{1j+1}^n) + a_2(U_{2j+1}^n, U_{2j+1}^n)]$
$\le c_5 + \Delta t \sum_{j=0}^{l-1} \| \vec{U}_{j+1}^n \|_0^2$, where $c_5 = l\Delta t c_3 + c_4 + c_{4(}\gamma_1 + \gamma_2)$, and $c_4 = c_1^2 + c_{12}^2$ (22)
Since the 2nd and the 3rd terms in the L.H.S. are nonnegative (from assumptions A(ii)), then
$\|\vec{U}_l^n\|_0^2 \le c_5 + \Delta t \sum_{j=0}^{l-1} \|\vec{U}_{j+1}^n\|_0^2$, with $\Delta t \le 1/2$.
Applying the discrete Bellman- Gromwell inequality, to get(for any arbitrary index l):
$\|\vec{U}_l^n\|_0^2 \le d_1^2$, hence $\|\vec{U}_j^n\|_0^2 \le d_1^2$, $\forall j = 0,1,\ldots\ldots,TN$, which gives
$\Delta t \sum_{j=0}^{NT-1} \|\vec{U}_j^n\|_0^2 \le d_2^2$, where $d_2^2 = \Delta t NT d_1^2$.
Again from (22) with $l = NT$, since the 1st & the 3rd terms in the L.H.S. are nonnegative, then
$\sum_{j=0}^{NT-1} \| \vec{U}_{j+1}^n - \vec{U}_j^n \|_0^2 \le d_3^2$, $d_3^2 = d_2^2 + c_5$.
Also, since the 1st and the 2nd terms in L.H.S.of (22) are nonnegative, then we have from A(ii)
$2\Delta t\gamma \sum_{j=0}^{NT-1} \| \vec{U}_{j+1}^n\|_1^2 \le d_3^2$, then
$\Delta t \sum_{j=0}^{NT-1} \| \vec{U}_{j+1}^n\|_1^2 \le d_4^2$, where $d_4^2 = \frac{d_3^2}{2\gamma}$ and $\gamma = \min(\overline{\gamma_1}, \overline{\gamma_2})$.

6.2 Convergence:
The following definitions for the functions "almost everywhere" on I are useful in the proof of next theorem, so let
$\vec{U}_-^n(t) := \vec{U}_j^n$, and $\vec{U}_+^n(t) := \vec{U}_{j+1}^n$ for $\in I_j^n$, $\forall j = 0,1,\ldots.,NT-1$.
also $\vec{U}_\wedge^n(t) := \vec{U}_j^n$ be a continuous affine linear function on I_j^n, $\forall j = 0,1,\ldots.,Nt-1$,

6.2.1 Theorem: The discrete solutions $\vec{U}_-^n(t)$, $\vec{U}_+^n(t)$, and $\vec{U}_\wedge^n(t)$ are converges strongly in $L^2(\Omega)$, where $n \longrightarrow \infty$.

Proof: From Lemma (6.1), $\|\vec{U}_j^n\|_1^2 \le d_1^2$, $\forall j = 0,1,\ldots.,Nt-1$, then
$\|\vec{U}_-^n\|_{L^2(I,V)}^2$, $\|\vec{U}_+^n\|_{L^2(I,V)}^2$, and $\|\vec{U}_\wedge^n\|_{L^2(I,V)}^2$ are bounded.
And also $\Delta t \sum_{j=0}^{Nt-1} \|\vec{U}_{j+1}^n - \vec{U}_j^n\|_0^2 \le \Delta t d_1^2, \longrightarrow 0$, as $\Delta t \longrightarrow 0$, gives $\vec{\vec{U}}_+^n \longrightarrow \vec{\vec{U}}_-^n$ strongly in $L^2(I,V)$ and then in $L^2(\Omega)$.
Then by Alaoglu theorem, there exist subsequences of $\{\vec{U}_-^n\}$, $\{\vec{U}_+^n\}$, $\{\vec{U}_\wedge^n\}$) use again the same notations, s.t. $\vec{U}_-^n \longrightarrow \vec{U}$, $\vec{U}_+^n \longrightarrow \vec{U}$, $\vec{U}_\wedge^n \longrightarrow \vec{U}$ weakly in $L^2(I,V)$.
In this point the first compactness theorem [12] is used, to get that $\vec{U}_\wedge^n \longrightarrow \vec{U}$ strongly in $L^2(\Omega)$, then $\vec{U}_+^n \longrightarrow \vec{U}$ and $\vec{U}_-^n \longrightarrow \vec{U}$ strongly in $L^2(\Omega)$.
Now, let $\{\vec{V}_N\}_{N=1}^{\infty}$ be a sequence of subspaces of $\vec{V}$. Then by the Galerkin approach for each $\vec{v} = (v_1, v_2) \in \vec{V}$, there is a sequence$\{\vec{v}_N = (v_{1N}, v_{2N})\} \in \vec{V}_N$, $\forall N$, s.t.$\vec{v}_N \longrightarrow \vec{v}$ ST in $L^2(\Omega)$.
Consider that $\vec{\varphi}(t) = (\varphi_1, \varphi_2) \in (C^2[0,T])^2$, for which $\vec{\varphi}(T) = \vec{\varphi}'(T) = 0$ and $\vec{\varphi}(0) = \vec{\varphi}'(0) \ne 0$, let $\vec{\varphi}^n(t)$ continuous piecewise interpolation of $\vec{\varphi}(t)$ with respect to I_j^n, and let $\vec{\zeta} = (v_1\varphi_1(t), v_2\varphi_2(t))$, with $\vec{\zeta}^n = (v_{1N}\varphi_1{}^n(t), v_{2N}\varphi_2{}^n(t))$, with
$\vec{\zeta}_-^n := (v_{1N}\varphi_1{}_-^n(t), v_{2N}\varphi_2{}_-^n(t))$, $t \in I_j^n$, $j = 0,1,\ldots.,Nt-1$, $\vec{v}_N \in \vec{V}_N$,
$\vec{\zeta}_+^n := (v_{1N}\varphi_1{}_+^n(t), v_{2N}\varphi_2{}_+^n(t))$, $t \in I_j^n$, $j = 0,1,\ldots.,Nt-1$, $\vec{v}_N \in \vec{V}_N$,

$\vec{\zeta}^{n}_{\wedge} := (v_{1N}\varphi_1{}^{n}(t), v_{2N}\varphi_2{}^{n}(t)),\ t \in I,\ \ \vec{v}_N \in \vec{V}_N,$

Setting $\vec{v} = \vec{\zeta}^{n}_{j+1}$ in equation ((19)-(20)), and summing both sides of the obtained equation for $j = 0$, to $j = NT - 1$, then using the discrete integration by parts for the 1st term in the L.H.S., once can get that

$$-\int_0^T (U^n_{1-}, (\zeta^n_{1\wedge})')\,dt + \int_0^T [a_1(U^n_{1+}, \zeta^n_{1+}) - (g(t^n_+)\,U^n_{2+}, \zeta^n_{1+})]\,dt =$$
$$\int_0^T (f_1(t^n_+, U^n_{1+}), \zeta^n_{1+})\,dt + (U^n_{10}, v_{1N})\,\varphi_1{}^{n}(0)$$
$$-\int_0^T (U^n_{2-}, (\zeta^n_{2\wedge})')\,dt + \int_0^T [a_2(U^n_{2+}, \zeta^n_{2+}) + (g(t^n_+)\,U^n_{1+}, \zeta^n_{2+})]\,dt =$$
$$\int_0^T (f_2(t^n_+, U^n_{2+}), \zeta^n_{2+})\,dt + (U^n_{20}, v_{2N})\,\varphi_2{}^{n}(0)$$

In this point, the same steps which used in the proof of theorem 3.1 in [13], can be used here to get the APPS $\vec{U}^n_-(t)$, $\vec{U}^n_+(t)$, $\vec{U}^n_\wedge(t)$ are converges strongly to $\vec{U}$ in $L^2(I, V)$, where $n \longrightarrow \infty$. Thus, the limit $\vec{U}$ satisfies the WF ((7)-(10)).

7. Numerical Examples

In this section numerical examples are carried out to show the efficiency for the presented method in this paper.

7.1. Example:- Let $I = [0,1]$, then the CLPBVPVC are given as

$$U_{1t} - \frac{\partial}{\partial x_1}\left[(x_1^2+1)\frac{\partial U_1}{\partial x_1}\right] - \frac{\partial}{\partial x_2}\left[(x_2^2+1)\frac{\partial U_1}{\partial x_2}\right] + (x_1^2x_2^2+1)U_1 - (x_1^4+x_2^2+1)U_2 = w_1(\vec{x},t),$$
in Q

$$U_{2t} - \frac{\partial}{\partial x_1}\left[(x_2^2+1)\frac{\partial U_2}{\partial x_1}\right] - \frac{\partial}{\partial x_2}\left[(x_1^2+1)\frac{\partial U_2}{\partial x_2}\right] + (x_1x_2^2+1)U_2 + (x_1^4+x_2^2+1)U_1 =$$
$w_2(\vec{x},t)$, in Q

$$U_1(\vec{x},0) = U_1^0(\vec{x}) = x_1x_2(1-x_1)(1-x_2), in\ \Omega$$
$$U_2(\vec{x},0) = U_2^0(\vec{x}) = x_1x_2(1-x_1)(1-x_2), in\ \Omega$$
$$U_1(\vec{x},t) = 0,\ on\ \partial\Omega \times I$$
$$U_2(\vec{x},t) = 0,\ on\ \partial\Omega \times I$$

Such that the right hand term $w_1(\vec{x},t)$ and $w_2(\vec{x},t)$ are given as

$$w_1(\vec{x},t) = x_1x_2(x_1-1)(x_2-1)[e^{-t}(x_1^2x_2^2-4) - (x_1^2+x_2^2+1)e^{-2t}]$$
$$-2x_1e^{-t}[x_1x_2(x_2-1) + (x_1-1)(x_2^2+1) + x_2^2(x_1-1)] - 2x_2e^{-t}(x_2-1)(x_1^2+1)$$
$$w_2(\vec{x},t) = e^{-2t}\left[x_1^2\left(x_2^3(x_2-1)\right) - x_2^2(x_2^2+1) + \left(x_2 - 2(x_1^2+x_1+1)\right) - 2x_2^2(x_2^2+x_2+1) - 2(x_1+x_2)\right] + e^{-t}[(x_2^2-x_2)(x_1^6-x_1^5) + x_1^2(x_2^4-x_2) - x_1(x_2^4-x_2^3+x_2^2+x_2)]$$

The exact solution (EXS) of the above CLPBVPVC is

$$U_1(\vec{x},t) = x_1x_2(1-x_1)(1-x_2)e^{-t}$$
$$U_2(\vec{x},t) = x_1x_2(1-x_1)(1-x_2)e^{-2t}$$

This problem is solved using the IGFEM for $M = 9, NT = 20\ and\ T = 1$, then the APPS $\vec{U}^n$ and the EXS $\vec{U}\ at\ x_1\ and\ x_2$ are given at the time $t = 0.5$ in the Table (1) and are shown in Figure (1), the absolute maximum error is (0.0024).

Table1: Comparison between the EXS and APPS.

x_1	x_2	EXS U_1	APPS U_1	Absolute error	x_1	x_2	EXS U_2	APPS U_2	Absolute error
0.1	0.1	0.0049	0.0045	0.0004	0.1	0.1	0.0030	0.0032	0.0002
0.3	0.1	0.0115	0.0105	0.0010	0.3	0.1	0.0070	0.0077	0.0007
0.5	0.1	0.0136	0.0125	0.0012	0.5	0.1	0.0083	0.0097	0.0014
0.7	0.1	0.0115	0.0105	0.0009	0.7	0.1	0.0070	0.0086	0.0017
0.9	0.1	0.0049	0.0046	0.0003	0.9	0.1	0.0030	0.0038	0.0009
0.1	0.3	0.0115	0.0104	0.0011	0.1	0.3	0.0070	0.0072	0.0002
0.3	0.3	0.0267	0.0240	0.0028	0.3	0.3	0.0162	0.0174	0.0012
0.5	0.3	0.0318	0.0286	0.0033	0.5	0.3	0.0193	0.0220	0.0027
0.7	0.3	0.0267	0.0243	0.0025	0.7	0.3	0.0162	0.0198	0.0036
0.9	0.3	0.0115	0.0106	0.0009	0.9	0.3	0.0070	0.0088	0.0019
0.1	0.5	0.0136	0.0122	0.0015	0.1	0.5	0.0083	0.0081	0.0001
0.3	0.5	0.0318	0.0282	0.0036	0.3	0.5	0.0193	0.0195	0.0002
0.5	0.5	0.0379	0.0337	0.0042	0.5	0.5	0.0230	0.0247	0.0017
0.7	0.5	0.0318	0.0286	0.0032	0.7	0.5	0.0193	0.0223	0.0029
0.9	0.5	0.0136	0.0125	0.0011	0.9	0.5	0.0083	0.0100	0.0017
0.1	0.7	0.0115	0.0102	0.0012	0.1	0.7	0.0070	0.0064	0.0005
0.3	0.7	0.0267	0.0237	0.0030	0.3	0.7	0.0162	0.0153	0.0009
0.5	0.7	0.0318	0.0284	0.0035	0.5	0.7	0.0193	0.0193	0.0000
0.7	0.7	0.0267	0.0241	0.0026	0.7	0.7	0.0162	0.0174	0.0012
0.9	0.7	0.0115	0.0105	0.0009	0.9	0.7	0.0070	0.0078	0.0009
0.1	0.9	0.0049	0.0044	0.0005	0.1	0.9	0.0030	0.0026	0.0003
0.3	0.9	0.0115	0.0103	0.0011	0.3	0.9	0.0070	0.0063	0.0007
0.5	0.9	0.0136	0.0124	0.0013	0.5	0.9	0.0083	0.0078	0.0004
0.7	0.9	0.0115	0.0105	0.0010	0.7	0.9	0.0070	0.0070	0.0001
0.9	0.9	0.0049	0.0046	0.0003	0.9	0.9	0.0030	0.0032	0.0002

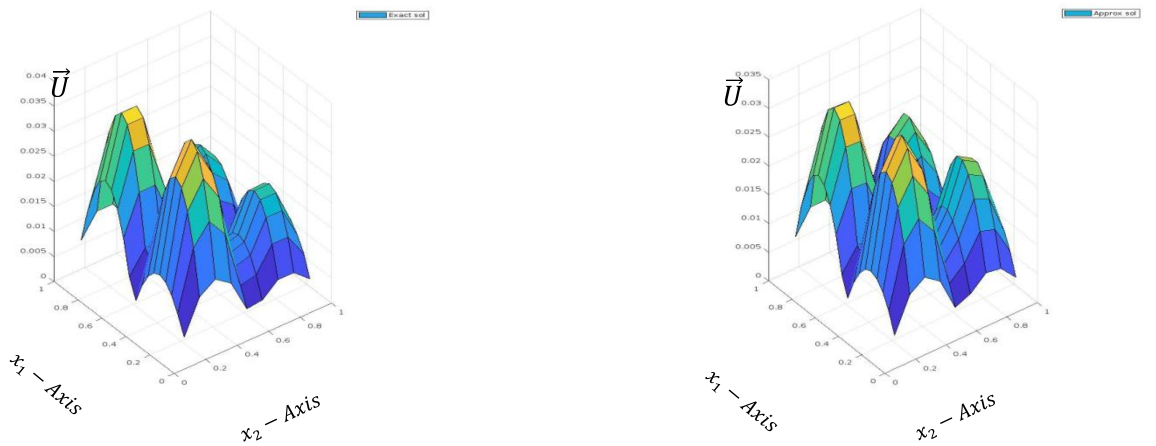

Figure1: Shows the EXS and shows the APPS

7.2. Example:- Let $I = [0,1]$, the CLPBVPVC is given as

$U_{1t} - \frac{\partial}{\partial x_1}\left[(x_1^2 - 2x_2 + 7)\frac{\partial U_1}{\partial x_1}\right] - \frac{\partial}{\partial x_2}\left[(x_1 + 1)\frac{\partial U_1}{\partial x_2}\right] + (x_1x_2 + 1)U_1 - (2x_1^2 + 5x_2 + 11)U_2 = w_1(\vec{x}, t)$, in Q

$U_{2t} - \frac{\partial}{\partial x_1}\left[(x_2e^{x_1})\frac{\partial U_2}{\partial x_1}\right] - \frac{\partial}{\partial x_2}\left[(x_2 + 1)\frac{\partial U_2}{\partial x_2}\right] + (x_1^2x_2^2)U_2 + (2x_1^2 + 5x_2 + 11)U_1 = w_2(\vec{x}, t)$, in Q

$U_1(\vec{x}, 0) = U_1^0(\vec{x}) = 2.7x_1(1 - x_1)(1 - x_2)\sin\left(\frac{x_2}{9}\right)$, $in\ \Omega$,

$U_2(\vec{x}, 0) = U_2^0(\vec{x}) = 0,\ in\ \Omega$,

$U_1(\vec{x}, t) = 0,\ on\ \partial\Omega \times I$,

$U_2(\vec{x}, t) = 0,\ on\ \partial\Omega \times I$,

Such that the right hand term $w_1(\vec{x}, t)$ and $w_2(\vec{x}, t)$ are given as

$$w_1(\vec{x}, t) = \sin\left(\frac{x_2}{9}\right) e^{\cos(t/9)}(x_2 - 1)\left[(x_1 - 1)\left(x_1(x_1x_2 + 1) - 2x_1 + x_1x_2\frac{e^{x_1}}{81} - \frac{x_1}{9}\sin(t/9)\right) - (4x_1^2 - 4x_2 + 14)\right] - x_1(x_1 - 1)e^{\cos(t/9)}\left[e^{x_1}\left(2x_2\cos\left(\frac{x_2}{9}\right) + \sin\left(\frac{x_2}{9}\right)\right) - \frac{(x_2-1)}{9}\cos\left(\frac{x_2}{9}\right)\right] - \left[\frac{\pi}{3}x_1(x_2 - 1)(2x_1^2 + 5x_2 + 11)\sin(x_1 - 1)\tan(t/9)\right]$$

$$w_2(\vec{x}, t) = \frac{2}{3}\pi\tan(t/9)\left[\frac{1}{9}x_1(x_2 + 1)\sin\left(\frac{x_2}{9}\right) sin(1 - x_1) - (x_1 + 1)(x_2 - 1)cos(x_1 - 1)\left(\cos\left(\frac{x_2}{9}\right) - 1\right)\right] + \pi x_1(x_2 - 1)\tan(t/9)\left[sin(x_1 - 1)\left(\frac{1}{243}(x_2 + 1)\cos\left(\frac{x_2}{9}\right) + \frac{1}{3}(x_1 + 1)\left(\cos\left(\frac{x_2}{9}\right) - 1\right)\right) + \left(\frac{1}{27}\sin\left(\frac{x_2}{9}\right) sin(x_1 - 1) - cos(x_1 - 1)\left(\cos\left(\frac{x_2}{9}\right) - 1\right)\right)\right] - \pi\, sin(x_1 - 1)\left(\cos\left(\frac{x_2}{9}\right) - 1\right)\left[\tan(t/9)(x_1 + x_2 + 1) - \frac{1}{3}x_1(x_2 - 1)\left(\sqrt[9]{\tan^2(t/9)} + x_1^2x_2^2\tan(t/9) + \frac{1}{9}\right)\right] + x_1(x_1 - 1)(x_2 - 1)e^{\cos(t/9)}\sin\left(\frac{x_2}{9}\right)(2x_1^2 + 5x_2 + 11).$$

The EXS of the above CLPBVPVC is

$U_1(\vec{x}, t) = x_1(1 - x_1)(1 - x_2)\sin\left(\frac{x_2}{9}\right) e^{\cos(-t/9)}$

$U_2(\vec{x}, t) = \frac{1}{3}\pi x_1(1 - x_2)\sin(1 - x_1)\left(1 - \cos\left(\frac{x_2}{9}\right)\right)\tan(-t/9)$

This problem is solved using the IGFEM for $M = 9, NT = 20\ and\ T = 1$, then the APPS $\vec{U}^n$ and the EXS $\vec{U}\ at\ x_1\ and\ x_2$ are given at the time $t = 0.5$ in the Table (2) and are shown in Figure (2), the absolute maximum error is (0.0002)

Table2: Comparison between the EXS and APPS.

x_1	x_2	EXS U_1	APPS U_1	Absolute error $1 * e^{-3}$	x_1	x_2	EXS U_2	APPS U_2	Absolute error $e * 10^{-3}$
0.1	0.1	0.0024	0.0024	0.0261	0.1	0.1	0.0000	0.0001	0.0520
0.3	0.1	0.0057	0.0056	0.0599	0.3	0.1	0.0000	0.0001	0.1012
0.5	0.1	0.0068	0.0067	0.0804	0.5	0.1	0.0000	0.0001	0.1031
0.7	0.1	0.0057	0.0056	0.0792	0.7	0.1	0.0000	0.0001	0.0721
0.9	0.1	0.0024	0.0024	0.0382	0.9	0.1	0.0000	0.0000	0.0226
0.7	0.2	0.0101	0.0101	0.0613	0.7	0.2	0.0000	0.0001	0.1242
0.1	0.3	0.0057	0.0057	0.0079	0.1	0.3	0.0000	0.0001	0.0957
0.3	0.3	0.0133	0.0133	0.0147	0.3	0.3	0.0000	0.0002	0.1982
0.5	0.3	0.0158	0.0158	0.0114	0.5	0.3	0.0000	0.0002	0.2106
0.7	0.3	0.0133	0.0133	0.0424	0.7	0.3	0.0000	0.0002	0.1561

0.9	0.3	0.0057	0.0057	0.0298	0.9	0.3	0.0000	0.0001	0.0565
0.1	0.5	0.0068	0.0068	0.0258	0.1	0.5	0.0000	0.0001	0.0903
0.3	0.5	0.0158	0.0159	0.0542	0.3	0.5	0.0000	0.0002	0.1964
0.5	0.5	0.0188	0.0189	0.0263	0.5	0.5	0.0000	0.0002	0.2183
0.7	0.5	0.0158	0.0158	0.0205	0.7	0.5	0.0000	0.0002	0.1706
0.9	0.5	0.0068	0.0068	0.0240	0.9	0.5	0.0000	0.0001	0.0669
0.1	0.7	0.0057	0.0057	0.0223	0.1	0.7	0.0000	0.0001	0.0580
0.3	0.7	0.0133	0.0133	0.0466	0.3	0.7	0.0000	0.0001	0.1377
0.5	0.7	0.0158	0.0158	0.0217	0.5	0.7	0.0000	0.0002	0.1629
0.7	0.7	0.0133	0.0133	0.0198	0.7	0.7	0.0000	0.0001	0.1357
0.9	0.7	0.0057	0.0057	0.0224	0.9	0.7	0.0000	0.0001	0.0579
0.1	0.9	0.0024	0.0024	0.0034	0.1	0.9	0.0000	0.0000	0.0162
0.3	0.9	0.0057	0.0057	0.0092	0.3	0.9	0.0000	0.0000	0.0487
0.5	0.9	0.0068	0.0068	0.0017	0.5	0.9	0.0000	0.0001	0.0633
0.7	0.9	0.0057	0.0057	0.0179	0.7	0.9	0.0000	0.0001	0.0574
0.9	0.9	0.0024	0.0024	0.0150	0.9	0.9	0.0000	0.0000	0.0280

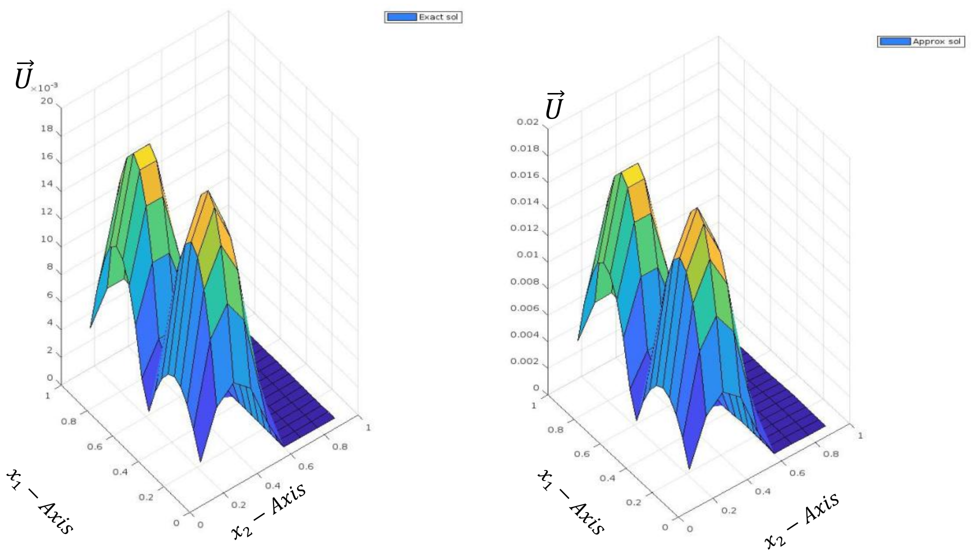

Figure 2: Shows the EXS and shows the APPS

8. Conclusions

In this article, the approximate method IGFEM has been proposed for solving CLPBVPVC. Two examples have been solved numerically to demonstrate the efficiency and accuracy of the method. Based on the obtained results, we can point out the following conclusions:

1. Depending on the result in Tables 1 and 2, the absolute maximum error between the EXS and the APPS for the considered problems show the efficiency of the method, although the space variable is discretized only for ten grid ($M = 9$) and $NT = 20$.
2. The transformed system of equations can be solved by the ChDeM, this method is very fast than the gauss elimination method for solving LAS.
3. Although, the obtained results from solving the two examples were for all the values of the time on the interval [0,1] with $NT = 20$, gave a good and accurate results but we shown the results at the time $t = 0.5$ t=0.5 to save the space.

4. The APPS in the two examples were obtained at all the discrete points for the space variable but they indicated half of these values to abbreviate the space.

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