DOI: 10.24996/ijs.2024.65.6.32

ISSN: 0067-2904

Efficient Computational Methods for Solving the One-Dimensional Parabolic Equation with Nonlocal Initial and Boundary Conditions

Othman Mahdi Salih, Majeed A. AL-Jawary*
Department of Mathematics, College of Education for Pure Science (Ibn AL-Haitham), University of Baghdad, Baghdad, Iraq.

Received: 11/3/2023 Accepted: 8/6/2023 Published: 30/6/2024

Abstract
The primary objective of the current paper is to suggest and implement effective computational methods (DECMs) to calculate analytic and approximate solutions to the nonlocal one-dimensional parabolic equation which is utilized to model specific real-world applications. The powerful and elegant methods that are used orthogonal basis functions to describe the solution as a double power series have been developed, namely the Bernstein, Legendre, Chebyshev, Hermite, and Bernoulli polynomials. Hence, a specified partial differential equation is reduced to a system of linear algebraic equations that can be solved by using Mathematica®12. The techniques of effective computational methods (DECMs) have been applied to solve some specific cases of time-dependent diffusion equations. Moreover, the maximum absolute error ($MAbsR_n$) is determined to demonstrate the accuracy of the proposed techniques.

Keywords: Nonlocal one-dimensional parabolic equation; Novel analytic approximate solution methods; Orthogonal basis functions; Power series.

طرق حسابية فعالة لحل معادلة القطع المكافئ أحادية البعد بشروط ابتدائية وحدودية غير محلية

عثمان مهدي صالح، مجيد احمد الجواري*
قسم الرياضيات، كلية التربية للعلوم الصرفة – ابن الهيثم، جامعة بغداد، بغداد، العراق.

الخلاصة
الهدف الرئيسي من البحث الحالي هو اقتراح وتنفيذ طرق حسابية فعالة (DECMs) لحساب الحلول التحليلية والتقريبية لمعادلة القطع المكافئ أحادية البعد في ظل الشروط الابتدائية والحدودية غير المحلية، والتي تستخدم لنمذجة تطبيقات محددة في العالم الحقيقي. تم تطوير نهج قوي وأنيق يعتمد على تمثيل سلسلة القوة المزدوجة للحل من خلال دوال الأساس المتعامدة المناسبة، مثل متعددات الحدود بيرنشتاين، ليجندر، تشيبشيف، هيرمت، برنولي. وبالتالي، يتم اختزال المعادلة التفاضلية الجزئية المكافئة إلى نظام من المعادلات الجبرية الخطية، والتي يمكن بعد ذلك حلها باستخدام برنامج ماثماتيكا®١٢. تم تطبيق تقنيات الأساليب الحسابية الفعالة (DECMs) لحل بعض الحالات المحددة لمعادلات الانتشار المعتمدة على الوقت. علاوة على ذلك، تم تحديد الخطأ المطلق الأقصى ($MAbsR_n$) لإثبات دقة الطرق المقترحة.

*Email: majeed.a.w@ihcoedu.uobaghdad.edu.iq

1. Introduction

In applied sciences and engineering, partial differential equations (PDEs) are used to model various natural processes [1]. Partial differential equations are also used to study fluid mechanics, flow in porous media, heat conduction in solids, diffusive transport of chemicals in porous media, and solid mechanics problems [2]. In addition, the equations arising from modeling spatial and temporal processes in nature or engineering are of particular interest [3]. Thus, many mathematicians tried numerous methods to solve these problems. For more detail, see [4-8].

Science and engineering models include an integral term over the spatial domain in some or all boundaries. Such problems are classified as non-local boundary value problems [9]. In 1963, these problems initially appeared on the scene, one of the quickest development fields acrosses different applications. Their measurement is typically more precise than that gives a local state which leads to a more positive outcome [10, 11]. Numerous physical phenomena are represented in recent years using nonlocal mathematical models. For example, problems in thermodynamics [12], heat conduction, and plasma physics [13] can be reduced to non-local problems with integral conditions.

Parabolic partial differential equations with nonlocal initial-boundary conditions simulate various physical and industrial problems because of this equation is essential in science and technology [14]. Researchers worked hard to perfect effective methods for solving parabolic PDEs such as; the Adomian decomposition method [15], finite difference methods [16], radial basis functions method [9], Legendre collocation method [17], spectral collocation methods [18], reproducing kernel method [19], Bernstein polynomials basis method [20]. Recently, the operational matrices method based on orthogonal polynomials has garnered significant interest from authors because it helps to address a variety of approximation theories and numerical analysis problems [21]. In addition, orthogonal polynomials like the Bernstein, Legendre, Chebyshev, Hermite, and Bernoulli polynomials are critical in the least squares approximation problems across finite domains [22]. Orthogonal polynomials also reduce the solution by translating non-linear differential equations into systems of non-linear algebraic equations using the operational matrix technique, which simplifies the equations and allows any modern software to solve them [23, 24].

On the other hand, the parabolic equations of the one-dimensional time-dependent diffusion type can describe significant engineering and industrial problems. The microwave heating process, spontaneous ignition, and mass movement in groundwater are only a few examples from the literature. The time-dependent diffusion equations have practical and exact solutions that interest engineers and mathematicians [25, 26]. A double Walsh series was first introduced in 1978 as a primary research project to approximately describe the functions of two independent variables. It then investigates single and simultaneous first-order PDEs [27]. The numerical techniques for resolving PDEs have significantly improved recently due to mathematics and computer science developments. These techniques include the Collocation, Galerkin, Tau, and Least square methods for more details, see [28-32]. In all these methods, the approximate solution is expressed in a linear combination of trial functions with indeterminate coefficients, where the indeterminate coefficients indicate the corresponding algebraic system solution [32].

Furthermore, Turkyilmazoglu [33] proposed an effective computational method (ECM) that relies on appropriate base functions based on the standard polynomials $[1, x, x^2, \dots]$ to handle many types of problems; for additional details, see [34-37]. In addition, when the solutions are polynomials, the exact solutions are obtained.

This paper extends and develops efficient analytic approximate solution methods based on the Turkyilmazoglu [14] to create a novel and accurate collection of the DECMs proposed methods using orthogonal base functions like Bernstein, Legendre, Chebyshev, Hermite, and Bernoulli polynomials with corresponding operational matrices. These orthogonal polynomials are substituted in the definition of the function $u(x,t)$ with derivatives for converting the differential equations into the matrix equation, then the inner product of these orthogonal base functions with both the left and right sides of the matrix equation is computed. Through these steps, we get the system of linear algebraic equations. By solving the obtained system, accurate novel approximate solutions to the parabolic PDEs with nonlocal initial-boundary conditions can be obtained. The solution to the parabolic equation appears as linear combinations of double power series of orthogonal basis functions. The coefficients of orthogonal polynomials are determined numerically or analytically using modern computing software.

The following is the structure of this paper: Section two gives the time-dependent diffusion equation formulation. Section three discusses the fundamental concepts underlying the proposed methods. Section four provides the application of the proposed methods to solve some examples for the parabolic type with nonlocal initial and boundary conditions and explains numerical results. Finally, section five presents the conclusions.

2. The time-dependent diffusion equation

The one-dimensional time-dependent diffusion equation with an integral condition can simulate a variety of physical processes in the contexts of thermoelasticity, heat conduction process, chemical engineering, population dynamics, aerodynamics and hydrodynamics, such as subsonic and supersonic mixed flows, medical science, control theory, and the life sciences [14, 38].

The time-dependent diffusion equation is given as follows [14]:

$$a(x,t)\frac{\partial u}{\partial t} = b(x,t)\frac{\partial^2 u}{\partial x^2} + c(x,t)u + Q(x,t),\ 0 < x \le 1,\ 0 < t \le 1, \tag{1}$$

with the following initial nonlocal conditions:

$$u(x,0) = \alpha u(x,1) + g(x), \qquad 0 \le x \le 1, \tag{2}$$

subject to the integral restrictions' nonlocal boundary conditions:

$$u(0,t) = \int_0^1 \rho(x)u(x,t)dx + f(t), \qquad 0 < t \le 1, \tag{3}$$

$$u(1,t) = \int_0^1 \psi(x)u(x,t)dx + h(t), \qquad 0 < t \le 1. \tag{4}$$

where $a, b, c, Q, f, g, h, \rho$ and ψ are known functions, $u(x,t)$ is the desired solution and α is a constant.

Various techniques have been utilized to solve this equation such as the finite difference method [39], the Galerkin technique [29], the collocation approaches [18], the radial basis functions method [9], the Bernstein polynomials basis method [20], the reproducing kernel method [40], and the Tau schemes [41]. Other approaches can be found in [42-46].

3. The fundamental concepts of the proposed methods

This section describes the fundamental concepts underlying the proposed methods. In addition, the orthogonal polynomials and associated operational matrices will be discussed as tools for developing and expanding the effective computational method technique to provide accurate novel analytic approximate solutions to the one-dimensional time-dependent diffusion problem.

3.1 The effective computational method and their operational matrices

The primary hypothesis is that the system of parabolic PDEs, Eqs. (1-4), has a unique solution. Now, we assume that the solution $u(x,t)$ to the considered problem is approximated as a linear combination of n^{th}- degree functional double power series based on standard polynomial as follows [14]:

$$u(x,t) = \sum_{k=0}^{n} \sum_{l=0}^{n} a_{kl} x^k t^l. \tag{5}$$

Where a_{kl} are the unknown standard polynomials coefficients whose values will be determined later. Now we define:

$$\boldsymbol{\Psi}(x) = [1\ \ x\ \ x^2\ x^3\ \dots\ x^n], \boldsymbol{\Upsilon}(t) = [1\ \ t\ \ t^2\ t^3\ \dots\ t^n]^T, \text{and } \boldsymbol{A} = \begin{bmatrix} a_{00} & a_{01} & \cdots & a_{0n} \\ a_{10} & a_{11} & \cdots & a_{1n} \\ \vdots & \vdots & \cdots & \vdots \\ a_{n0} & a_{n1} & \cdots & a_{nn} \end{bmatrix}.$$

The approximate solution to the n^{th} -degree Eq. (5) can be written in matrix form using the following dot product:

$$u(x,t) = \boldsymbol{\Psi}(x)\,.\boldsymbol{A}\,.\boldsymbol{\Upsilon}(t), \tag{6}$$

Moreover, we can obtain the following n^{th}-order partial derivatives for $\boldsymbol{\Psi}(x)$ and $\boldsymbol{\Upsilon}(t)$:

$$\frac{\boldsymbol{\partial^n \Psi}(x)}{\boldsymbol{\partial x^n}} = \boldsymbol{\Psi}(x)\,.(\boldsymbol{B}^*)^n, \qquad \frac{\boldsymbol{\partial^n \Upsilon}(t)}{\boldsymbol{\partial t^n}} = (\boldsymbol{B}^{*\boldsymbol{n}})^T\,.\boldsymbol{\Upsilon}(t)\,.$$

That $\boldsymbol{B}^*{}_{(n+1)\times(n+1)}$ is the operational matrix whose values are as follows [33]:

$$\boldsymbol{B}^* = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & 0 & 4 & \cdots & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \ddots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & n-1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & n \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}_{(n+1)\times(n+1)}$$

Hence, the partial derivatives for function $u(x,t)$ that are derived from Eq. (6) can be defined using the following forms:

$$\begin{aligned} \frac{\partial^n u(x,t)}{\partial x^n} &= \boldsymbol{\Psi}(x)\,.\boldsymbol{B}^{*n}.\mathbf{A}\,.\boldsymbol{\Upsilon}(t), \qquad n \geq 1, \\ \frac{\partial^m u(x,t)}{\partial t^m} &= \boldsymbol{\Psi}(x)\,.\boldsymbol{A}.(\boldsymbol{B}^{*\boldsymbol{m}})^T\,.\boldsymbol{\Upsilon}(t), \qquad m \geq 1. \end{aligned} \tag{7}$$

Consequently, substituting the Eqs. (6) and (7) into the Eqs. (1-4), the following matrix equations are obtained:

$$\begin{aligned} & a(x,t)\boldsymbol{\Psi}(x)\,.\boldsymbol{A}.(\boldsymbol{B}^*)^T\,.\boldsymbol{\Upsilon}(t) \\ & \qquad = b(x,t)\boldsymbol{\Psi}(x)\,.(\boldsymbol{B}^*)^2.\mathbf{A}\,.\boldsymbol{\Upsilon}(t) + c(x,t)\boldsymbol{\Psi}(x)\,.\boldsymbol{A}\,.\boldsymbol{\Upsilon}(t) + Q(x,t), \end{aligned} \tag{8}$$

with, $$\boldsymbol{\Psi}(x)\,.\boldsymbol{A}\,.\boldsymbol{\Upsilon}(0) = \alpha\,\boldsymbol{\Psi}(x)\,.\boldsymbol{A}\,.\boldsymbol{\Upsilon}(1) + g(x), \tag{9}$$

and, $$\boldsymbol{\Psi}(0)\,.\boldsymbol{A}\,.\boldsymbol{\Upsilon}(t) = \int_0^1 \rho(x)\,\boldsymbol{\Psi}(x)\,.\boldsymbol{A}\,.\boldsymbol{\Upsilon}(t)dx + f(t), \tag{10}$$

$$\boldsymbol{\Psi}(1)\,.\boldsymbol{A}\,.\boldsymbol{\Upsilon}(t) = \int_0^1 \psi(x)\,\boldsymbol{\Psi}(x)\,.\boldsymbol{A}\,.\boldsymbol{\Upsilon}(t)dx + h(t). \tag{11}$$

In the Hilbert space $H = L^2([0,1] \times [0,1])$, the inner product is given as follows:

$$\langle f,g\rangle = \int_0^1\int_0^1 f(x,t)g(x,t)dxdt, \tag{12}$$

In addition, the set of functions $\boldsymbol{X} = \{\boldsymbol{X}_0, \boldsymbol{X}_1 \dots, \boldsymbol{X}_n\}$, and $\boldsymbol{T} = \{\boldsymbol{T}_0, \boldsymbol{T}_1 \dots, \boldsymbol{T}_m\}$ are linearly independent in H, where $\boldsymbol{X}_i = x^i, 0 \le i \le n$, and $\boldsymbol{T}_j = t^j, 0 \le j \le m$ are the base functions of the standard polynomials [33].

Hence, implementing Eq. (12) to set the base functions $\boldsymbol{X}$ and $\boldsymbol{T}$ with the left and right sides of Eq. (8) results in the matrix equation which is shown as follows:

$$\boldsymbol{K} = \boldsymbol{R}.$$

Where the matrix $\boldsymbol{K}$ contains the coefficients $\boldsymbol{A}$, while the matrix $\boldsymbol{R}$ represents the known values in Eq. (8) as follows:

$$\begin{aligned}&\langle x^i t^j, a(x,t)\boldsymbol{\Psi}(x)\,.\boldsymbol{A}.(\boldsymbol{B}^*)^T\,.\boldsymbol{Y}(t) - b(x,t)\boldsymbol{\Psi}(x)\,.(\boldsymbol{B}^*)^2.\mathbf{A}\,.\boldsymbol{Y}(t)\\ &\qquad - c(x,t)\boldsymbol{\Psi}(x)\,.\boldsymbol{A}\,.\boldsymbol{Y}(t)\,\rangle,\ \ \langle x^i t^j, Q(x,t)\rangle,\ \ 0 \le i,j \le n.\end{aligned} \tag{13}$$

As a result, by replacing the Eqs. (9), (10), and (11) into the Eq. (13), some entries in the matrix equation will be adjusted. We construct an $(n+1 \times n+1)$ linear algebraic equation system with coefficients $\boldsymbol{A}$. To obtain the coefficients $\boldsymbol{A}$, this system can be numerically solved using the Mathematica®12. Finally, these values are substituted into Eq. (6) to provide an approximate solution of Eq. (1).

3.2 The operational matrices for the Bernstein polynomials

The definition of the n^{th}-degree Bernstein polynomials $\boldsymbol{B}_{i,n}(x)$ on [0,1] is as follows [20]:

$$\boldsymbol{B}_{i,n}(x) = \frac{n!\ x^i\ (1-x)^{n-i}}{i!\ (n-i)!}\ ,\ \ 0 \le i \le n \tag{14}$$

Hence, we assume that a linear combination of the Bernstein polynomials can approximatively describe $u(x,t)$ as follows:

$$u(x,t) = \sum_{k=0}^{n}\sum_{l=0}^{n} a_{kl}\boldsymbol{B}_{k,n}(x)\boldsymbol{B}_{l,n}(t) = \boldsymbol{A}^T\,.\boldsymbol{\Psi}(x)\,.\boldsymbol{Y}(t), \tag{15}$$

where,

$\boldsymbol{\Psi}(x) = \left[\boldsymbol{B}_{0,n}(x), \boldsymbol{B}_{1,n}(x), \boldsymbol{B}_{2,n}(x)\,, \dots, \boldsymbol{B}_{n,n}(x)\right]$,

$\boldsymbol{Y}(t) = \left[\boldsymbol{B}_{0,n}(t), \boldsymbol{B}_{1,n}(t), \boldsymbol{B}_{2,n}(t), \dots, \boldsymbol{B}_{n,n}(t)\right]^T$, and $\boldsymbol{A} = \begin{bmatrix} a_{00} & a_{01} & \cdots & a_{0n}\\ a_{10} & a_{11} & \cdots & a_{1n}\\ \vdots & \vdots & \cdots & \vdots\\ a_{n0} & a_{n1} & \cdots & a_{nn}\end{bmatrix}^T$,

such that $a_{kl}, k, l = 0, \dots, n$ are the unknown Bernstein polynomials coefficients whose values will be determined later.

The matrix form expressions for the partial derivatives of the function $u(x,t)$ for x and t are as follows:

$$\begin{aligned}\frac{\partial^n u(x,t)}{\partial x^n} &= \boldsymbol{A}^T.\boldsymbol{Bn^*}^n.\boldsymbol{\Psi}(x)\,.\boldsymbol{Y}(t) \qquad n \ge 1,\\ \frac{\partial^m u(x,t)}{\partial t^m} &= \boldsymbol{A}^T.\boldsymbol{\Psi}(x)\,.(\boldsymbol{Bn^{*m}})^T.\boldsymbol{Y}(t) \qquad m \ge 1.\end{aligned} \tag{16}$$

such that, ${\boldsymbol{Bn^*}^n}_{(n+1)\times(n+1)}$, which is the operational matrix of partial derivatives, has the following definitions [47]: $\boldsymbol{Bn^*}^n = \boldsymbol{D\,V\,Q}$

where, $\boldsymbol{D} = \begin{bmatrix} (-1)^0\binom{n}{0} & (-1)^1\binom{n}{0}\binom{n-0}{1} & \dots & (-1)^{n-0}\binom{n}{0}\binom{n-0}{n-0}\\ 0 & (-1)^0\binom{n}{j} & \dots & (-1)^{n-j}\binom{n}{j}\binom{n-j}{n-j}\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \dots & (-1)^0\binom{n}{n}\end{bmatrix}_{(n+1)\times(n+1).}$,

$$\boldsymbol{V}=\begin{bmatrix}0&0&0&\dots&0\\1&0&0&\dots&0\\0&2&0&\cdots&0\\\vdots&\vdots&\vdots&\ddots&\vdots\\0&0&0&\cdots&n\end{bmatrix}_{(n+1)\times n}, \text{ and } \boldsymbol{Q}=\begin{bmatrix}D_1^{-1}\\D_2^{-1}\\D_3^{-1}\\\vdots\\D_n^{-1}\end{bmatrix}_{n\times(n+1)}.$$

Thus, substituting Eqs. (15) and (16) into Eqs. (1-4), then the following matrix equations are obtained:

$$a(x,t)\boldsymbol{A^T}.\boldsymbol{\Psi}(x).(\boldsymbol{Bn^*})^{\boldsymbol{T}}.\boldsymbol{\Upsilon}(t)$$
$$=b(x,t)\boldsymbol{A^T}.\boldsymbol{Bn^{*2}}.\boldsymbol{\Psi}(x).\boldsymbol{\Upsilon}(t)+c(x,t)\boldsymbol{A^T}.\boldsymbol{\Psi}(x).\boldsymbol{\Upsilon}(t)+Q(x,t),$$

with $\boldsymbol{A^T}.\boldsymbol{\Psi}(x).\boldsymbol{\Upsilon}(0)=\alpha\,\boldsymbol{A^T}.\boldsymbol{\Psi}(x).\boldsymbol{\Upsilon}(1)+g(x),$

and $\boldsymbol{A^T}.\boldsymbol{\Psi}(0).\boldsymbol{\Upsilon}(t)=\int_0^1\rho(x)\,\boldsymbol{A^T}.\boldsymbol{\Psi}(x).\boldsymbol{\Upsilon}(t)dx+f(t),$

$$\boldsymbol{A^T}.\boldsymbol{\Psi}(1).\boldsymbol{\Upsilon}(t)=\int_0^1\psi(x)\,\boldsymbol{A^T}.\boldsymbol{\Psi}(x).\boldsymbol{\Upsilon}(t)dx+h(t).$$

3.3 The operational matrices for the Legendre polynomials

The definition of the n^{th}-degree Legendre polynomials $\boldsymbol{P}_n(x)$ on [-1,1] is given as follows [48]:

$$\boldsymbol{P}_n(x)=\sum_{k=0}^{n}(-1)^{n+k}\frac{(n+k)!}{2^k(n-k)!\,(k!)^2}\,(x+1)^k. \qquad (17)$$

Hence, $\boldsymbol{P}_0(x)=1,\ \boldsymbol{P}_1(x)=x,\ \dots,\ \boldsymbol{P}_{n+1}(x)=\frac{x\,\boldsymbol{P}_n(x)(2\,n+1)-n\,\boldsymbol{P}_{n-1}(x)}{n+1},\ n=1,2,\dots.$

Furthermore, we assume that the solution $u(x,t)$ is approximated by the double series based on the Legendre polynomials as follows [49]:

$$u(x,t)=\sum_{k=0}^{n}\sum_{l=0}^{n}a_{kl}\boldsymbol{P}_k(x)\boldsymbol{P}_l(t)=\boldsymbol{A}^T.\boldsymbol{\Psi}(x).\boldsymbol{\Upsilon}(t), \qquad (18)$$

where $\boldsymbol{\Psi}(x)=[\boldsymbol{P}_0(x),\boldsymbol{P}_1(x),\dots,\boldsymbol{P}_n(x)]$, $\boldsymbol{\Upsilon}(t)=[\boldsymbol{P}_0(t),\boldsymbol{P}_1(t),\dots,\boldsymbol{P}_n(t)]^T$, and

$$\boldsymbol{A}=\begin{bmatrix}a_{00}&a_{01}&\cdots&a_{0n}\\a_{10}&a_{11}&\cdots&a_{1n}\\\vdots&\vdots&\cdots&\vdots\\a_{n0}&a_{n1}&\cdots&a_{nn}\end{bmatrix}^T,$$

such that $a_{kl},k,l=0,\dots,n$ are the unknown coefficients of the Legendre polynomials whose values will be calculated later.

Using the matrix form, we can write the partial derivatives of $u(x,t)$ for x and t as follows:

$$\frac{\partial^n u(x,t)}{\partial x^n}=\boldsymbol{A}^T.\boldsymbol{L^{*n}}.\boldsymbol{\Psi}(x).\boldsymbol{\Upsilon}(t) \qquad n\geq 1,$$
$$\frac{\partial^m u(x,t)}{\partial t^m}=\boldsymbol{A}^T.\boldsymbol{\Psi}(x).(\boldsymbol{L^{*m}})^T.\boldsymbol{\Upsilon}(t) \qquad m\geq 1. \qquad (19)$$

where $\boldsymbol{L}^*_{(n+1)\times(n+1)}$ is the derivatives' operational matrix which is given as follows:

$$\boldsymbol{L}^*=\begin{cases}(2k-1), & k=j-n, \text{ where, } \begin{cases}n=1,3,\dots,i, & \text{if } i \text{ odd},\\ n=1,3,\dots,i-1, & \text{if } i \text{ even}\end{cases},\\ 0 & \text{Otherwise.}\end{cases}$$

Hence, substituting Eqs. (18) and (19) into Eqs. (1-4), then the following matrix equations are obtained:

$$a(x,t)\boldsymbol{A^T}.\boldsymbol{\Psi}(x).(\boldsymbol{L}^*)^{\boldsymbol{T}}.\boldsymbol{\Upsilon}(t)$$
$$=b(x,t)\boldsymbol{A^T}.\boldsymbol{L^{*2}}.\boldsymbol{\Psi}(x).\boldsymbol{\Upsilon}(t)+c(x,t)\boldsymbol{A^T}.\boldsymbol{\Psi}(x).\boldsymbol{\Upsilon}(t)+Q(x,t),$$

with $\boldsymbol{A^T}.\boldsymbol{\Psi}(x).\boldsymbol{\Upsilon}(0)=\alpha\,\boldsymbol{A^T}.\boldsymbol{\Psi}(x).\boldsymbol{\Upsilon}(1)+g(x),$

and $\boldsymbol{A^T}.\boldsymbol{\Psi}(0).\boldsymbol{\Upsilon}(t)=\int_0^1\rho(x)\,\boldsymbol{A^T}.\boldsymbol{\Psi}(x).\boldsymbol{\Upsilon}(t)dx+f(t),$

$$\boldsymbol{A^T}\,.\,\boldsymbol{\Psi}(1)\,.\,\boldsymbol{\Upsilon}(t) = \int_0^1 \psi(x)\,\boldsymbol{A^T}\,.\,\boldsymbol{\Psi}(x)\,.\,\boldsymbol{\Upsilon}(t)dx + h(t).$$

3.4 The operational matrices for the Chebyshev polynomials

The definition of the first kind of degree n Chebyshev polynomials $\boldsymbol{T}_n(x)$ is given as follows:

$$\boldsymbol{T}_n(x) = \sum_{k=0}^{n} (-1)^{n-k}\, 2^k \frac{(n+k-1)!}{(n-k)!\,(2k)!}(x+1)^k. \tag{20}$$

In addition, we assume that the following approximation for the solution $u(x,t)$ using the double series based on the Chebyshev polynomials of the first kind [50]:

$$u(x,t) = \sum_{k=0}^{n}\sum_{l=0}^{n} a_{kl}\boldsymbol{T}_k(x)\boldsymbol{T}_l(t) = \boldsymbol{A}^T\,.\,\boldsymbol{\Psi}(x)\,.\,\boldsymbol{\Upsilon}(t), \tag{21}$$

where $\boldsymbol{\Psi}(x) = [\boldsymbol{T}_0(x), \boldsymbol{T}_1(x), \dots, \boldsymbol{T}_n(x)]$, $\boldsymbol{\Upsilon}(t) = [\boldsymbol{T}_0(t), \boldsymbol{T}_1(t), \dots, \boldsymbol{T}_n(t)]^T$, and

$$\boldsymbol{A} = \begin{bmatrix} a_{00} & a_{01} & \cdots & a_{0n} \\ a_{10} & a_{11} & \cdots & a_{1n} \\ \vdots & \vdots & \cdots & \vdots \\ a_{n0} & a_{n1} & \cdots & a_{nn} \end{bmatrix}^T,$$

such that $a_{kl}, k, l = 0, \dots, n$ are the unknown coefficients of the first kind of Chebyshev polynomials whose values will be calculated later.

The partial derivatives of the function $u(x,t)$ for x and t are expressed in matrix form as follows:

$$\begin{aligned} \frac{\partial^n u(x,t)}{\partial x^n} &= \boldsymbol{A}^T.\boldsymbol{V}^{*n}.\boldsymbol{\Psi}(x)\,.\,\boldsymbol{\Upsilon}(t) \qquad n \geq 1, \\ \frac{\partial^m u(x,t)}{\partial t^m} &= \boldsymbol{A}^T.\boldsymbol{\Psi}(x)\,.\,(\boldsymbol{V}^{*m})^T.\boldsymbol{\Upsilon}(t) \qquad m \geq 1. \end{aligned} \tag{22}$$

where $\boldsymbol{V}^*_{(n+1)\times(n+1)}$ is the operational matrix of derivatives and has the following definition:

$$\boldsymbol{V}^* = (d_{i,j}) = \begin{cases} \dfrac{2i}{\mu_j}, & \text{for } j = i-k, \\ 0 & \text{otherwise,} \end{cases}$$

where $k = 1, 3, 5, \dots, n$ if n is odd, or $k = 1, 3, 5, \dots, n-1$ if n is even, $\mu_0 = 2$, and $\mu_k = 1$ for all $k \geq 1$.

Consequently, substituting Eqs. (21) and (22) into Eqs. (1-4), then the following matrix equations are obtained:

$$\begin{aligned} & a(x,t)\boldsymbol{A^T}.\boldsymbol{\Psi}(x)\,.\,(\boldsymbol{V}^*)^{\boldsymbol{T}}.\boldsymbol{\Upsilon}(t) \\ &\quad = b(x,t)\boldsymbol{A^T}.\boldsymbol{V}^{*\boldsymbol{2}}.\boldsymbol{\Psi}(x)\,.\,\boldsymbol{\Upsilon}(t) + c(x,t)\boldsymbol{A^T}\,.\,\boldsymbol{\Psi}(x)\,.\,\boldsymbol{\Upsilon}(t) + Q(x,t), \end{aligned}$$

with $\boldsymbol{A^T}\,.\,\boldsymbol{\Psi}(x)\,.\,\boldsymbol{\Upsilon}(0) = \alpha\,\boldsymbol{A^T}\,.\,\boldsymbol{\Psi}(x)\,.\,\boldsymbol{\Upsilon}(1) + g(x),$

and $\boldsymbol{A^T}\,.\,\boldsymbol{\Psi}(0)\,.\,\boldsymbol{\Upsilon}(t) = \int_0^1 \rho(x)\,\boldsymbol{A^T}\,.\,\boldsymbol{\Psi}(x)\,.\,\boldsymbol{\Upsilon}(t)dx + f(t),$

$$\boldsymbol{A^T}\,.\,\boldsymbol{\Psi}(1)\,.\,\boldsymbol{\Upsilon}(t) = \int_0^1 \psi(x)\,\boldsymbol{A^T}\,.\,\boldsymbol{\Psi}(x)\,.\,\boldsymbol{\Upsilon}(t)dx + h(t).$$

3.5 The operational matrices for the Hermite polynomials

The definition of the n^{th}-degree Hermite polynomials $\boldsymbol{H}_n(x)$ on $(-\infty, \infty)$ is given as follows [51]:

$$\boldsymbol{H}_n(x) = n!\sum_{i=0}^{k}(-1)^i \frac{(2\,x)^{n-2i}}{i\,!\,(n-2\,i)!}, \tag{23}$$

where $k = \frac{n}{2}$ if n is even and $k = \frac{n-1}{2}$ if n is odd.

Moreover, we assume that the double series based on the Hermite polynomials can be used to approximate the function $u(x,t)$ as follows:

$$u(x,t)=\sum_{k=0}^{n}\sum_{l=0}^{n} a_{kl}\boldsymbol{H}_k(x)\boldsymbol{H}_l(t)=\boldsymbol{\Psi}(x)\,.\,\boldsymbol{A}\,.\,\boldsymbol{Y}(t), \tag{24}$$

where $\boldsymbol{\Psi}(x)=[\boldsymbol{H}_0(x),\boldsymbol{H}_1(x),\dots,\boldsymbol{H}_n(x)]$, $\boldsymbol{Y}(t)=[\boldsymbol{H}_0(t),\boldsymbol{H}_1(t),\dots,\boldsymbol{H}_n(t)]^T$, and

$$\boldsymbol{A}=\begin{bmatrix} a_{00} & a_{01} & \cdots & a_{0n}\\ a_{10} & a_{11} & \cdots & a_{1n}\\ \vdots & \vdots & \cdots & \vdots\\ a_{n0} & a_{n1} & \cdots & a_{nn}\end{bmatrix}.$$

such that $a_{kl}, k,l=0,\dots,n$ are the unknown coefficients of the Hermite polynomials whose values will be determined later.

In addition, the $\boldsymbol{\Psi}(x)$ can be expressed by the relevant matrix relation as follows:

$$\boldsymbol{\Psi}(x)=\boldsymbol{\Lambda}(x).(\boldsymbol{E}^{-1})^T.$$

where $\boldsymbol{\Lambda}(x)=[1,x,\dots,x^n]$, and the matrix $\boldsymbol{E}$ is defined for odd n as follows [52]:

$$\boldsymbol{E}=\begin{pmatrix} 1 & 0 & 0 & 0 & \dots & 0\\ 0 & \frac{1}{2} & 0 & 0 & \cdots & 0\\ \frac{1}{2} & 0 & \frac{1}{4} & 0 & \cdots & 0\\ 0 & \frac{3}{4} & 0 & \frac{1}{8} & \cdots & 0\\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots\\ 0 & \frac{n!}{2^n\left(\frac{n-1}{2}\right)!1!} & 0 & \frac{n!}{2^n\left(\frac{n-1}{2}-1\right)!3!} & \cdots & 0 \end{pmatrix},$$

and if n is even, then the matrix $\boldsymbol{E}$ is defined as follows [52]:

$$\boldsymbol{E}=\begin{pmatrix} 1 & 0 & 0 & 0 & \dots & 0\\ 0 & \frac{1}{2} & 0 & 0 & \dots & 0\\ \frac{1}{2} & 0 & \frac{1}{4} & 0 & \dots & 0\\ 0 & \frac{3}{4} & 0 & \frac{1}{8} & \dots & 0\\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots\\ \frac{n!}{2^n\left(\frac{n}{2}\right)!0!} & 0 & \frac{n!}{2^n\left(\frac{n}{2}-1\right)!2!} & 0 & \dots & \frac{n!}{2^n(0)!n!} \end{pmatrix}.$$

Consequently, the derivatives of the $\boldsymbol{\Psi}(x)$ can be represented as follows:

$$\boldsymbol{\Psi}(x)^{(n)}=\boldsymbol{\Lambda}(x).\boldsymbol{N}^n.(\boldsymbol{E}^{-1})^T.$$

where, $\boldsymbol{N}=\begin{bmatrix} 0 & 1 & 0 & & 0\\ 0 & 0 & 2 & \cdots & 0\\ 0 & 0 & 0 & & 0\\ & \vdots & & \ddots & \vdots\\ 0 & 0 & 0 & & n\\ 0 & 0 & 0 & \dots & 0\end{bmatrix}_{(n+1)\times(n+1)}$.

Consequently, using the matrix form, we can write the partial derivatives of $u(x,t)$ for x and t as follows:

$$\begin{aligned}\frac{\partial^n u(x,t)}{\partial x^n}&=\boldsymbol{\Lambda}(x).\boldsymbol{N}^n\,.(\boldsymbol{E}^{-1})^T.\boldsymbol{A}\,.\boldsymbol{Y}(t)\,, \qquad n\geq 1,\\ \frac{\partial^m u(x,t)}{\partial t^m}&=\boldsymbol{\Psi}(x)\,.\boldsymbol{A}.\boldsymbol{\Lambda}(t).(\boldsymbol{N}^m)^T.(\boldsymbol{E}^{-1})^T, \qquad m\geq 1.\end{aligned} \tag{25}$$

Thus, substituting the Eqs. (24) and (25) into the Eqs. (1-4), the following matrix equations are obtained:

$$a(x,t)\boldsymbol{\Psi}(x)\,.\boldsymbol{A}.\boldsymbol{\Lambda}(t).(N)^T.(\boldsymbol{E}^{-1})^T$$
$$= b(x,t)\boldsymbol{\Lambda}(x).\boldsymbol{N}^2\,.(\boldsymbol{E}^{-1})^T.\boldsymbol{A}\,.\boldsymbol{\Upsilon}(t) + c(x,t)\boldsymbol{\Psi}(x)\,.\boldsymbol{A}\,.\boldsymbol{\Upsilon}(t) + Q(x,t),$$

with $$\boldsymbol{\Psi}(x)\,.\boldsymbol{A}\,.\boldsymbol{\Upsilon}(0) = \alpha\,\boldsymbol{\Psi}(x)\,.\boldsymbol{A}\,.\boldsymbol{\Upsilon}(1) + g(x),$$

and $$\boldsymbol{\Psi}(0)\,.\boldsymbol{A}\,.\boldsymbol{\Upsilon}(t) = \int_0^1 \rho(x)\,\boldsymbol{\Psi}(x)\,.\boldsymbol{A}\,.\boldsymbol{\Upsilon}(t)dx + f(t),$$

$$\boldsymbol{\Psi}(1)\,.\boldsymbol{A}\,.\boldsymbol{\Upsilon}(t) = \int_0^1 \psi(x)\,\boldsymbol{\Psi}(x)\,.\boldsymbol{A}\,.\boldsymbol{\Upsilon}(t)dx + h(t).$$

3.6 The operational matrices for the Bernoulli polynomials

The definition of the n^{th}-degree Bernoulli polynomials $\boldsymbol{B}_n(x)$ is as follows:

$$\boldsymbol{B}_n(x) = \sum_{k=0}^{n} \frac{n!\;\boldsymbol{b}_k}{k\,!\;(n-k)!\;2^{n-k}}\;(x+1)^{n-k}, \tag{26}$$

where $\boldsymbol{b}_k = \boldsymbol{B}_k(0)$ for each $k \geq 0$ is referred to as the Bernoulli number. These numbers are computed using the identity that is given below [53]:

$$\frac{x}{e^x - 1} = \sum_{k=0}^{\infty} \boldsymbol{b}_k \frac{x^k}{k!},$$

The following are some of the first Bernoulli numbers:

$\boldsymbol{b}_0 = 1, \boldsymbol{b}_1 = -\frac{1}{2}, \boldsymbol{b}_2 = \frac{1}{6}, \boldsymbol{b}_4 = -\frac{1}{30}, \ldots$, and $\boldsymbol{b}_{2k+1} = 0$ for $k \geq 1$.

In addition, we assume the following approximation for the solution $u(x,t)$ using the double series based on the Bernoulli polynomials [32]:

$$u(x,t) = \sum_{k=0}^{n}\sum_{l=0}^{n} a_{kl}\boldsymbol{B}_k(x)\boldsymbol{B}_l(t) = \boldsymbol{A}^T\,.\boldsymbol{\Psi}(x)\,.\boldsymbol{\Upsilon}(t), \tag{27}$$

where, $\boldsymbol{\Psi}(x) = [\boldsymbol{B}_0(x), \boldsymbol{B}_1(x), \ldots, \boldsymbol{B}_n(x)]$, $\boldsymbol{\Upsilon}(t) = [\boldsymbol{B}_0(t), \boldsymbol{B}_1(t), \ldots, \boldsymbol{B}_n(t)]^T$, and

$$\boldsymbol{A} = \begin{bmatrix} a_{00} & a_{01} & \cdots & a_{0n} \\ a_{10} & a_{11} & \cdots & a_{1n} \\ \vdots & \vdots & \cdots & \vdots \\ a_{n0} & a_{n1} & \cdots & a_{nn} \end{bmatrix}^T.$$

such that $a_{kl}, k, l = 0, \ldots, n$ are the unknown coefficients of the Bernoulli polynomials whose values will be computed later.

Moreover, we can write the partial derivatives of $u(x,t)$ for x and t in matrix form as follows:

$$\begin{aligned} \frac{\partial^n u(x,t)}{\partial x^n} &= \boldsymbol{A}^T.\boldsymbol{Bi}^{*n}.\boldsymbol{\Psi}(x)\,.\boldsymbol{\Upsilon}(t) \qquad n \geq 1, \\ \frac{\partial^m u(x,t)}{\partial t^m} &= \boldsymbol{A}^T.\boldsymbol{\Psi}(x)\,.(\boldsymbol{Bi}^{*m})^T.\boldsymbol{\Upsilon}(t) \qquad m \geq 1. \end{aligned} \tag{28}$$

where $\boldsymbol{Bi}^*_{(n+1)\times(n+1)}$ is the derivatives' operational matrix and is given as follows [54]:

$$\boldsymbol{Bi}^* = \begin{bmatrix} 0 & 0 & 0 & \ldots & 0 & 0 \\ 1 & 0 & 0 & \ldots & 0 & 0 \\ 0 & 2 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & n & 0 \end{bmatrix}_{(n+1)\times(n+1)}.$$

Therefore, substituting the Eqs. (27) and (28) into the Eqs. (1-4), the following matrix equations are obtained:

$$a(x,t)\boldsymbol{A^T}.\boldsymbol{\Psi}(x)\,.(\boldsymbol{Bi}^*)^{\boldsymbol{T}}.\boldsymbol{\Upsilon}(t)$$
$$= b(x,t)\boldsymbol{A^T}.\boldsymbol{Bi}^{*\boldsymbol{2}}.\boldsymbol{\Psi}(x)\,.\boldsymbol{\Upsilon}(t) + c(x,t)\boldsymbol{A^T}\,.\boldsymbol{\Psi}(x)\,.\boldsymbol{\Upsilon}(t) + Q(x,t),$$

with $$\boldsymbol{A^T}\,.\boldsymbol{\Psi}(x)\,.\boldsymbol{\Upsilon}(0) = \alpha\,\boldsymbol{A^T}\,.\boldsymbol{\Psi}(x)\,.\boldsymbol{\Upsilon}(1) + g(x),$$

and $$\boldsymbol{A^T}\,.\boldsymbol{\Psi}(0)\,.\boldsymbol{\Upsilon}(t) = \int_0^1 \rho(x)\,\boldsymbol{A^T}\,.\boldsymbol{\Psi}(x)\,.\boldsymbol{\Upsilon}(t)dx + f(t),$$
$$\boldsymbol{A^T}\,.\boldsymbol{\Psi}(1)\,.\boldsymbol{\Upsilon}(t) = \int_0^1 \psi(x)\,\boldsymbol{A^T}\,.\boldsymbol{\Psi}(x)\,.\boldsymbol{\Upsilon}(t)dx + h(t).$$

4. The application of the proposed methods and numerical results

In this paper, the proposed methods DECMs are based on the base functions of orthogonal polynomials such as the Bernstein, the Legendre, the Chebyshev, the Hermite, and the Bernoulli polynomials which have been implemented to solve some specific cases of the time-dependent diffusion equations that are considered nonlocal parabolic partial differential equations to demonstrate the application of the suggested approaches and their performance. The base functions of the orthogonal polynomials are executed in two steps of the DECMs proposed methods to extend and improve the ECM method. First, define the unknown function $u(x,t)$ and its derivatives, then compute the inner product to obtain the matrix equation explained in Eq. (13).

In addition, the nonlocal initial and boundary conditions given in Eqs. (9-11) are substituted into the matrix equation, and then some entries of Eq. (13) are modified. These operations convert the considered problem to an $(n+1 \times n+1)$ linear algebraic equation system with coefficients $\boldsymbol{A}$. Consequently, we solve this system using Mathematica®12 and obtain novel approximate and analytic solutions to the time-dependent diffusion equations.

Example 1. Consider the time-dependent diffusion equation of the following form [14]:
$$x\frac{\partial u}{\partial t} - t\frac{\partial^2 u}{\partial x^2} = (-1+2t)x^3(-1+x^2) - 2t(1-t+t^2)(-1+6x^2), \tag{29}$$
Subjected to the nonlocal boundary conditions:
$$\begin{aligned} &u(x,0) = u(x,1), && 0 \le x \le 1,\\ &u(0,t) = u(1,t) = 0, && 0 \le t \le 1. \end{aligned} \tag{30}$$
This problem's exact solution is $u(x,t) = x^2(x^2-1)(-t+t^2+1)$.
Turkyilmazoglu in [14] solved this problem for $n \ge 4$, where n is the order of approximation using the efficient analytic approximate method, namely ECM based on the standard polynomials, and obtained the exact solution.
Furthermore, we can apply the novel DECMs proposed methods to solve this problem in the current study for $n = 2$ as follows:
First: Applying the DECMs proposed methods based on the Bernstein polynomials.
By inserting the Eqs. (15) and (16) into the Eqs. (29) and (30), we change the function $u(x,t)$ and its partial derivatives into matrices. Therefore, the following results are obtained:
$$\begin{aligned} &x\left(\boldsymbol{A}^T.\boldsymbol{\Psi}(x)\,.(\boldsymbol{Bn}^*)^T.\boldsymbol{\Upsilon}(t)\right) - t\left(\boldsymbol{A}^T.\boldsymbol{Bn}^{*2}.\boldsymbol{\Psi}(x)\,.\boldsymbol{\Upsilon}(t)\right)\\ &\qquad = (-1+2t)x^3(-1+x^2) - 2t(1-t+t^2)(-1+6x^2), \end{aligned} \tag{31}$$
$$\begin{aligned} &\boldsymbol{A}^T\,.\boldsymbol{\Psi}(x)\,.\boldsymbol{\Upsilon}(0) = \boldsymbol{A}^T\,.\boldsymbol{\Psi}(x)\,.\boldsymbol{\Upsilon}(1),\\ &\boldsymbol{A}^T\,.\boldsymbol{\Psi}(0)\,.\boldsymbol{\Upsilon}(t) = \boldsymbol{A}^T\,.\boldsymbol{\Psi}(1)\,.\boldsymbol{\Upsilon}(t) = 0. \end{aligned} \tag{32}$$
The technique has been applied as shown in Eq. (13), which results in:
$$\begin{aligned} &\langle \boldsymbol{B}_{i,2}(x)\boldsymbol{B}_{j,2}(t), x\left(\boldsymbol{A}^T.\boldsymbol{\Psi}(x)\,.(\boldsymbol{Bn}^*)^T.\boldsymbol{\Upsilon}(t)\right)\\ &\qquad - t\left(\boldsymbol{A}^T.\boldsymbol{Bn}^{*2}.\boldsymbol{\Psi}(x)\,.\boldsymbol{\Upsilon}(t)\right)\rangle,\ \ \langle \boldsymbol{B}_{i,2}(x)\boldsymbol{B}_{j,2}(t), (-1+2t)x^3(-1+x^2)\\ &\qquad - 2t(1-t+t^2)(-1+6x^2)\rangle,\ \ 0 \le i,j \le 2. \end{aligned} \tag{33}$$
Then, applying the inner product to solve the left and right sides of the matrix equation given in Eq. (33), with substituting the initial conditions Eq. (32), we obtain the following linear algebraic system for coefficients $a_{00}, a_{01}, a_{02}, a_{10}, a_{11}, a_{12}, a_{20}, a_{21}$ and a_{22}:

$$\begin{bmatrix} \frac{1}{5} & 0 & -\frac{1}{5} & \frac{1}{10} & 0 & -\frac{1}{10} & \frac{1}{30} & 0 & -\frac{1}{30} \\ \frac{1}{10} & \frac{2}{15} & \frac{1}{10} & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{30} & \frac{1}{10} & \frac{1}{5} & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{10} & 0 & -\frac{1}{10} & \frac{2}{15} & 0 & -\frac{2}{15} & \frac{1}{10} & 0 & -\frac{1}{10} \\ -\frac{1}{30} & -\frac{2}{45} & -\frac{1}{30} & \frac{1}{45} & \frac{4}{45} & \frac{1}{9} & -\frac{2}{45} & -\frac{2}{45} & -\frac{1}{45} \\ -\frac{1}{60} & -\frac{1}{18} & -\frac{17}{180} & \frac{1}{90} & \frac{1}{15} & \frac{23}{90} & -\frac{1}{45} & -\frac{1}{15} & -\frac{7}{90} \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{5} & \frac{1}{10} & \frac{1}{30} \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{10} & \frac{2}{15} & \frac{1}{10} \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{30} & \frac{1}{10} & \frac{1}{5} \end{bmatrix} \begin{bmatrix} a_{00} \\ a_{01} \\ a_{02} \\ a_{10} \\ a_{11} \\ a_{12} \\ a_{20} \\ a_{21} \\ a_{22} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ -\frac{16}{225} \\ -\frac{169}{1400} \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

Therefore, the following coefficient values are calculated for this system using Mathematica®12:

$a_{00} = 0, a_{01} = 0, a_{02} = 0, a_{10} = -\frac{283}{700}, a_{11} = -\frac{271}{1400}, a_{12} = -\frac{283}{700}, a_{20} = 0, a_{21} =$ 0 and $a_{22} = 0$.

Finally, by entering the values of these coefficients into Eq. (15), the results are as follows:

$$\begin{aligned} u(x,t) = {} & (1-2t+t^2)\big((1-2x+x^2)a_{00} + (2x-2x^2)a_{10} + x^2 a_{20}\big) \\ & + (2t-2t^2)\big((1-2x+x^2)a_{01} + (2x-2x^2)a_{11} + x^2 a_{21}\big) \\ & + t^2\big((1-2x+x^2)a_{02} + (2x-2x^2)a_{12} + x^2 a_{22}\big). \end{aligned}$$

Then, the following approximate solution is produced for $n = 2$:

$$u(x,y) = \frac{1}{350}(283 + 295(-1+t)t)(-1+x)x.$$

Moreover, by proceeding in this way for $n = 3$, the following approximate solution is obtained:

$$\begin{aligned} u(x,y) = {} & \frac{429979x}{2143575} - \frac{8999tx}{40830} + \frac{10199t^2x}{40830} - \frac{40t^3x}{1361} - \frac{3669097x^2}{1667225} + \frac{30443tx^2}{13610} \\ & - \frac{655603t^2x^2}{285810} + \frac{1630t^3x^2}{28581} + \frac{6002404x^3}{3001005} - \frac{8233tx^3}{4083} + \frac{58421t^2x^3}{28581} \\ & - \frac{790t^3x^3}{28581}. \end{aligned}$$

Furthermore, for all $n \geq 4$, the exact solution is obtained as follows:

$$u(x,t) = x^2(x^2-1)(-t+t^2+1).$$

Second: Using the DECMs proposed methods based on the Legendre polynomials.

By substituting Eqs. (18) and (19) into Eqs. (29) and (30), yielding the following:

$$\begin{aligned} & x\big(\boldsymbol{A}^T.\boldsymbol{\Psi}(x).(\boldsymbol{L}^*)^T.\boldsymbol{\Upsilon}(t)\big) - t\left(\boldsymbol{A}^T.\boldsymbol{L}^{*2}.\boldsymbol{\Psi}(x).\boldsymbol{\Upsilon}(t)\right) \\ & \qquad = (-1+2t)x^3(-1+x^2) - 2t(1-t+t^2)(-1+6x^2), \end{aligned} \tag{34}$$

$$\begin{aligned} & \boldsymbol{A}^T.\boldsymbol{\Psi}(x).\boldsymbol{\Upsilon}(0) = \boldsymbol{A}^T.\boldsymbol{\Psi}(x).\boldsymbol{\Upsilon}(1), \\ & \boldsymbol{A}^T.\boldsymbol{\Psi}(0).\boldsymbol{\Upsilon}(t) = \boldsymbol{A}^T.\boldsymbol{\Psi}(1).\boldsymbol{\Upsilon}(t) = 0. \end{aligned} \tag{35}$$

Then, the technique has been applied as shown in Eq. (13), which results in:

$$\langle \boldsymbol{P}_i(x)\boldsymbol{P}_j(t), x\left(\boldsymbol{A}^T.\boldsymbol{\Psi}(x).(\boldsymbol{L}^*)^T.\boldsymbol{Y}(t)\right) - t\left(\boldsymbol{A}^T.\boldsymbol{L}^{*2}.\boldsymbol{\Psi}(x).\boldsymbol{Y}(t)\right)\rangle,\ \ \langle \boldsymbol{P}_i(x)\boldsymbol{P}_j(t), (-1+2t)x^3(-1+x^2) - 2t(1-t+t^2)(-1+6x^2)\rangle,\ \ 0 \le i,j \le 2. \tag{36}$$

Also, by applying the same procedure mentioned above for the Eqs. (36) and (35), we obtain the following linear algebraic system for coefficients $a_{00}, a_{01}, a_{02}, a_{10}, a_{11}, a_{12}, a_{20}, a_{21}$ and a_{22}:

$$\begin{bmatrix} 0 & -1 & -\frac{3}{2} & 0 & -\frac{1}{2} & -\frac{3}{4} & 0 & 0 & 0 \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{8} & 0 & 0 & 0 & -\frac{1}{4} & -\frac{1}{6} & -\frac{1}{16} \\ 0 & \frac{1}{8} & \frac{1}{5} & 0 & 0 & 0 & 0 & -\frac{1}{16} & -\frac{1}{10} \\ 0 & -\frac{1}{2} & -\frac{3}{4} & 0 & -\frac{1}{3} & -\frac{1}{2} & 0 & -\frac{1}{8} & -\frac{3}{16} \\ 0 & \frac{1}{6} & \frac{1}{3} & 0 & \frac{1}{8} & \frac{1}{4} & -\frac{1}{2} & -\frac{37}{120} & -\frac{1}{15} \\ 0 & 0 & \frac{1}{8} & 0 & 0 & \frac{3}{32} & -\frac{3}{16} & -\frac{1}{5} & -\frac{11}{80} \\ 1 & \frac{1}{2} & 0 & 1 & \frac{1}{2} & 0 & 1 & \frac{1}{2} & 0 \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{8} & \frac{1}{2} & \frac{1}{3} & \frac{1}{8} & \frac{1}{2} & \frac{1}{3} & \frac{1}{8} \\ 0 & \frac{1}{8} & \frac{1}{5} & 0 & \frac{1}{8} & \frac{1}{5} & 0 & \frac{1}{8} & \frac{1}{5} \end{bmatrix} \begin{bmatrix} a_{00} \\ a_{01} \\ a_{02} \\ a_{10} \\ a_{11} \\ a_{12} \\ a_{20} \\ a_{21} \\ a_{22} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ -\frac{121}{210} \\ -\frac{26}{105} \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

By solving this system numerically using Mathematica®12, the values of coefficients are given as follows:

$$a_{00} = \frac{4906}{5985}, a_{01} = -\frac{212}{399}, a_{02} = \frac{424}{1197}, a_{10} = -\frac{4906}{1995}, a_{11} = \frac{212}{133}, a_{12} = -\frac{424}{399}, a_{20} = \frac{9812}{5985}, \quad a_{21} = -\frac{424}{399}, \quad a_{22} = \frac{848}{1197}.$$

Finally, by inserting the values of these coefficients into Eq. (18), the results are as follows:

$$u(x,t) = a_{00} + xa_{10} + \left(-\frac{1}{2} + \frac{3x^2}{2}\right)a_{20} + t\left(a_{01} + xa_{11} + \left(-\frac{1}{2} + \frac{3x^2}{2}\right)a_{21}\right) + \left(-\frac{1}{2} + \frac{3t^2}{2}\right)\left(a_{02} + xa_{12} + \left(-\frac{1}{2} + \frac{3x^2}{2}\right)a_{22}\right).$$

Then, the following approximate solution is produced, for $n = 2$:

$$u(x,t) = \frac{2}{665}(641 + 530(-1+t)t)(-1+x)x.$$

Additionally, by proceeding in this way for $n = 3$, the following approximate solution is obtained:

$$u(x,t) = \frac{43119369493x}{315658049175} - \frac{261947587tx}{1002089045} + \frac{443785127t^2x}{1002089045} - \frac{36367508t^3x}{200417809} - \frac{1431965499991x^2}{631316098350} + \frac{4760726501tx^2}{2004178090} - \frac{5094246971t^2x^2}{2004178090} + \frac{33352047t^3x^2}{200417809} + \frac{269145352201x^3}{126263219670} - \frac{4236831327tx^3}{2004178090} + \frac{4206676717t^2x^3}{2004178090} + \frac{3015461t^3x^3}{200417809}.$$

Furthermore, the exact solution mentioned above is obtained for all $n \geq 4$.

Third: Implementing the DECMs proposed methods based on the Chebyshev polynomials of the first kind. By inserting the Eqs. (21) and (22) into the Eqs. (29) and (30), we obtain:

$$x\left(\boldsymbol{A}^T.\boldsymbol{\Psi}(x).(\boldsymbol{V}^*)^T.\boldsymbol{\Upsilon}(t)\right) - t\left(\boldsymbol{A}^T.\boldsymbol{V}^{*2}.\boldsymbol{\Psi}(x).\boldsymbol{\Upsilon}(t)\right) = (-1+2t)x^3(-1+x^2) - 2t(1-t+t^2)(-1+6x^2), \quad (37)$$

$$\begin{aligned} &\boldsymbol{A}^T.\boldsymbol{\Psi}(x).\boldsymbol{\Upsilon}(0) = \boldsymbol{A}^T.\boldsymbol{\Psi}(x).\boldsymbol{\Upsilon}(1), \\ &\boldsymbol{A}^T.\boldsymbol{\Psi}(0).\boldsymbol{\Upsilon}(t) = \boldsymbol{A}^T.\boldsymbol{\Psi}(1).\boldsymbol{\Upsilon}(t) = 0. \end{aligned} \quad (38)$$

Then, the procedure has been applied as shown in Eq. (13), which results in:

$$\left\langle \boldsymbol{T}_i(x)\boldsymbol{T}_j(t), x\left(\boldsymbol{A}^T.\boldsymbol{\Psi}(x).(\boldsymbol{V}^*)^T.\boldsymbol{\Upsilon}(t)\right) - t\left(\boldsymbol{A}^T.\boldsymbol{V}^{*2}.\boldsymbol{\Psi}(x).\boldsymbol{\Upsilon}(t)\right)\right\rangle, \quad \left\langle \boldsymbol{T}_i(x)\boldsymbol{T}_j(t), (-1+2t)x^3(-1+x^2) - 2t(1-t+t^2)(-1+6x^2)\right\rangle, \quad 0 \leq i,j \leq 2. \quad (39)$$

Applying the same procedure mentioned above for the Eqs. (39) and (38), we obtain the following linear algebraic system for coefficients $a_{00}, a_{01}, a_{02}, a_{10}, a_{11}, a_{12}, a_{20}, a_{21}$ and a_{22}:

$$\begin{bmatrix} 0 & -1 & -2 & 0 & -\frac{1}{2} & -1 & 0 & -\frac{1}{3} & \frac{2}{3} \\ \frac{1}{2} & \frac{1}{3} & 0 & 0 & 0 & 0 & -\frac{1}{2} & -\frac{1}{3} & 0 \\ -\frac{1}{3} & 0 & \frac{7}{15} & 0 & 0 & 0 & \frac{1}{3} & 0 & -\frac{7}{15} \\ 0 & -\frac{1}{2} & -1 & 0 & -\frac{1}{3} & -\frac{2}{3} & 0 & 0 & 0 \\ 0 & \frac{1}{6} & \frac{4}{9} & 0 & \frac{1}{8} & \frac{1}{3} & -\frac{2}{3} & -\frac{7}{15} & -\frac{2}{45} \\ 0 & -\frac{1}{9} & 0 & 0 & -\frac{1}{12} & 0 & 0 & -\frac{7}{45} & -\frac{1}{3} \\ 1 & \frac{1}{2} & -\frac{1}{3} & 1 & \frac{1}{2} & -\frac{1}{3} & 1 & \frac{1}{2} & -\frac{1}{3} \\ \frac{1}{2} & \frac{1}{3} & 0 & \frac{1}{2} & \frac{1}{3} & 0 & \frac{1}{2} & \frac{1}{3} & 0 \\ -\frac{1}{3} & 0 & \frac{7}{15} & -\frac{1}{3} & 0 & \frac{7}{15} & -\frac{1}{3} & 0 & \frac{7}{15} \end{bmatrix} \begin{bmatrix} a_{00} \\ a_{01} \\ a_{02} \\ a_{10} \\ a_{11} \\ a_{12} \\ a_{20} \\ a_{21} \\ a_{22} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ -\frac{121}{210} \\ -\frac{11}{210} \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

Thus, solving this system numerically using Mathematica®12, the values of coefficients are given as follows:

$$a_{00} = \frac{759}{560}, a_{01} = -\frac{11}{14}, a_{02} = \frac{11}{28}, a_{10} = -\frac{759}{280}, a_{11} = \frac{11}{7}, a_{12} = -\frac{11}{14}, a_{20} = \frac{759}{560}, a_{21} = -\frac{11}{14}, a_{22} = \frac{11}{28}.$$

Finally, by inserting the values of these coefficients into Eq. (21), the results are as follows:

$$u(x,t) = a_{00} + xa_{10} + (-1+2x^2)a_{20} + t(a_{01} + xa_{11} + (-1+2x^2)a_{21}) + (-1+2t^2)(a_{02} + xa_{12} + (-1+2x^2)a_{22}).$$

Then, the following approximate solution is produced, for $n = 2$:

$$u(x,t) = \frac{11}{280}(49 + 40(-1+t)t)(-1+x)x.$$

Also, by proceeding in this way for $n = 3$, the following approximate solution is obtained:

$$u(x,t)=\frac{49746579x}{482054720}-\frac{15067tx}{63295}+\frac{469884t^2x}{1076015}-\frac{42749t^3x}{215203}-\frac{2652568957x^2}{1205136800}+\frac{2491841tx^2}{1076015}-\frac{2688846t^2x^2}{1076015}+\frac{39401t^3x^2}{215203}+\frac{5056405019x^3}{2410273600}-\frac{2235702tx^3}{1076015}+\frac{2218962t^2x^3}{1076015}+\frac{3348t^3x^3}{215203}.$$

Furthermore, the exact solution mentioned above is obtained for all $n \geq 4$.
Fourth: Utilizing the DECMs proposed methods based on the Hermite polynomials.
The following findings are obtained by inserting the Eqs. (24) and (25) into the Eqs. (29) and (30):

$$x\left(\boldsymbol{\Psi}(x).\boldsymbol{A}.\boldsymbol{\Lambda}(t).(\boldsymbol{N})^T.(\boldsymbol{E}^{-1})^T\right)-t\left(\boldsymbol{\Lambda}(x).\boldsymbol{N}^2.(\boldsymbol{E}^{-1})^T.\boldsymbol{A}.\boldsymbol{\Upsilon}(t)\right)=(-1+2t)x^3(-1+x^2)-2t(1-t+t^2)(-1+6x^2), \quad (40)$$

$$\begin{gathered}\boldsymbol{\Psi}(x).\boldsymbol{A}.\boldsymbol{\Upsilon}(0)=\boldsymbol{\Psi}(x).\boldsymbol{A}.\boldsymbol{\Upsilon}(1),\\ \boldsymbol{\Psi}(0).\boldsymbol{A}.\boldsymbol{\Upsilon}(t)=\boldsymbol{\Psi}(1).\boldsymbol{A}.\boldsymbol{\Upsilon}(t)=0.\end{gathered} \quad (41)$$

Then, the technique has been applied as shown in Eq. (13), which results in:

$$\langle \boldsymbol{H}_i(x)\boldsymbol{H}_j(t), x\left(\boldsymbol{\Psi}(x).\boldsymbol{A}.\boldsymbol{\Lambda}(t).(\boldsymbol{N})^T.(\boldsymbol{E}^{-1})^T\right)-t\left(\boldsymbol{\Lambda}(x).\boldsymbol{N}^2.(\boldsymbol{E}^{-1})^T.\boldsymbol{A}.\boldsymbol{\Upsilon}(t)\right)\rangle,\ \langle \boldsymbol{H}_i(x)\boldsymbol{H}_j(t),(-1+2t)x^3(-1+x^2)-2t(1-t+t^2)(-1+6x^2)\rangle,\ 0\leq i,j\leq 2. \quad (42)$$

Applying the same technique mentioned above for the Eqs. (42) and (41), we obtain the following linear algebraic system for coefficients $a_{00}, a_{01}, a_{02}, a_{10}, a_{11}, a_{12}, a_{20}, a_{21}$ and a_{22}:

$$\begin{bmatrix} 0 & -2 & -4 & 0 & -2 & -4 & 0 & \frac{4}{3} & \frac{8}{3} \\ 1 & \frac{4}{3} & 0 & 0 & 0 & 0 & -2 & -\frac{8}{3} & 0 \\ -\frac{2}{3} & 0 & \frac{28}{15} & 0 & 0 & 0 & \frac{4}{3} & 0 & -\frac{56}{15} \\ 0 & -2 & -4 & 0 & -\frac{8}{3} & -\frac{16}{3} & 0 & 0 & 0 \\ 0 & \frac{4}{3} & \frac{32}{9} & 0 & 2 & \frac{16}{3} & -\frac{16}{3} & -\frac{112}{15} & -\frac{32}{45} \\ 0 & -\frac{8}{9} & 0 & 0 & -\frac{4}{3} & 0 & 0 & -\frac{112}{45} & -\frac{16}{3} \\ 1 & 1 & -\frac{2}{3} & 2 & 2 & -\frac{4}{3} & 2 & 2 & -\frac{4}{3} \\ 1 & \frac{4}{3} & 0 & 2 & \frac{8}{3} & 0 & 2 & \frac{8}{3} & 0 \\ -\frac{2}{3} & 0 & \frac{28}{15} & -\frac{4}{3} & 0 & \frac{56}{15} & -\frac{4}{3} & 0 & \frac{56}{15} \end{bmatrix}\begin{bmatrix} a_{00} \\ a_{01} \\ a_{02} \\ a_{10} \\ a_{11} \\ a_{12} \\ a_{20} \\ a_{21} \\ a_{22} \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ -\frac{242}{105} \\ -\frac{22}{105} \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

In addition, by solving this system numerically using Mathematica®12, the values of coefficients are given as follows:

$$a_{00}=\frac{759}{560}, a_{01}=-\frac{11}{28}, a_{02}=\frac{11}{56}, a_{10}=-\frac{759}{560}, a_{11}=\frac{11}{28}, a_{12}=-\frac{11}{56}, a_{20}=\frac{759}{1120}, a_{21}=-\frac{11}{56}, a_{22}=\frac{11}{112}.$$

Finally, by inserting the values of these coefficients into Eq. (24), the results are as follows:

$$u(x,t)=a_{00}+2xa_{10}+(-2+4x^2)a_{20}+2t(a_{01}+2xa_{11}+(-2+4x^2)a_{21})+(-2+4t^2)(a_{02}+2xa_{12}+(-2+4x^2)a_{22})$$

Then, the following approximate solution is produced, for $n=2$:

$$u(x,t)=\frac{11}{280}(49+40(-1+t)t)(-1+x)x.$$

Moreover, by proceeding in this way for $n=3$, the following approximate solution is obtained:

$$u(x,t)=\frac{49746579x}{482054720}-\frac{15067tx}{63295}+\frac{469884t^2x}{1076015}-\frac{42749t^3x}{215203}-\frac{2652568957x^2}{1205136800}+\frac{2491841tx^2}{1076015}-\frac{2688846t^2x^2}{1076015}+\frac{39401t^3x^2}{215203}+\frac{5056405019x^3}{2410273600}-\frac{2235702tx^3}{1076015}+\frac{2218962t^2x^3}{1076015}+\frac{3348t^3x^3}{215203}.$$

Furthermore, the exact solution mentioned above is obtained for all $n\geq 4$.

Fifth: Applying the DECMs proposed methods based on the Bernoulli polynomials.

By substituting the Eqs. (27) and (28) into the Eqs. (29) and (30) produce the following results:

$$x\left(\boldsymbol{A}^T.\boldsymbol{\Psi}(x).(\boldsymbol{Bi}^*)^T.\boldsymbol{Y}(t)\right)-t\left(\boldsymbol{A}^T.\boldsymbol{Bi}^{*2}.\boldsymbol{\Psi}(x).\boldsymbol{Y}(t)\right)=(-1+2t)x^3(-1+x^2)-2t(1-t+t^2)(-1+6x^2), \quad (43)$$

$$\begin{gathered}\boldsymbol{A}^T.\boldsymbol{\Psi}(x).\boldsymbol{Y}(0)=\boldsymbol{A}^T.\boldsymbol{\Psi}(x).\boldsymbol{Y}(1),\\ \boldsymbol{A}^T.\boldsymbol{\Psi}(0).\boldsymbol{Y}(t)=\boldsymbol{A}^T.\boldsymbol{\Psi}(1).\boldsymbol{Y}(t)=0.\end{gathered} \quad (44)$$

Then, the procedure has been applied as shown in Eq. (13), which results in:

$$\langle \boldsymbol{B}_i(x)\boldsymbol{B}_j(t), x\left(\boldsymbol{A}^T.\boldsymbol{\Psi}(x).(\boldsymbol{Bi}^*)^T.\boldsymbol{Y}(t)\right)-t\left(\boldsymbol{A}^T.\boldsymbol{Bi}^{*2}.\boldsymbol{\Psi}(x).\boldsymbol{Y}(t)\right)\rangle,\ \ \langle \boldsymbol{B}_i(x)\boldsymbol{B}_j(t), (-1+2t)x^3(-1+x^2)-2t(1-t+t^2)(-1+6x^2)\rangle,\ \ 0\leq i,j\leq 2. \quad (45)$$

Applying the same procedure mentioned above for the Eqs. (45) and (44), we obtain the following linear algebraic system for coefficients $a_{00}, a_{01}, a_{02}, a_{10}, a_{11}, a_{12}, a_{20}, a_{21}$ and a_{22}:

$$\begin{bmatrix} 0 & -\frac{1}{2} & -\frac{1}{4} & 0 & -\frac{1}{8} & -\frac{1}{16} & 0 & 0 & 0\\ \frac{1}{4} & \frac{1}{12} & \frac{1}{96} & 0 & 0 & 0 & -\frac{1}{48} & -\frac{1}{144} & -\frac{1}{1152}\\ 0 & \frac{1}{96} & \frac{1}{180} & 0 & 0 & 0 & 0 & -\frac{1}{1152} & -\frac{1}{2160}\\ 0 & -\frac{1}{8} & -\frac{1}{16} & 0 & -\frac{1}{24} & -\frac{1}{48} & 0 & -\frac{1}{192} & -\frac{1}{384}\\ 0 & \frac{1}{48} & \frac{1}{72} & 0 & \frac{1}{128} & \frac{1}{192} & -\frac{1}{48} & -\frac{37}{5760} & -\frac{1}{2160}\\ 0 & 0 & \frac{1}{576} & 0 & 0 & \frac{1}{1536} & -\frac{1}{384} & -\frac{1}{720} & -\frac{11}{34560}\\ 1 & \frac{1}{4} & 0 & \frac{1}{2} & \frac{1}{8} & 0 & \frac{1}{6} & \frac{1}{24} & 0\\ \frac{1}{4} & \frac{1}{12} & \frac{1}{96} & \frac{1}{8} & \frac{1}{24} & \frac{1}{192} & \frac{1}{24} & \frac{1}{72} & \frac{1}{576}\\ 0 & \frac{1}{96} & \frac{1}{180} & 0 & \frac{1}{192} & \frac{1}{360} & 0 & \frac{1}{576} & \frac{1}{1080} \end{bmatrix}\begin{bmatrix} a_{00}\\ a_{01}\\ a_{02}\\ a_{10}\\ a_{11}\\ a_{12}\\ a_{20}\\ a_{21}\\ a_{22}\end{bmatrix}=\begin{bmatrix}0\\0\\0\\0\\-\frac{121}{840}\\-\frac{13}{630}\\0\\0\\0\end{bmatrix}.$$

Therefore, by solving this system numerically using Mathematica®12, the values of coefficients are given as follows:

$$a_{00}=\frac{4906}{5985}, a_{01}=-\frac{424}{399}, a_{02}=\frac{848}{399}, a_{10}=-\frac{9812}{1995}, a_{11}=\frac{848}{133}, a_{12}=-\frac{1696}{133}, a_{20}=\frac{19624}{1995}, a_{21}=-\frac{1696}{133}, a_{22}=\frac{3392}{133}.$$

Ultimately, by inserting the values of these coefficients into Eq. (27), the results are as follows:

$$u(x,t) = a_{00} + \frac{1}{2}xa_{10} + \left(-\frac{1}{12} + \frac{x^2}{4}\right)a_{20} + \frac{1}{2}t\left(a_{01} + \frac{1}{2}xa_{11} + \left(-\frac{1}{12} + \frac{x^2}{4}\right)a_{21}\right) + \left(-\frac{1}{12} + \frac{t^2}{4}\right)\left(a_{02} + \frac{1}{2}xa_{12} + \left(-\frac{1}{12} + \frac{x^2}{4}\right)\, a_{22}\right).$$

Thus, the following approximate solution is produced, for $n = 2$:

$$u(x,t) = \frac{2}{665}(641 + 530(-1+t)t)(-1+x)x.$$

Moreover, by proceeding in this way for $n = 3$, the following approximate solution is obtained:

$$u(x,t) = \frac{43119369493x}{315658049175} - \frac{261947587tx}{1002089045} + \frac{443785127t^2x}{1002089045} - \frac{36367508t^3x}{200417809} - \frac{1431965499991x^2}{631316098350} + \frac{4760726501tx^2}{2004178090} - \frac{5094246971t^2x^2}{2004178090} + \frac{33352047t^3x^2}{200417809} + \frac{269145352201x^3}{126263219670} - \frac{4236831327tx^3}{2004178090} + \frac{4206676717t^2x^3}{2004178090} + \frac{3015461t^3x^3}{200417809}.$$

Furthermore, the exact solution mentioned above is obtained for all $n \geq 4$.

To explain the efficiency of the DECMs proposed methods to solve Example 1 referred to in Eqs. (29) and (30), the maximum absolute error ($MAbsR_n$) is computed as follows:

$$MAbsR_n = \max_{0 \le x,t \le 1} |u_e(x,t) - u(x,t)|. \tag{46}$$

Where $u_e(x,t)$ is the exact solution and $u(x,t)$ is the approximate solution achieved.

Table (1) lists $MAbsR_n$ for the approximate solution obtained by applying the DECMs proposed methods with approximation order $n = 2$ and 3. Moreover, we can demonstrate the accuracy of the proposed methods by comparing the error values for $n = 2$ and 3, as the error becomes smaller as the value of n increases.

Table 1: The comparison between the $MAbsR_n$ for the example (1) by the DECMs proposed methods.

n	DECMs Bernstein	DECMs Legendre	DECMs Chebyshev	DECMs Hermite	DECMs Bernoulli
2	0.0857417	0.269846	0.270249	0.270249	0.269846
3	0.00951975	0.0360467	0.0393046	0.0393046	0.0360467

Example 2. Consider the time-dependent diffusion equation of the following form [14]:

$$\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} - u(x,t) = (7 - 3t)tx. \tag{47}$$

Subjected to the nonlocal boundary conditions:

$$\begin{aligned} &u(x,0) = -u(x,1), \qquad 0 \le x \le 1, \\ &u(0,t) = 0, \qquad u(1,t) = 2\int_0^1 u(x,t)dx\,, \qquad 0 \le t \le 1. \end{aligned} \tag{48}$$

This problem's exact solution is $u(x,t) = x(3t^2 - t - 1)$.

Turkyilmazoglu in [14] solved this problem for $n \geq 2$ using the efficient analytic approximate method, namely ECM based on the standard polynomial, and obtained the exact solution.

Furthermore, now we can use the same procedure in the previous example to apply the novel DECMs proposed methods to solve this problem for $n = 2$ as follows:

First: Applying the DECMs proposed methods based on the Bernstein polynomials.

By inserting the Eqs. (15) and (16) into the Eqs. (47) and (48), we convert the function $u(x,t)$ and its partial derivatives into matrices. Therefore, the following results are obtained:

$$\boldsymbol{A}^T.\boldsymbol{\Psi}(x).(\boldsymbol{Bn}^*)^T.\boldsymbol{\Upsilon}(t)-\boldsymbol{A}^T.\boldsymbol{Bn}^{*2}.\boldsymbol{\Psi}(x).\boldsymbol{\Upsilon}(t)-\boldsymbol{A}^T.\boldsymbol{\Psi}(x).\boldsymbol{\Upsilon}(t)=(7-3t)tx, \quad (49)$$

$$\boldsymbol{A}^T.\boldsymbol{\Psi}(x).\boldsymbol{\Upsilon}(0)=-\boldsymbol{A}^T.\boldsymbol{\Psi}(x).\boldsymbol{\Upsilon}(1),$$
$$\boldsymbol{A}^T.\boldsymbol{\Psi}(0).\boldsymbol{\Upsilon}(t)=0,\ \ \boldsymbol{A}^T.\boldsymbol{\Psi}(1).\boldsymbol{\Upsilon}(t)=2\int_0^1\boldsymbol{A}^T.\boldsymbol{\Psi}(x).\boldsymbol{\Upsilon}(t)\,dx. \quad (50)$$

The technique has been applied as shown in Eq. (13), which results in:

$$\langle \boldsymbol{B}_{i,2}(x)\boldsymbol{B}_{j,2}(t),\boldsymbol{A}^T.\boldsymbol{\Psi}(x).(\boldsymbol{Bn}^*)^T.\boldsymbol{\Upsilon}(t)-\boldsymbol{A}^T.\boldsymbol{Bn}^{*2}.\boldsymbol{\Psi}(x).\boldsymbol{\Upsilon}(t)-\boldsymbol{A}^T.\boldsymbol{\Psi}(x).\boldsymbol{\Upsilon}(t)\rangle,\ \ \langle \boldsymbol{B}_{i,2}(x)\boldsymbol{B}_{j,2}(t),(7-3t)tx\rangle,\ \ 0\le i,j\le 2. \quad (51)$$

Moreover, we use the same procedure in the previous example for solving Eq. (51). After generating a linear algebraic system of equations, we use Mathematica®12 to determine the values of the coefficients $\boldsymbol{A}$ as follows,

$$\boldsymbol{A}=\begin{bmatrix}a_{00}&a_{01}&a_{02}\\a_{10}&a_{11}&a_{12}\\a_{20}&a_{21}&a_{22}\end{bmatrix}=\begin{bmatrix}0&0&0\\-\frac{1}{2}&-\frac{3}{4}&\frac{1}{2}\\-1&-\frac{3}{2}&1\end{bmatrix}.$$

Finally, by substituting the coefficients $\boldsymbol{A}$ in the Eq. (15), we get the following exact solution:

$$u(x,t)=x(3t^2-t-1).$$

Second: Using the DECMs proposed methods based on the Legendre polynomials.
By substituting Eqs. (18) and (19) into Eqs. (47) and (48), yielding the following:

$$\boldsymbol{A}^T.\boldsymbol{\Psi}(x).(\boldsymbol{L}^*)^T.\boldsymbol{\Upsilon}(t)-\boldsymbol{A}^T.\boldsymbol{L}^{*2}.\boldsymbol{\Psi}(x).\boldsymbol{\Upsilon}(t)-\boldsymbol{A}^T.\boldsymbol{\Psi}(x).\boldsymbol{\Upsilon}(t)=(7-3t)tx, \quad (52)$$

$$\boldsymbol{A}^T.\boldsymbol{\Psi}(x).\boldsymbol{\Upsilon}(0)=-\boldsymbol{A}^T.\boldsymbol{\Psi}(x).\boldsymbol{\Upsilon}(1),$$
$$\boldsymbol{A}^T.\boldsymbol{\Psi}(0).\boldsymbol{\Upsilon}(t)=0,\ \ \boldsymbol{A}^T.\boldsymbol{\Psi}(1).\boldsymbol{\Upsilon}(t)=2\int_0^1\boldsymbol{A}^T.\boldsymbol{\Psi}(x).\boldsymbol{\Upsilon}(t)\,dx. \quad (53)$$

Then, the technique has been applied as it is shown in Eq. (13) which results in the following:

$$\langle \boldsymbol{P}_{i}(x)\boldsymbol{P}_{j}(t),\boldsymbol{A}^T.\boldsymbol{\Psi}(x).(\boldsymbol{L}^*)^T.\boldsymbol{\Upsilon}(t)-\boldsymbol{A}^T.\boldsymbol{L}^{*2}.\boldsymbol{\Psi}(x).\boldsymbol{\Upsilon}(t)-\boldsymbol{A}^T.\boldsymbol{\Psi}(x).\boldsymbol{\Upsilon}(t)\rangle,\ \ \langle \boldsymbol{P}_{i}(x)\boldsymbol{P}_{j}(t),(7-3t)tx\rangle,\ \ 0\le i,j\le 2. \quad (54)$$

Moreover, we use Mathematica®12 to solve Eq. (54), and determine the values of the coefficients $\boldsymbol{A}$ as follows,

$$\boldsymbol{A}=\begin{bmatrix}a_{00}&a_{01}&a_{02}\\a_{10}&a_{11}&a_{12}\\a_{20}&a_{21}&a_{22}\end{bmatrix}=\begin{bmatrix}0&0&0\\0&-1&2\\0&0&0\end{bmatrix}.$$

Finally, by substituting the coefficients $\boldsymbol{A}$ in the Eq. (18), we get the exact solution mentioned above.

Third: Implementing the DECMs proposed methods based on the Chebyshev polynomials of the first kind. By inserting the Eqs. (21) and (22) into the Eqs. (47) and (48), we obtain:

$$\boldsymbol{A}^T.\boldsymbol{\Psi}(x).(\boldsymbol{V}^*)^T.\boldsymbol{\Upsilon}(t)-\boldsymbol{A}^T.\boldsymbol{V}^{*2}.\boldsymbol{\Psi}(x).\boldsymbol{\Upsilon}(t)-\boldsymbol{A}^T.\boldsymbol{\Psi}(x).\boldsymbol{\Upsilon}(t)=(7-3t)tx, \quad (55)$$

$$\boldsymbol{A}^T.\boldsymbol{\Psi}(x).\boldsymbol{\Upsilon}(0)=-\boldsymbol{A}^T.\boldsymbol{\Psi}(x).\boldsymbol{\Upsilon}(1),$$
$$\boldsymbol{A}^T.\boldsymbol{\Psi}(0).\boldsymbol{\Upsilon}(t)=0,\ \ \boldsymbol{A}^T.\boldsymbol{\Psi}(1).\boldsymbol{\Upsilon}(t)=2\int_0^1\boldsymbol{A}^T.\boldsymbol{\Psi}(x).\boldsymbol{\Upsilon}(t)\,dx. \quad (56)$$

Then, the technique has been applied as shown in Eq. (13) which results in the:

$$\langle \boldsymbol{T}_{i}(x)\boldsymbol{T}_{j}(t),\boldsymbol{A}^T.\boldsymbol{\Psi}(x).(\boldsymbol{V}^*)^T.\boldsymbol{\Upsilon}(t)-\boldsymbol{A}^T.\boldsymbol{V}^{*2}.\boldsymbol{\Psi}(x).\boldsymbol{\Upsilon}(t)-\boldsymbol{A}^T.\boldsymbol{\Psi}(x).\boldsymbol{\Upsilon}(t)\rangle,\ \ \langle \boldsymbol{T}_{i}(x)\boldsymbol{T}_{j}(t),(7-3t)tx\rangle,\ \ 0\le i,j\le 2. \quad (57)$$

Additionally, we use Mathematica®12 to solve Eq. (57), and determine the values of the coefficients $\boldsymbol{A}$ as follows:

$$\boldsymbol{A}=\begin{bmatrix} a_{00} & a_{01} & a_{02} \\ a_{10} & a_{11} & a_{12} \\ a_{20} & a_{21} & a_{22} \end{bmatrix}=\begin{bmatrix} 0 & 0 & 0 \\ \frac{1}{2} & -1 & \frac{3}{2} \\ 0 & 0 & 0 \end{bmatrix}.$$

Therefore, by substituting the coefficients $\boldsymbol{A}$ in the Eq. (21), we get the exact solution mentioned above.

Fourth: Utilizing the DECMs proposed methods based on the Hermite polynomials.

The following findings are obtained by inserting the Eqs. (24) and (25) into the Eqs. (47) and (48):

$$\boldsymbol{\Psi}(x)\,.\boldsymbol{A}.\boldsymbol{\Lambda}(t).(\boldsymbol{N})^T.(\boldsymbol{E}^{-1})^T-\boldsymbol{\Lambda}(x).\boldsymbol{N}^2\,.(\boldsymbol{E}^{-1})^T.\boldsymbol{A}\,.\boldsymbol{\Upsilon}(t)-\boldsymbol{\Psi}(x)\,.\boldsymbol{A}\,.\boldsymbol{\Upsilon}(t) \\ =(7-3t)tx, (58)$$

$$\boldsymbol{\Psi}(x)\,.\boldsymbol{A}\,.\boldsymbol{\Upsilon}(0)=-\boldsymbol{\Psi}(x)\,.\boldsymbol{A}\,.\boldsymbol{\Upsilon}(1),$$
$$\boldsymbol{\Psi}(0)\,.\boldsymbol{A}\,.\boldsymbol{\Upsilon}(t)=0, \qquad \boldsymbol{\Psi}(1)\,.\boldsymbol{A}\,.\boldsymbol{\Upsilon}(t)=2\int_0^1 \boldsymbol{\Psi}(x)\,.\boldsymbol{A}\,.\boldsymbol{\Upsilon}(t)\,dx. \tag{59}$$

Then, the technique has been applied as shown in Eq. (13) which results in:

$$\langle \boldsymbol{H}_i(x)\boldsymbol{H}_j(t), \boldsymbol{\Psi}(x)\,.\boldsymbol{A}.\boldsymbol{\Lambda}(t).(\boldsymbol{N})^T.(\boldsymbol{E}^{-1})^T-\boldsymbol{\Lambda}(x).\boldsymbol{N}^2\,.(\boldsymbol{E}^{-1})^T.\boldsymbol{A}\,.\boldsymbol{\Upsilon}(t) \\ -\boldsymbol{\Psi}(x)\,.\boldsymbol{A}\,.\boldsymbol{\Upsilon}(t)\,\rangle,\ \langle \boldsymbol{H}_i(x)\boldsymbol{H}_j(t),(7-3t)tx\rangle,\ 0\le i,j\le 2. \tag{60}$$

Furthermore, we use Mathematica®12 to solve Eq. (60), and determine the values of the coefficients $\boldsymbol{A}$ as follows:

$$\boldsymbol{A}=\begin{bmatrix} a_{00} & a_{01} & a_{02} \\ a_{10} & a_{11} & a_{12} \\ a_{20} & a_{21} & a_{22} \end{bmatrix}=\begin{bmatrix} 0 & 0 & 0 \\ \frac{1}{4} & -\frac{1}{4} & \frac{3}{8} \\ 0 & 0 & 0 \end{bmatrix}.$$

Consequently, by substituting the coefficients $\boldsymbol{A}$ in the Eq. (24), we get the exact solution mentioned above.

Fifth: Applying the DECMs proposed methods based on the Bernoulli polynomials.

By substituting the Eqs. (27) and (28) into the Eqs. (47) and (48) produce the following results:

$$\boldsymbol{A}^T.\boldsymbol{\Psi}(x)\,.(\boldsymbol{Bi}^*)^T.\boldsymbol{\Upsilon}(t)-\boldsymbol{A}^T.\boldsymbol{Bi}^{*2}.\boldsymbol{\Psi}(x)\,.\boldsymbol{\Upsilon}(t)-\boldsymbol{A}^T\,.\boldsymbol{\Psi}(x)\,.\boldsymbol{\Upsilon}(t)=(7-3t)tx, \tag{61}$$

$$\boldsymbol{A}^T\,.\boldsymbol{\Psi}(x)\,.\boldsymbol{\Upsilon}(0)=-\boldsymbol{A}^T\,.\boldsymbol{\Psi}(x)\,.\boldsymbol{\Upsilon}(1),$$
$$\boldsymbol{A}^T\,.\boldsymbol{\Psi}(0)\,.\boldsymbol{\Upsilon}(t)=0,\ \ \boldsymbol{A}^T\,.\boldsymbol{\Psi}(1)\,.\boldsymbol{\Upsilon}(t)=2\int_0^1 \boldsymbol{A}^T\,.\boldsymbol{\Psi}(x)\,.\boldsymbol{\Upsilon}(t)\,dx. \tag{62}$$

Then, the technique has been applied as shown in Eq. (13) which results in the:

$$\langle \boldsymbol{B}_i(x)\boldsymbol{B}_j(t), \boldsymbol{A}^T.\boldsymbol{\Psi}(x)\,.(\boldsymbol{Bi}^*)^T.\boldsymbol{\Upsilon}(t)-\boldsymbol{A}^T.\boldsymbol{Bi}^{*2}.\boldsymbol{\Psi}(x)\,.\boldsymbol{\Upsilon}(t) \\ -\boldsymbol{A}^T\,.\boldsymbol{\Psi}(x)\,.\boldsymbol{\Upsilon}(t)\,\rangle,\ \langle \boldsymbol{B}_i(x)\boldsymbol{B}_j(t),(7-3t)tx\rangle,\ 0\le i,j\le 2. \tag{63}$$

Additionally, we use Mathematica®12 to solve Eq. (63), and determine the values of the coefficients $\boldsymbol{A}$ as follows,

$$\boldsymbol{A}=\begin{bmatrix} a_{00} & a_{01} & a_{02} \\ a_{10} & a_{11} & a_{12} \\ a_{20} & a_{21} & a_{22} \end{bmatrix}=\begin{bmatrix} 0 & 0 & 0 \\ 0 & -4 & 24 \\ 0 & 0 & 0 \end{bmatrix}.$$

Thus, by substituting the coefficients $\boldsymbol{A}$ in the Eq. (27), we get the exact solution mentioned above.

Example 3. Consider the time-dependent diffusion equation of the following form [14, 20]:

$$u_t(x,t)-u_{xx}(x,t)=-(2+x^2)e^{-t}. \tag{64}$$

Subjected to the nonlocal boundary conditions:

$$u(x,0)=x^2, \qquad 0\le x\le 1,$$
$$u(0,t)=0, \qquad u(1,t)=3\int_0^1 u(x,t)dx,\ \ 0\le t\le 1. \tag{65}$$

This problem's exact solution is $u(x,t) = x^2 e^{-t}$.
Turkyilmazoglu in [14] solved this problem for $n = 5$ using the efficient analytic approximate method, namely ECM based on the standard polynomials, and produced the following approximate solution:

$$\begin{aligned} u(x,t) \approx\ & t(-0.0000106x - 0.999977x^2 + 0.0000057x^3 - 0.0000205x^4 + 0.0000029x^5) \\ & +t^2(0.0001035x + 0.499678x^2 - 0.0000366x^3 + 0.0001757x^4 \\ & -0.0000187x^5) + t^3(-0.0003798x - 0.165163x^2 - 0.0000083x^3 \\ & -0.0004624x^4 - 0.0000057x^5) + t^4(0.0005608x + 0.0385037x^2 \\ & +0.0001756x^3 + 0.0004746x^4 + 0.0000933x^5) + t^5(-0.0003255x \\ & -0.0051564x^2 - 0.0002198x^3 - 0.0001256x^4 - 0.0001151x^5). \end{aligned}$$

Furthermore, now we can use the same procedure in the previous examples to apply the DECMs proposed methods to solve this problem as follows:
First: Applying the DECMs proposed methods based on the Bernstein polynomials. By substituting the Eqs. (15) and (16) into the Eqs. (64) and (65), we convert the function $u(x,t)$ and its partial derivatives into matrices. Therefore, the following results are obtained:

$$\boldsymbol{A}^T.\boldsymbol{\Psi}(x).(\boldsymbol{Bn}^*)^T.\boldsymbol{\Upsilon}(t) - \boldsymbol{A}^T.\boldsymbol{Bn}^{*2}.\boldsymbol{\Psi}(x).\boldsymbol{\Upsilon}(t) = -(2+x^2)e^{-t}, \tag{66}$$

$$\boldsymbol{A}^T.\boldsymbol{\Psi}(x).\boldsymbol{\Upsilon}(0) = x^2,$$
$$\boldsymbol{A}^T.\boldsymbol{\Psi}(0).\boldsymbol{\Upsilon}(t) = 0,\ \ \boldsymbol{A}^T.\boldsymbol{\Psi}(1).\boldsymbol{\Upsilon}(t) = 3\int_0^1 \boldsymbol{A}^T.\boldsymbol{\Psi}(x).\boldsymbol{\Upsilon}(t)\,dx. \tag{67}$$

The technique has been applied as shown in Eq. (13) which results in:

$$\begin{aligned} &\langle \boldsymbol{B}_{i,n}(x)\boldsymbol{B}_{j,n}(t), \boldsymbol{A}^T.\boldsymbol{\Psi}(x).(\boldsymbol{Bn}^*)^T.\boldsymbol{\Upsilon}(t) - \boldsymbol{A}^T.\boldsymbol{Bn}^{*2}.\boldsymbol{\Psi}(x).\boldsymbol{\Upsilon}(t)\rangle, \\ &\langle \boldsymbol{B}_{i,n}(x)\boldsymbol{B}_{j,n}(t), -(2+x^2)e^{-t}\rangle,\ \ 0 \le i,j \le n. \end{aligned} \tag{68}$$

Moreover, we use the same procedure in the previous examples for solving Eq. (68). After generating a linear algebraic system of equations for $n = 5$ as solved in [14], we use Mathematica®12 to determine the values of the coefficients $\boldsymbol{A}$ as follows:

$$\boldsymbol{A} = \begin{bmatrix} 1.5931\times10^{-7} & -3.9827\times10^{-7} & 5.3103\times10^{-7} & -3.9827\times10^{-7} & 1.5931\times10^{-7} & -2.6551\times10^{-8} \\ -9.5585\times10^{-7} & 2.9579\times10^{-6} & -4.2052\times10^{-6} & 3.3281\times10^{-6} & -1.3795\times10^{-6} & 2.1206\times10^{-7} \\ 0.10000 & 0.079996 & 0.065004 & 0.053336 & 0.044148 & 0.036788 \\ 0.30000 & 0.24001 & 0.19498 & 0.16003 & 0.13243 & 0.11036 \\ 0.60000 & 0.48000 & 0.38999 & 0.32004 & 0.26488 & 0.22073 \\ 1.0000 & 0.80001 & 0.64997 & 0.53340 & 0.44146 & 0.36788 \end{bmatrix}.$$

Finally, by substituting the coefficients $\boldsymbol{A}$ in the Eq. (15), we get the following approximate solution for $n = 5$:

$$\begin{aligned} u(x,t) \approx\ & 1.59309\times10^{-7} - 5.57581\times10^{-6}x + 1.00004\,x^2 - 0.00013382\,x^3 \\ & + 0.000167274\,x^4 - 0.0000736007\,x^5 \\ & + t^4(0.0000334549 - 0.00148504\,x + 0.047857\,x^2 - 0.0283518\,x^3 \\ & + 0.0354723\,x^4 - 0.0152071\,x^5) \\ & + t^2(0.0000148688 - 0.000628183\,x + 0.503839\,x^2 - 0.0125701\,x^3 \\ & + 0.0157746\,x^4 - 0.00684482\,x^5) \\ & + t\,(-2.78791\times10^{-6} + 0.000111782\,x - 1.00076\,x^2 + 0.00235022\,x^3 \\ & - 0.00294969\,x^4 + 0.00128853\,x^5) \\ & + t^5(-0.0000122668 + 0.000552275\,x - 0.00859203\,x^2 + 0.0103951\,x^3 \\ & - 0.0129788\,x^4 + 0.00553251\,x^5) \\ & + t^3(-0.0000334549 + 0.00145593\,x - 0.17452\,x^2 + 0.0283307\,x^3 \\ & - 0.0355084\,x^4 + 0.0153124\,x^5). \end{aligned}$$

Second: Using the DECMs proposed methods based on the Legendre polynomials.
By substituting Eqs. (18) and (19) into Eqs. (64) and (65), this yields the following:

$$\boldsymbol{A}^T.\boldsymbol{\Psi}(x).(\boldsymbol{L}^*)^T.\boldsymbol{\Upsilon}(t) - \boldsymbol{A}^T.\boldsymbol{L}^{*2}.\boldsymbol{\Psi}(x).\boldsymbol{\Upsilon}(t) = -(2+x^2)e^{-t}, \tag{69}$$

$$\boldsymbol{A}^T.\boldsymbol{\Psi}(x).\boldsymbol{\Upsilon}(0)=x^2,$$
$$\boldsymbol{A}^T.\boldsymbol{\Psi}(0).\boldsymbol{\Upsilon}(t)=0,\ \ \boldsymbol{A}^T.\boldsymbol{\Psi}(1).\boldsymbol{\Upsilon}(t)=3\int_0^1 \boldsymbol{A}^T.\boldsymbol{\Psi}(x).\boldsymbol{\Upsilon}(t)\,dx. \tag{70}$$

Also, the technique has been applied as shown in Eq. (13) which results in:

$$\langle \boldsymbol{P}_i(x)\boldsymbol{P}_j(t), \boldsymbol{A}^T.\boldsymbol{\Psi}(x).(\boldsymbol{L}^*)^T.\boldsymbol{\Upsilon}(t)-\boldsymbol{A}^T.\boldsymbol{L}^{*2}.\boldsymbol{\Psi}(x).\boldsymbol{\Upsilon}(t)\rangle,$$
$$\langle \boldsymbol{P}_i(x)\boldsymbol{P}_j(t), -(2+x^2)e^{-t}\rangle,\ \ 0\le i,j\le n. \tag{71}$$

Moreover, we use Mathematica®12 to solve Eq. (71), and determine the values of the coefficients $\boldsymbol{A}$ as follows:

$$\boldsymbol{A}=\begin{bmatrix} 0.42294 & -0.44279 & 0.19791 & -0.075885 & 0.025443 & -0.0051188 \\ -0.075795 & 0.18204 & -0.19128 & 0.12770 & -0.054133 & 0.011784 \\ 0.86282 & -0.92628 & 0.43859 & -0.18032 & 0.062988 & -0.012872 \\ -0.053682 & 0.12893 & -0.13547 & 0.090432 & -0.038334 & 0.0083433 \\ 0.023020 & -0.055288 & 0.058095 & -0.038781 & 0.016440 & -0.0035785 \\ -0.0051073 & 0.012266 & -0.012888 & 0.0086022 & -0.0036459 & 0.00079316 \end{bmatrix}.$$

Finally, by substituting the coefficients $\boldsymbol{A}$ in the Eq. (18), we get the following approximate solution for $n=5$:

$$\begin{aligned} u(x,t)\approx\ & 9.55854\times10^{-7}-0.0000286756\,x+1.0002\,x^2-0.000535278\,x^3 \\ & +0.000602188\,x^4-0.000240875\,x^5 \\ & +t^4(0.000499265\ -0.0151775\,x+0.143642\,x^2-0.279703\,x^3 \\ & +0.314671\,x^4-0.125613\,x^5) \\ & +t^2(0.000176955\ -0.0053522\,x+0.536921\,x^2-0.0991218\,x^3 \\ & +0.11153\,x^4-0.0445689\,x^5) \\ & +t\,(-0.0000265583+0.000800449\,x-1.00556\,x^2+0.0148749\,x^3 \\ & -0.0167364\,x^4+0.00669141\,x^5) \\ & +t^5(-0.000195643+0.00595839\,x-0.0463702\,x^2+0.109605\,x^3 \\ & -0.123289\,x^4+0.0491881\,x^5) \\ & +t^3(-0.000455628+0.0138197\,x-0.261089\,x^2+0.255245\,x^3 \\ & -0.287185\,x^4+0.114703\,x^5). \end{aligned}$$

Third: Implementing the DECMs proposed methods based on the Chebyshev polynomials of the first kind.

By inserting the Eqs. (21) and (22) into the Eqs. (64) and (65), we obtain:

$$\boldsymbol{A}^T.\boldsymbol{\Psi}(x).(\boldsymbol{V}^*)^T.\boldsymbol{\Upsilon}(t)-\boldsymbol{A}^T.\boldsymbol{V}^{*2}.\boldsymbol{\Psi}(x).\boldsymbol{\Upsilon}(t)=-(2+x^2)e^{-t}, \tag{72}$$
$$\boldsymbol{A}^T.\boldsymbol{\Psi}(x).\boldsymbol{\Upsilon}(0)=x^2,$$
$$\boldsymbol{A}^T.\boldsymbol{\Psi}(0).\boldsymbol{\Upsilon}(t)=0,\ \ \boldsymbol{A}^T.\boldsymbol{\Psi}(1).\boldsymbol{\Upsilon}(t)=3\int_0^1 \boldsymbol{A}^T.\boldsymbol{\Psi}(x).\boldsymbol{\Upsilon}(t)\,dx. \tag{73}$$

Also, the technique has been applied as shown in Eq. (13) which results in:

$$\langle \boldsymbol{T}_i(x)\boldsymbol{T}_j(t), \boldsymbol{A}^T.\boldsymbol{\Psi}(x).(\boldsymbol{V}^*)^T.\boldsymbol{\Upsilon}(t)-\boldsymbol{A}^T.\boldsymbol{V}^{*2}.\boldsymbol{\Psi}(x).\boldsymbol{\Upsilon}(t)\rangle,$$
$$\langle \boldsymbol{T}_i(x)\boldsymbol{T}_j(t), -(2+x^2)e^{-t}\rangle,\ \ 0\le i,j\le n. \tag{74}$$

Additionally, we use Mathematica®12 to solve Eq. (74), and determine the values of the coefficients $\boldsymbol{A}$ as follows:

$$\boldsymbol{A}=\begin{bmatrix} 0.72838 & -0.73420 & 0.25220 & -0.082395 & 0.024152 & -0.0044284 \\ -0.17072 & 0.30274 & -0.20871 & 0.10811 & -0.038574 & 0.0075689 \\ 0.75023 & -0.77294 & 0.27891 & -0.096230 & 0.029089 & -0.0053969 \\ -0.061430 & 0.10893 & -0.075100 & 0.038898 & -0.013878 & 0.0027228 \\ 0.022129 & -0.039241 & 0.027054 & -0.014013 & 0.0049999 & -0.00098100 \\ -0.0044185 & 0.0078351 & -0.0054014 & 0.0027974 & -0.00099799 & 0.00019570 \end{bmatrix}.$$

Finally, by substituting the coefficients $\boldsymbol{A}$ in the Eq. (21), we get the following approximate solution for $n = 5$:

$$\begin{aligned} u(x,t) \approx\ & 9.55854 \times 10^{-7} - 0.0000286756\,x + 1.0002\,x^2 - 0.000535278\,x^3 \\ & + 0.000602188\,x^4 - 0.000240875\,x^5 \\ & + t^4(0.000507711 - 0.0154298\,x + 0.145424\,x^2 - 0.284433\,x^3 \\ & + 0.319993\,x^4 - 0.127743\,x^5) \\ & + t^2(0.000179073 - 0.00541516\,x + 0.537368\,x^2 - 0.100308\,x^3 \\ & + 0.112864\,x^4 - 0.0451029\,x^5) \\ & + t\,(-0.0000267617 + 0.000806472\,x - 1.00561\,x^2 + 0.0149887\,x^3 \\ & - 0.0168645\,x^4 + 0.00674269\,x^5) \\ & + t^5(-0.00019926 + 0.00606666\,x - 0.0471338\,x^2 + 0.111631\,x^3 \\ & - 0.125569\,x^4 + 0.0501002\,x^5) \\ & + t^3(-0.000462392 + 0.0140213\,x - 0.262516\,x^2 + 0.259032\,x^3 \\ & - 0.291445\,x^4 + 0.116408\,x^5). \end{aligned}$$

Fourth: Utilizing the DECMs proposed methods based on the Hermite polynomials.
The following findings are obtained by inserting the Eqs. (24) and (25) into the Eqs. (64) and (65):

$$\boldsymbol{\Psi}(x)\,.\boldsymbol{A}.\boldsymbol{\Lambda}(t).(\boldsymbol{N})^T.(\boldsymbol{E}^{-1})^T - \boldsymbol{\Lambda}(x).\boldsymbol{N}^2\,.(\boldsymbol{E}^{-1})^T.\boldsymbol{A}\,.\boldsymbol{\Upsilon}(t) \quad = -(2 + x^2)e^{-t}, \tag{75}$$

$$\boldsymbol{\Psi}(x)\,.\boldsymbol{A}\,.\boldsymbol{\Upsilon}(0) = x^2,$$

$$\boldsymbol{\Psi}(0)\,.\boldsymbol{A}\,.\boldsymbol{\Upsilon}(t) = 0, \qquad \boldsymbol{\Psi}(1)\,.\boldsymbol{A}\,.\boldsymbol{\Upsilon}(t) = 3\int_0^1 \boldsymbol{\Psi}(x)\,.\boldsymbol{A}\,.\boldsymbol{\Upsilon}(t)\,dx. \tag{76}$$

Moreover, the technique has been applied as it is shown in Eq. (13) which results in:

$$\begin{aligned} &\langle \boldsymbol{H}_i(x)\boldsymbol{H}_j(t), \boldsymbol{\Psi}(x)\,.\boldsymbol{A}.\boldsymbol{\Lambda}(t).(\boldsymbol{N})^T.(\boldsymbol{E}^{-1})^T - \boldsymbol{\Lambda}(x).\boldsymbol{N}^2\,.(\boldsymbol{E}^{-1})^T.\boldsymbol{A}\,.\boldsymbol{\Upsilon}(t)\rangle, \\ &\qquad \langle \boldsymbol{H}_i(x)\boldsymbol{H}_j(t), -(2 + x^2)e^{-t}\rangle, \;\; 0 \le i,j \le n. \end{aligned} \tag{77}$$

Furthermore, we use Mathematica®12 to solve Eq. (77), and determine the values of the coefficients $\boldsymbol{A}$ as follows:

$$\boldsymbol{A} = \begin{bmatrix} 0.92188 & -0.75728 & 0.33227 & -0.12159 & 0.020269 & -0.0038354 \\ -0.44268 & 0.69018 & -0.40013 & 0.17179 & -0.029871 & 0.0058813 \\ 0.57602 & -0.55808 & 0.27016 & -0.10546 & 0.017900 & -0.0034470 \\ -0.11092 & 0.17293 & -0.10026 & 0.043040 & -0.0074844 & 0.0014734 \\ 0.019221 & -0.029969 & 0.017374 & -0.0074600 & 0.0012971 & -0.00025543 \\ -0.0038377 & 0.0059825 & -0.0034686 & 0.0014889 & -0.00025893 & 0.000050962 \end{bmatrix}.$$

Hence, by substituting the coefficients $\boldsymbol{A}$ in the Eq. (24), we get the following approximate solution for $n = 5$:

$$\begin{aligned} u(x,t) \approx\ & 9.55854 \times 10^{-7} - 0.0000286756\,x + 1.0002\,x^2 - 0.000535278\,x^3 \\ & + 0.000602188\,x^4 - 0.000240875\,x^5 \\ & + t^4(0.000526853 - 0.0160013\,x + 0.149464\,x^2 - 0.295154\,x^3 \\ & + 0.332053\,x^4 - 0.13257\,x^5) \\ & + t^2(0.000183716 - 0.00555308\,x + 0.538348\,x^2 - 0.102907\,x^3 \\ & + 0.115788\,x^4 - 0.0462736\,x^5) \\ & + t\,(-0.0000271906 + 0.000819164\,x - 1.0057\,x^2 + 0.0152288\,x^3 \\ & - 0.0171344\,x^4 + 0.0068508\,x^5) \\ & + t^5(-0.000207529 + 0.006314\,x - 0.0488791\,x^2 + 0.116263\,x^3 \\ & - 0.130779\,x^4 + 0.0521855\,x^5) \\ & + t^3(-0.000477527 + 0.0144721\,x - 0.26571\,x^2 + 0.267506\,x^3 \\ & - 0.300977\,x^4 + 0.120224\,x^5). \end{aligned}$$

Fifth: Applying the DECMs proposed methods based on the Bernoulli polynomials.
By substituting the Eqs. (27) and (28) into the Eqs. (64) and (65) produce the following results:

$$\boldsymbol{A}^T.\boldsymbol{\Psi}(x)\,.(\boldsymbol{Bi}^*)^T.\boldsymbol{\Upsilon}(t) - \boldsymbol{A}^T.\boldsymbol{Bi}^{*2}.\boldsymbol{\Psi}(x)\,.\boldsymbol{\Upsilon}(t) \; = -(2 + x^2)e^{-t}, \tag{78}$$

$$\boldsymbol{A}^T.\boldsymbol{\Psi}(x).\boldsymbol{Y}(0) = x^2,$$
$$\boldsymbol{A}^T.\boldsymbol{\Psi}(0).\boldsymbol{Y}(t) = 0,\ \ \boldsymbol{A}^T.\boldsymbol{\Psi}(1).\boldsymbol{Y}(t) = 3\int_0^1 \boldsymbol{A}^T.\boldsymbol{\Psi}(x).\boldsymbol{Y}(t)\,dx. \tag{79}$$

Then, the technique has been applied as shown in Eq. (13) which results in the:

$$\langle \boldsymbol{B}_i(x)\boldsymbol{B}_j(t), \boldsymbol{A}^T.\boldsymbol{\Psi}(x).(\boldsymbol{Bi}^*)^T.\boldsymbol{Y}(t) - \boldsymbol{A}^T.\boldsymbol{Bi}^{*2}.\boldsymbol{\Psi}(x).\boldsymbol{Y}(t)\ \rangle,$$
$$\langle \boldsymbol{B}_i(x)\boldsymbol{B}_j(t), -(2+x^2)e^{-t}\rangle,\ \ 0 \le i,j \le n. \tag{80}$$

Moreover, we use Mathematica®12 to solve Eq. (80), and determine the values of the coefficients $\boldsymbol{A}$ as follows:

$$\boldsymbol{A} = \begin{bmatrix} 0.42294 & -1.0476 & 1.6963 & -2.2343 & 1.7810 & -1.2899 \\ -0.26917 & 2.2835 & -7.9202 & 14.927 & -13.456 & 10.544 \\ 5.6373 & -17.340 & 36.895 & -57.982 & 49.471 & -37.497 \\ -1.7887 & 15.173 & -52.624 & 99.166 & -89.397 & 70.033 \\ 1.6114 & -13.671 & 47.416 & -89.363 & 80.556 & -63.124 \\ -1.2870 & 10.917 & -37.862 & 71.338 & -64.314 & 50.369 \end{bmatrix}.$$

Consequently, by substituting the coefficients $\boldsymbol{A}$ in the Eq. (27), we get the following approximate solution for $n = 5$:

$$\begin{aligned} u(x,t) \approx\ & 9.55854 \times 10^{-7} - 0.0000286756\,x + 1.0002\,x^2 - 0.000535278\,x^3 \\ & + 0.000602188\,x^4 - 0.000240875\,x^5 \\ & + t^4(0.000499265\ - 0.0151775\,x + 0.143642\,x^2 - 0.279703\,x^3 \\ & + 0.314671\,x^4 - 0.125613\,x^5) \\ & + t^2(0.000176955\ - 0.0053522\,x + 0.536921\,x^2 - 0.0991218\,x^3 \\ & + 0.11153\,x^4 - 0.0445689\,x^5) \\ & + t\,(-0.0000265583 + 0.000800449\,x - 1.00556\,x^2 + 0.0148749\,x^3 \\ & - 0.0167364\,x^4 + 0.00669141\,x^5) \\ & + t^5(-0.000195643 + 0.00595839\,x - 0.0463702\,x^2 + 0.109605\,x^3 \\ & - 0.123289\,x^4 + 0.0491881\,x^5) \\ & + t^3(-0.000455628 + 0.0138197\,x - 0.261089\,x^2 + 0.255245\,x^3 \\ & - 0.287185\,x^4 + 0.114703\,x^5). \end{aligned}$$

In addition, the $MAbsR_n$ referred to in Eq. (46) is computed to demonstrate the accuracy and reliability of the DECMs proposed methods to solve Example 3.

Table (2) lists $MAbsR_n$ for the approximate solution obtained by applying the ECM method described in [14] and by the DECMs proposed methods with approximation order $n = 7$, as explained in [14]. Moreover, we can demonstrate that the accuracy of the DECMs proposed methods is better than that of the ECM method [14]. In addition, the DECMs based on the Bernstein polynomial have slightly greater accuracy and less error than the other proposed methods.

Table 2: The comparison between the $MAbsR_7$ for the example (3) by the ECM method [14], and by the DECMs proposed methods.

ECM Standard [14]	**DECMs Bernstein**	**DECMs Legendre**	**DECMs Chebyshev**	**DECMs Hermite**	**DECMs Bernoulli**
3.55525×10^{-8}	1.14285×10^{-9}	1.21206×10^{-9}	1.21206×10^{-9}	1.21206×10^{-9}	1.21206×10^{-9}

Furthermore, Figures (1) and (2) illustrate the absolute error values for $0 \le x,t \le 1$ that is obtained using the proposed approaches with approximation order $n = 4$ and 7, respectively, as given in [14]. In reality, the accuracy improves as the approximation order increases.

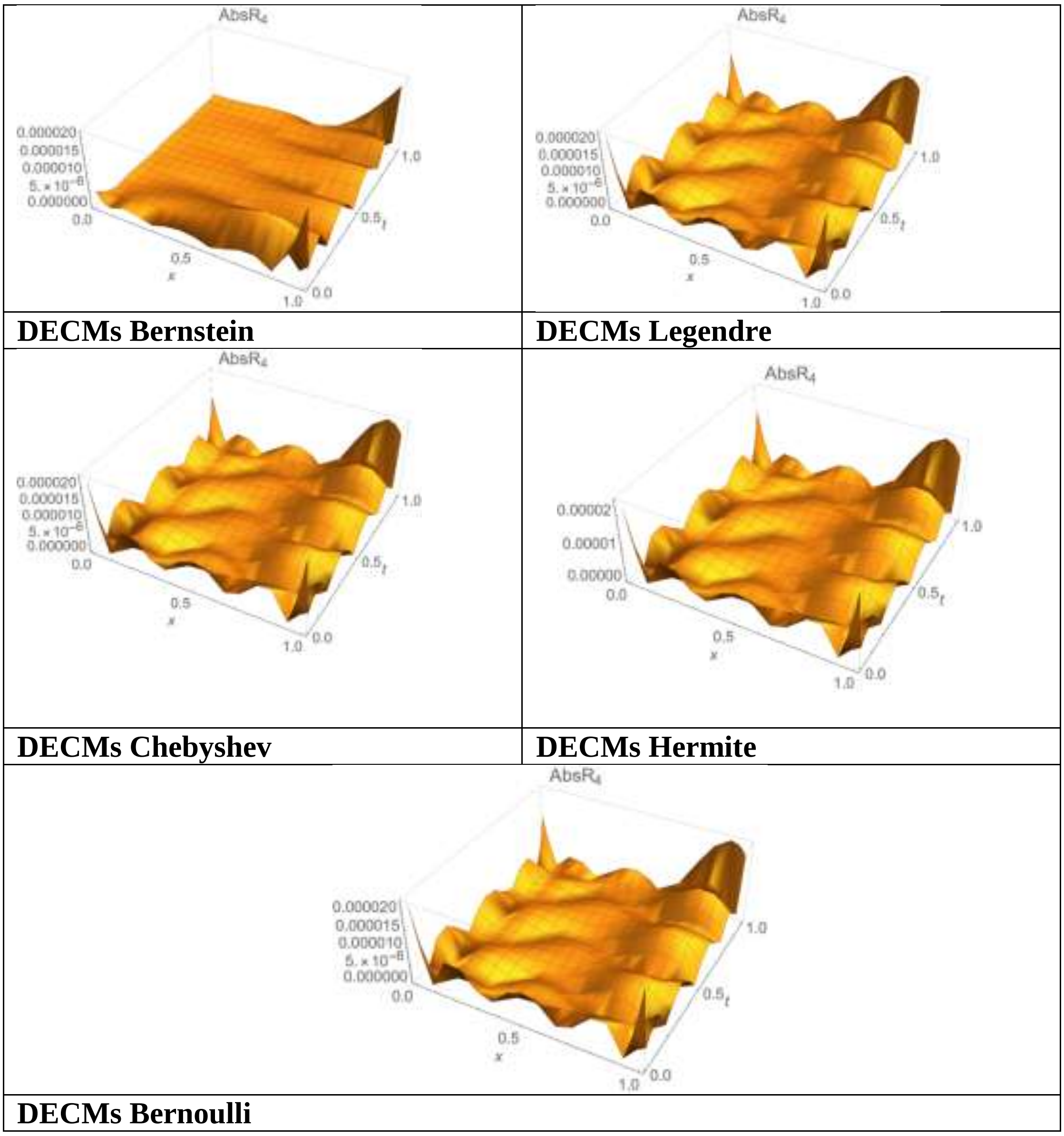


Figure 1: The absolute error values achieved by the proposed methods of the example (3) with $n = 4$.

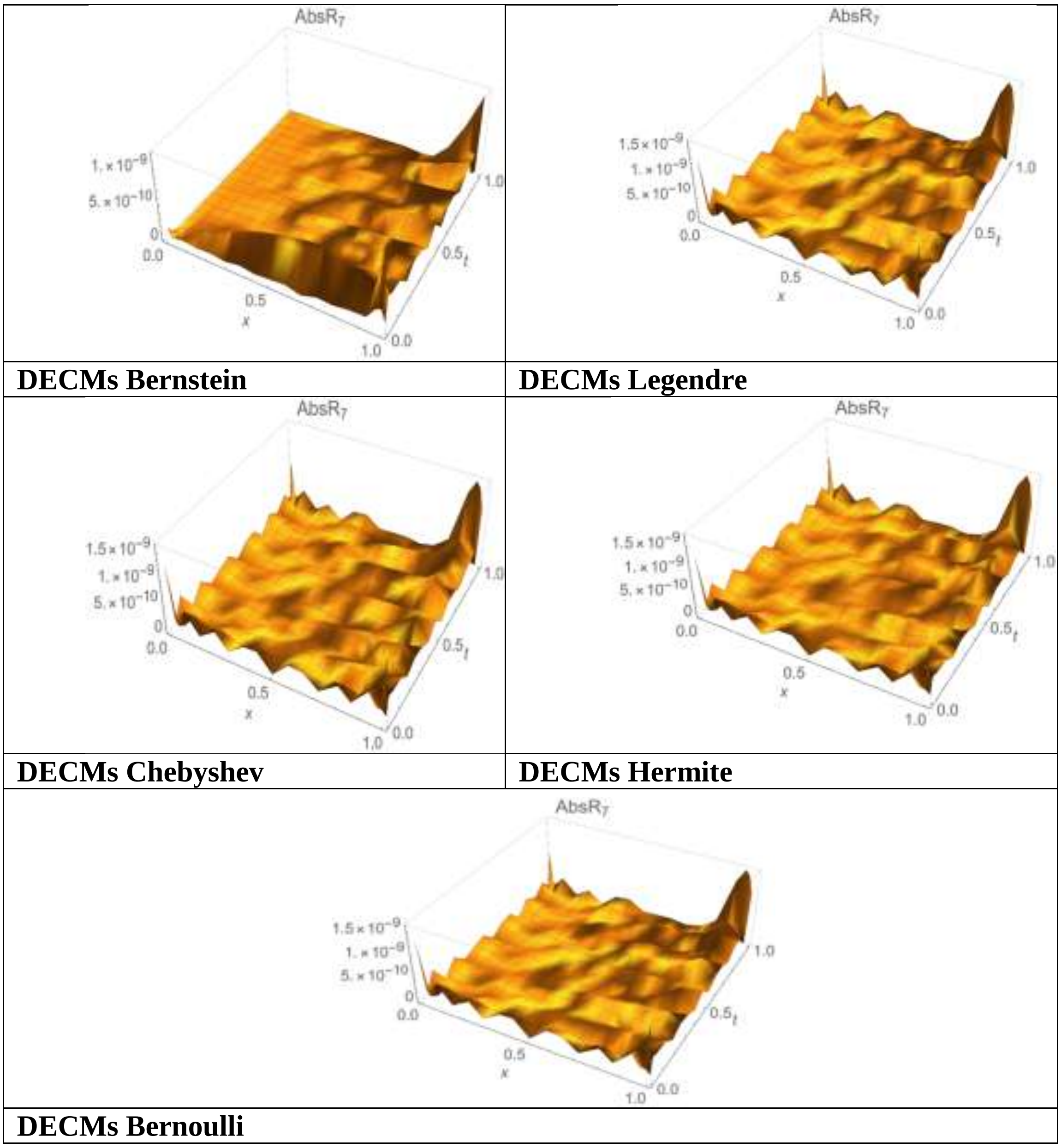


Figure 2: The absolute error values achieved by the proposed methods of the example (3) with $n = 7$

Example 4. Consider the time-dependent diffusion equation of the following form [14]:

$$(1 + e^{2x})\, u_t(x,t) - u_{xx}(x,t) + u(x,t) = e^{-t}x\big(-6 + 20x^2 + e^{2x}x^2(-1 + x^2) + 2e^t(6 - 21x^2 + x^4)\big). \quad (81)$$

Subjected to the nonlocal boundary conditions:

$$u(x,0) = \alpha u(x,1), \qquad 0 \le x \le 1,$$
$$u(0,t) = u(1,t) = 0, \qquad 0 \le t \le 1. \quad (82)$$

Where the value of $\alpha = \frac{e}{-1+2e}$, and this problem's exact solution is $u(x,t) = x^3(x^2 - 1)(2 - e^t)$.

Turkyilmazoglu in [14] solved this problem for $n = 5$ using the efficient analytic approximate method, namely ECM based on the standard polynomials, and produced the following approximate solution:

$$
\begin{aligned}
u(x,t) \approx\ & 0.00000023x - 0.0000018x^2 - 0.999994x^3 - 0.0000065x^4 + x^5 \\
& + t(0.0000085x - 0.0000378x^2 - 0.999827x^3 - 0.0001808x^4 + 1.00004x^5) \\
& + t^2(-0.0000662x + 0.0003904x^2 + 0.498235x^3 + 0.0016908x^4 \\
& - 0.50025x^5) + t^3(0.0001787x - 0.0011697x^2 - 0.160948x^3 - 0.0048825x^4 \\
& + 0.166822x^5) + t^4(-0.0002019x + 0.0013966x^2 + 0.0335542x^3 \\
& + 0.0057011x^4 - 0.0404499x^5) + t^5(0.0000811x - 0.0005806x^2 \\
& - 0.0031315x^3 - 0.0023327x^4 + 0.0059638x^5).
\end{aligned}
$$

Furthermore, now we can use the same procedure in the previous examples to apply the DECMs proposed methods to solve this problem as follows:

First: Applying the DECMs proposed methods based on the Bernstein polynomials.

By substituting the Eqs. (15) and (16) into the Eqs. (81) and (82), we convert the function $u(x,t)$ and its partial derivatives into matrices. Therefore, the following results are obtained:

$$
\begin{aligned}
(1+e^{2x})\boldsymbol{A}^T.\boldsymbol{\Psi}(x).(\boldsymbol{Bn}^*)^T.\boldsymbol{\Upsilon}(t) - \boldsymbol{A}^T.\boldsymbol{Bn}^{*2}.\boldsymbol{\Psi}(x).\boldsymbol{\Upsilon}(t) + \boldsymbol{A}^T.\boldsymbol{\Psi}(x).\boldsymbol{\Upsilon}(t) \\
= e^{-t}x(-6+20x^2+e^{2x}x^2(-1+x^2)+2e^t(6-21x^2+x^4)), \qquad (83)
\end{aligned}
$$

$$
\left.
\begin{aligned}
&\boldsymbol{A}^T.\boldsymbol{\Psi}(x).\boldsymbol{\Upsilon}(0) = \alpha\left(\boldsymbol{A}^T.\boldsymbol{\Psi}(x).\boldsymbol{\Upsilon}(1)\right), \\
&\boldsymbol{A}^T.\boldsymbol{\Psi}(0).\boldsymbol{\Upsilon}(t) = \boldsymbol{A}^T.\boldsymbol{\Psi}(1).\boldsymbol{\Upsilon}(t) = 0.
\end{aligned}
\right. \qquad (84)
$$

The technique has been applied as it is shown in Eq. (13) which results in:

$$
\begin{aligned}
\langle \boldsymbol{B}_{i,n}(x)\boldsymbol{B}_{j,n}(t), (1+e^{2x})\boldsymbol{A}^T.\boldsymbol{\Psi}(x).(\boldsymbol{Bn}^*)^T.\boldsymbol{\Upsilon}(t) - \boldsymbol{A}^T.\boldsymbol{Bn}^{*2}.\boldsymbol{\Psi}(x).\boldsymbol{\Upsilon}(t) \\
+ \boldsymbol{A}^T.\boldsymbol{\Psi}(x).\boldsymbol{\Upsilon}(t)\rangle,\ \langle \boldsymbol{B}_{i,n}(x)\boldsymbol{B}_{j,n}(t), e^{-t}x(-6+20x^2+e^{2x}x^2(-1+x^2) \\
+ 2e^t(6-21x^2+x^4))\rangle,\ 0 \le i,j \le n. \qquad (85)
\end{aligned}
$$

Moreover, we use the same procedure in the previous examples for solving Eq. (85). After generating a linear algebraic system of equations for $n = 5$, as solved in [14], we use Mathematica®12 to determine the values of the coefficients $\boldsymbol{A}$ as follows:

$$
\boldsymbol{A} = \begin{bmatrix}
-3.7999\times10^{-76} & 1.3231\times10^{-74} & -3.7455\times10^{-74} & 4.4131\times10^{-74} & -2.2394\times10^{-74} & 3.3249\times10^{-75} \\
1.6506\times10^{-8} & 2.8813\times10^{-7} & -3.2380\times10^{-7} & 3.4019\times10^{-7} & -1.2190\times10^{-7} & 2.6939\times10^{-8} \\
-1.9728\times10^{-8} & 1.5853\times10^{-7} & 2.0048\times10^{-7} & -1.9374\times10^{-7} & 2.0082\times10^{-7} & -3.2198\times10^{-8} \\
-0.10000 & -0.12000 & -0.13500 & -0.14666 & -0.15585 & -0.16321 \\
-0.40000 & -0.48000 & -0.54001 & -0.58664 & -0.62342 & -0.65285 \\
7.5998\times10^{-76} & -4.0417\times10^{-75} & 8.8780\times10^{-75} & -1.0156\times10^{-74} & 6.0971\times10^{-75} & -1.5545\times10^{-75}
\end{bmatrix}.
$$

Finally, by substituting the coefficients $\boldsymbol{A}$ in the Eq. (15), we get the following approximate solution for $n = 5$:

$$
\begin{aligned}
u(x,t) \approx\ & -7.34074\times10^{-75} + 8.25281\times10^{-8}x - 5.27392\times10^{-7}x^2 - 0.999998\,x^3 \\
& - 2.17806\times10^{-6}x^4 + 1.\,x^5 \\
& + t^2(-2.10904\times10^{-72} - 0.0000441777\,x + 0.000163079\,x^2 \\
& + 0.498926\,x^3 + 0.000878309\,x^4 - 0.499923\,x^5) \\
& + t^4(-6.67995\times10^{-72} - 0.000114037\,x + 0.000532387\,x^2 \\
& + 0.0361264\,x^3 + 0.0027073\,x^4 - 0.039252\,x^5) \\
& + t^5(2.70538\times10^{-72} + 0.0000435023\,x - 0.00021567\,x^2 \\
& - 0.00421196\,x^3 - 0.00107865\,x^4 + 0.00546277\,x^5) \\
& + t^3(5.8213\times10^{-72} + 0.000107974\,x - 0.000461881\,x^2 - 0.163071\,x^3 \\
& - 0.00240194\,x^4 + 0.165827\,x^5) \\
& + t\,(2.80071\times10^{-73} + 6.79056\times10^{-6}x - 0.0000182492\,x^2 \\
& - 0.999889\,x^3 - 0.000106406\,x^4 + 1.00001\,x^5).
\end{aligned}
$$

Second: Using the DECMs proposed methods based on the Legendre polynomials.

By substituting Eqs. (18) and (19) into Eqs. (81) and (82), this yields the following:

$$
\begin{aligned}
(1+e^{2x})\boldsymbol{A}^T.\boldsymbol{\Psi}(x).(\boldsymbol{L}^*)^T.\boldsymbol{\Upsilon}(t) - \boldsymbol{A}^T.\boldsymbol{L}^{*2}.\boldsymbol{\Psi}(x).\boldsymbol{\Upsilon}(t) + \boldsymbol{A}^T.\boldsymbol{\Psi}(x).\boldsymbol{\Upsilon}(t) \\
= e^{-t}x(-6+20x^2+e^{2x}x^2(-1+x^2)+2e^t(6-21x^2+x^4)), \qquad (86)
\end{aligned}
$$

$$
\left.
\begin{aligned}
&\boldsymbol{A}^T.\boldsymbol{\Psi}(x).\boldsymbol{\Upsilon}(0) = \alpha\left(\boldsymbol{A}^T.\boldsymbol{\Psi}(x).\boldsymbol{\Upsilon}(1)\right), \\
&\boldsymbol{A}^T.\boldsymbol{\Psi}(0).\boldsymbol{\Upsilon}(t) = \boldsymbol{A}^T.\boldsymbol{\Psi}(1).\boldsymbol{\Upsilon}(t) = 0.
\end{aligned}
\right. \qquad (87)
$$

Also, the technique has been applied as it is shown in Eq. (13) which results in:

$$\langle \boldsymbol{P}_i(x)\boldsymbol{P}_j(t), (1+e^{2x})\boldsymbol{A}^T.\boldsymbol{\Psi}(x).(\boldsymbol{L}^*)^T.\boldsymbol{\Upsilon}(t) - \boldsymbol{A}^T.\boldsymbol{L}^{*2}.\boldsymbol{\Psi}(x).\boldsymbol{\Upsilon}(t) + \boldsymbol{A}^T.\boldsymbol{\Psi}(x).\boldsymbol{\Upsilon}(t)\rangle,\ \langle \boldsymbol{P}_i(x)\boldsymbol{P}_j(t), e^{-t}x(-6+20x^2+e^{2x}x^2(-1+x^2) + 2e^t(6-21x^2+x^4))\rangle,\ 0 \le i,j \le n. \quad (88)$$

Moreover, we use Mathematica®12 to solve Eq. (88), and determine the values of the coefficients $\boldsymbol{A}$ as follows:

$$\boldsymbol{A} = \begin{bmatrix} 1.0162\times10^{-6} & -9.4949\times10^{-6} & 0.000026001 & -0.000035923 & 0.000028370 & -0.000012166 \\ -0.14157 & -0.18875 & 0.060792 & -0.011615 & 0.0014319 & -0.000080776 \\ 0.000014155 & -0.000052985 & 0.000098833 & -0.00011590 & 0.000084470 & -0.000034427 \\ 0.036687 & 0.048989 & -0.015859 & 0.0031237 & -0.00045230 & 0.000053748 \\ 0.000016163 & -0.000045327 & 0.000062440 & -0.000058737 & 0.000036974 & -0.000013458 \\ 0.10485 & 0.13987 & -0.045121 & 0.0087019 & -0.0011294 & 0.000087080 \end{bmatrix}.$$

Finally, by substituting the coefficients $\boldsymbol{A}$ in the Eq. (18), we get the following approximate solution for $n = 5$:

$$\begin{aligned} u(x,t) \approx\ & (4.99066\times10^{-8} - 1.21941\times10^{-7}t - 4.74508\times10^{-6}t^2 + 0.0000221306\,t^3 \\ & - 0.0000320084\,t^4 + 0.0000147764\,t^5)x \\ & + (-9.09067\times10^{-7} + 0.0000186785\,t - 0.0000840553\,t^2 \\ & + 0.000126286\,t^3 - 0.0000522606\,t^4 - 9.22353\times10^{-6}t^5)x^2 \\ & + (-0.999996 - 1.00004\,t + 0.499919\,t^2 - 0.16534\,t^3 + 0.0382885\,t^4 \\ & - 0.00494216\,t^5)x^3 \\ & + (-5.21355\times10^{-6} + 0.0000767543\,t - 0.000196836\,t^2 \\ & - 0.000127227\,t^3 + 0.000707697\,t^4 - 0.000463683\,t^5)x^4 \\ & + (1. + 0.999948\,t - 0.499634\,t^2 + 0.165319\,t^3 - 0.038912\,t^4 \\ & + 0.00540029\,t^5)x^5. \end{aligned}$$

Third: Implementing the DECMs proposed methods based on the Chebyshev polynomials of the first kind.

By inserting the Eqs. (21) and (22) into the Eqs. (81) and (82), we obtain:

$$(1+e^{2x})\boldsymbol{A}^T.\boldsymbol{\Psi}(x).(\boldsymbol{V}^*)^T.\boldsymbol{\Upsilon}(t) - \boldsymbol{A}^T.\boldsymbol{V}^{*2}.\boldsymbol{\Psi}(x).\boldsymbol{\Upsilon}(t) + \boldsymbol{A}^T.\boldsymbol{\Psi}(x).\boldsymbol{\Upsilon}(t) = e^{-t}x(-6+20x^2+e^{2x}x^2(-1+x^2) + 2e^t(6-21x^2+x^4)), \quad (89)$$

$$\begin{aligned} &\boldsymbol{A}^T.\boldsymbol{\Psi}(x).\boldsymbol{\Upsilon}(0) = \alpha\left(\boldsymbol{A}^T.\boldsymbol{\Psi}(x).\boldsymbol{\Upsilon}(1)\right), \\ &\boldsymbol{A}^T.\boldsymbol{\Psi}(0).\boldsymbol{\Upsilon}(t) = \boldsymbol{A}^T.\boldsymbol{\Psi}(1).\boldsymbol{\Upsilon}(t) = 0. \end{aligned} \quad (90)$$

Also, the technique has been applied as shown in Eq. (13), which results in:

$$\langle \boldsymbol{T}_i(x)\boldsymbol{T}_j(t), (1+e^{2x})\boldsymbol{A}^T.\boldsymbol{\Psi}(x).(\boldsymbol{V}^*)^T.\boldsymbol{\Upsilon}(t) - \boldsymbol{A}^T.\boldsymbol{V}^{*2}.\boldsymbol{\Psi}(x).\boldsymbol{\Upsilon}(t) + \boldsymbol{A}^T.\boldsymbol{\Psi}(x).\boldsymbol{\Upsilon}(t)\rangle,\ \langle \boldsymbol{T}_i(x)\boldsymbol{T}_j(t), e^{-t}x(-6+20x^2+e^{2x}x^2(-1+x^2) + 2e^t(6-21x^2+x^4))\rangle,\ 0 \le i,j \le n. \quad (91)$$

Additionally, we use Mathematica®12 to solve Eq. (91), and determine the values of the coefficients $\boldsymbol{A}$ as follows:

$$\boldsymbol{A} = \begin{bmatrix} 0.000015189 & -0.000036502 & 0.000043851 & -0.000042323 & 0.000026281 & -0.000010382 \\ -0.092001 & -0.14080 & 0.033547 & -0.0052813 & 0.00055220 & -0.000021205 \\ 0.000031428 & -0.000067963 & 0.000070804 & -0.000061758 & 0.000036349 & -0.000013799 \\ 0.045963 & 0.070484 & -0.016860 & 0.0027166 & -0.00032104 & 0.000027738 \\ 0.000016238 & -0.000031461 & 0.000026952 & -0.000019435 & 0.000010068 & "-3.4168"\times10^{-6} \\ 0.045975 & 0.070456 & -0.016829 & 0.0026882 & -0.00030386 & 0.000021066 \end{bmatrix}.$$

Finally, by substituting the coefficients $\boldsymbol{A}$ in the Eq. (21), we get the following approximate solution for $n = 5$:

$$u(x,t) \approx (4.64891\times 10^{-8} - 8.8072\times 10^{-8}t - 4.81882\times 10^{-6}t^2 + 0.0000220878\,t^3 - 0.0000317708\,t^4 + 0.0000146193\,t^5)x + (-8.83322\times 10^{-7} + 0.0000185412\,t - 0.0000852805\,t^2 + 0.000133145\,t^3 - 0.0000627425\,t^4 - 4.22163\times 10^{-6}t^5)x^2 + (-0.999996 - 1.00004\,t + 0.499932\,t^2 - 0.165384\,t^3 + 0.0383442\,t^4 - 0.00496603\,t^5)x^3 + (-5.17053\times 10^{-6} + 0.0000780914\,t - 0.000213084\,t^2 - 0.0000752439\,t^3 + 0.000644322\,t^4 - 0.000437353\,t^5)x^4 + (1. + 0.999948\,t - 0.499629\,t^2 + 0.165304\,t^3 - 0.038894\,t^4 + 0.00539298\,t^5)x^5.$$

Fourth: Utilizing the DECMs proposed methods based on the Hermite polynomials.

The following findings are obtained by inserting the Eqs. (24) and (25) into the Eqs. (81) and (82):

$$(1+e^{2x})\boldsymbol{\Psi}(x)\,.\boldsymbol{A}.\boldsymbol{\Lambda}(t).(\boldsymbol{N})^T.(\boldsymbol{E}^{-1})^T - \boldsymbol{\Lambda}(x).\boldsymbol{N}^2\,.(\boldsymbol{E}^{-1})^T.\boldsymbol{A}\,.\boldsymbol{\Upsilon}(t) + \boldsymbol{\Psi}(x)\,.\boldsymbol{A}\,.\boldsymbol{\Upsilon}(t) = e^{-t}x\big(-6+20x^2+e^{2x}x^2(-1+x^2)+2e^t(6-21x^2+x^4)\big), \quad (92)$$

$$\begin{aligned}&\boldsymbol{\Psi}(x)\,.\boldsymbol{A}\,.\boldsymbol{\Upsilon}(0) = \alpha\big(\boldsymbol{\Psi}(x)\,.\boldsymbol{A}\,.\boldsymbol{\Upsilon}(1)\big),\\ &\boldsymbol{\Psi}(0)\,.\boldsymbol{A}\,.\boldsymbol{\Upsilon}(t) = \boldsymbol{\Psi}(1)\,.\boldsymbol{A}\,.\boldsymbol{\Upsilon}(t) = 0.\end{aligned} \quad (93)$$

Moreover, the technique has been applied as it is shown in Eq. (13) which results in:

$$\langle \boldsymbol{H}_i(x)\boldsymbol{H}_j(t), (1+e^{2x})\boldsymbol{\Psi}(x)\,.\boldsymbol{A}.\boldsymbol{\Lambda}(t).(\boldsymbol{N})^T.(\boldsymbol{E}^{-1})^T - \boldsymbol{\Lambda}(x).\boldsymbol{N}^2\,.(\boldsymbol{E}^{-1})^T.\boldsymbol{A}\,.\boldsymbol{\Upsilon}(t) + \boldsymbol{\Psi}(x)\,.\boldsymbol{A}\,.\boldsymbol{\Upsilon}(t)\rangle,\ \ \langle \boldsymbol{H}_i(x)\boldsymbol{H}_j(t), e^{-t}x\big(-6+20x^2+e^{2x}x^2(-1+x^2) + 2e^t(6-21x^2+x^4)\big)\rangle,\ \ 0\le i,j\le n. \quad (94)$$

Furthermore, we use Mathematica®12 to solve Eq. (94), and determine the values of the coefficients $\boldsymbol{A}$ as follows:

$$\boldsymbol{A} = \begin{bmatrix} 0.00012616 & -0.00040435 & 0.00018824 & -0.00015998 & 0.000020509 & -8.6775\times 10^{-6}\\ 0.81110 & 0.71361 & -0.17346 & 0.027172 & -0.0027504 & 0.00019748\\ 0.00015340 & -0.00043748 & 0.00020977 & -0.00016563 & 0.000021844 & -8.7283\times 10^{-6}\\ 0.36051 & 0.31706 & -0.077046 & 0.012033 & -0.0012170 & 0.000085366\\ 0.000015054 & -0.000039218 & 0.000019275 & -0.000014273 & 1.9316\times 10^{-6} & -7.3160\times 10^{-7}\\ 0.022533 & 0.019813 & -0.0048138 & 0.00075056 & -0.000075880 & 5.2491\times 10^{-6}\end{bmatrix}.$$

Consequently, by substituting the coefficients $\boldsymbol{A}$ in the Eq. (24), we get the following approximate solution for $n=5$:

$$u(x,t) \approx (4.13292\times 10^{-8} - 3.56961\times 10^{-8}t - 4.94814\times 10^{-6}t^2 + 0.0000221775\,t^3 - 0.0000317107\,t^4 + 0.0000145432\,t^5)x + (-8.33215\times 10^{-7} + 0.0000182897\,t - 0.0000881145\,t^2 + 0.000148243\,t^3 - 0.0000854539\,t^4 + 6.50886\times 10^{-6}t^5)x^2 + (-0.999996 - 1.00005\,t + 0.499961\,t^2 - 0.165484\,t^3 + 0.0384733\,t^4 - 0.00502154\,t^5)x^3 + (-5.06971\times 10^{-6} + 0.0000808576\,t - 0.000249877\,t^2 + 0.0000458956\,t^3 + 0.000494497\,t^4 - 0.000374578\,t^5)x^4 + (1. + 0.999947\,t - 0.499618\,t^2 + 0.165268\,t^3 - 0.0388506\,t^4 + 0.00537507\,t^5)x^5.$$

Fifth: Applying the DECMs proposed methods based on the Bernoulli polynomials.

By substituting the Eqs. (27) and (28) into the Eqs. (81) and (82) produce the following results:

$$(1+e^{2x})\boldsymbol{A}^T.\boldsymbol{\Psi}(x)\,.(\boldsymbol{Bi}^*)^T.\boldsymbol{\Upsilon}(t) - \boldsymbol{A}^T.\boldsymbol{Bi}^{*2}.\boldsymbol{\Psi}(x)\,.\boldsymbol{\Upsilon}(t) + \boldsymbol{A}^T\,.\boldsymbol{\Psi}(x)\,.\boldsymbol{\Upsilon}(t) = e^{-t}x(-6+20x^2+e^{2x}x^2(-1+x^2)+2e^t(6-21x^2+x^4)), \quad (95)$$

$$\begin{aligned}&\boldsymbol{A}^T\,.\boldsymbol{\Psi}(x)\,.\boldsymbol{\Upsilon}(0) = \alpha\big(\boldsymbol{A}^T\,.\boldsymbol{\Psi}(x)\,.\boldsymbol{\Upsilon}(1)\big),\\ &\boldsymbol{A}^T\,.\boldsymbol{\Psi}(0)\,.\boldsymbol{\Upsilon}(t) = \boldsymbol{A}^T\,.\boldsymbol{\Psi}(1)\,.\boldsymbol{\Upsilon}(t) = 0.\end{aligned} \quad (96)$$

Then, the technique has been applied as it is shown in Eq. (13) which results in the:

$$\langle \boldsymbol{B}_i(x)\boldsymbol{B}_j(t), (1+e^{2x})\boldsymbol{A}^T.\boldsymbol{\Psi}(x).(\boldsymbol{Bi}^*)^T.\boldsymbol{\Upsilon}(t) - \boldsymbol{A}^T.\boldsymbol{Bi}^{*2}.\boldsymbol{\Psi}(x).\boldsymbol{\Upsilon}(t) + \boldsymbol{A}^T.\boldsymbol{\Psi}(x).\boldsymbol{\Upsilon}(t)\rangle, \ \langle \boldsymbol{B}_i(x)\boldsymbol{B}_j(t), e^{-t}x(-6+20x^2+e^{2x}x^2(-1+x^2) + 2e^t(6-21x^2+x^4))\rangle, \ 0 \le i,j \le n. \quad (97)$$

Moreover, we use Mathematica®12 to solve Eq. (97), and determine the values of the coefficients $\boldsymbol{A}$ as follows,

$$\boldsymbol{A} = \begin{bmatrix} 1.0162\times10^{-6} & -0.00011517 & 0.00072341 & -0.0024218 & 0.0019859 & -0.0030660 \\ -0.000062668 & 0.0015137 & -0.0082398 & 0.025237 & -0.020974 & 0.030266 \\ 0.00040819 & -0.0071406 & 0.035977 & -0.10400 & 0.087241 & -0.11988 \\ 15.412 & 43.710 & -43.148 & 27.472 & -11.702 & 3.3431 \\ 0.0011314 & -0.016453 & 0.077988 & -0.21412 & 0.18117 & -0.23741 \\ 26.422 & 74.923 & -73.915 & 46.930 & -19.923 & 5.5299 \end{bmatrix}.$$

Thus, by substituting the coefficients $\boldsymbol{A}$ in the Eq. (27), we get the following approximate solution for $n = 5$:

$$\begin{aligned} u(x,t) \approx{} & (4.99066\times10^{-8} - 1.21941\times10^{-7}t - 4.74508\times10^{-6}t^2 + 0.0000221306\,t^3 \\ & - 0.0000320084\,t^4 + 0.0000147764\,t^5)x \\ & + (-9.09067\times10^{-7} + 0.0000186785\,t - 0.0000840553\,t^2 \\ & + 0.000126286\,t^3 - 0.0000522606\,t^4 - 9.22353\times10^{-6}t^5)x^2 \\ & + (-0.999996 - 1.00004\,t + 0.499919\,t^2 - 0.16534\,t^3 + 0.0382885\,t^4 \\ & - 0.00494216\,t^5)x^3 \\ & + (-5.21355\times10^{-6} + 0.0000767543\,t - 0.000196836\,t^2 \\ & - 0.000127227\,t^3 + 0.000707697\,t^4 - 0.000463683\,t^5)x^4 \\ & + (1. + 0.999948\,t - 0.499634\,t^2 + 0.165319\,t^3 - 0.038912\,t^4 \\ & + 0.00540029\,t^5)x^5 \end{aligned}$$

Furthermore, the $MAbsR_n$ is calculated to demonstrate the accuracy and reliability of the suggested approaches for solving the problem given in Eqs. (81) and (82).
Table (3) presents the values of $MAbsR_n$ corresponding to the approximate solution derived from the ECM technique delineated in [14], and the DECMs proposed methods, with an approximation order of $n = 6$, as expounded in [14]. Furthermore, it can be demonstrated that the accuracy of the DECMs' proposed techniques slightly surpasses that of the ECM approach.

Table 3: The comparison between the $MAbsR_6$ for the example (4) by the ECM [14], and by the DECMs proposed methods.

ECM Standard [14]	**DECMs Bernstein**	**DECMs Legendre**	**DECMs Chebyshev**	**DECMs Hermite**	**DECMs Bernoulli**
3.6391×10^{-9}	3.64035×10^{-9}	3.13369×10^{-9}	3.17874×10^{-9}	3.40491×10^{-9}	3.13369×10^{-9}

Furthermore, Figures (3) and (4) demonstrate the absolute error values for the interval $0 \le x,t \le 1$, obtained from the proposed techniques with approximation orders of $n =$ 4 and 6, respectively. In reality, the precision of the solution improves, and the error is less with the ascending value of n.

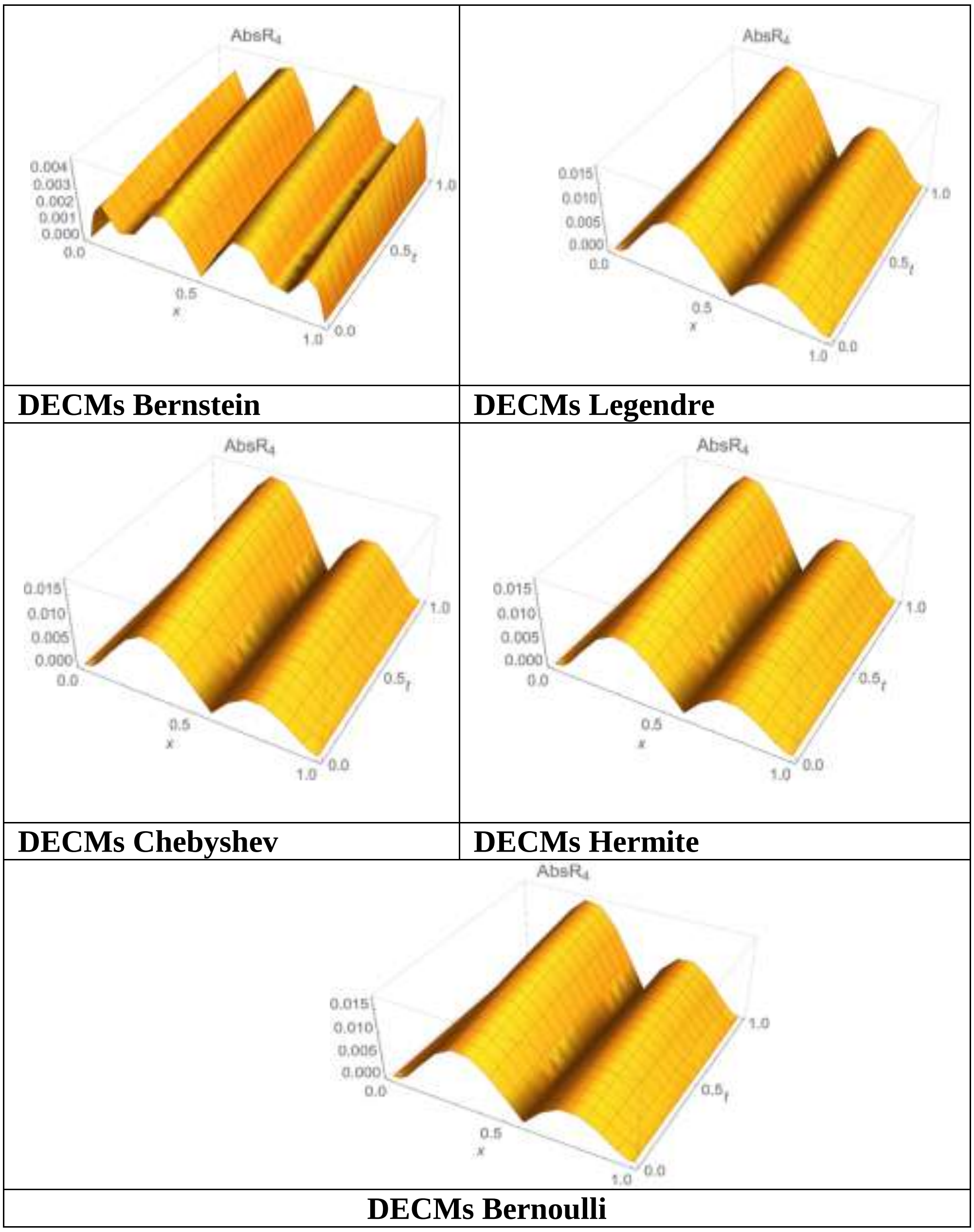
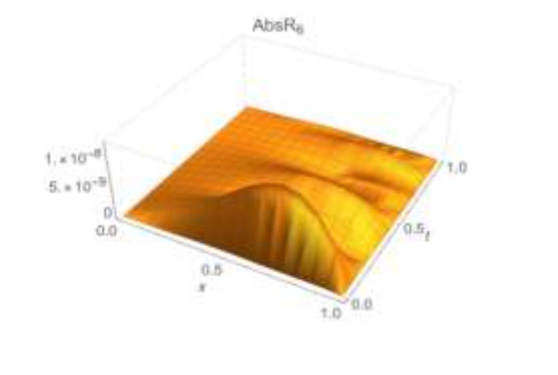
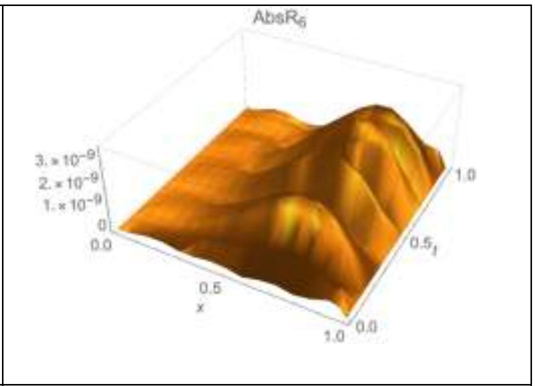
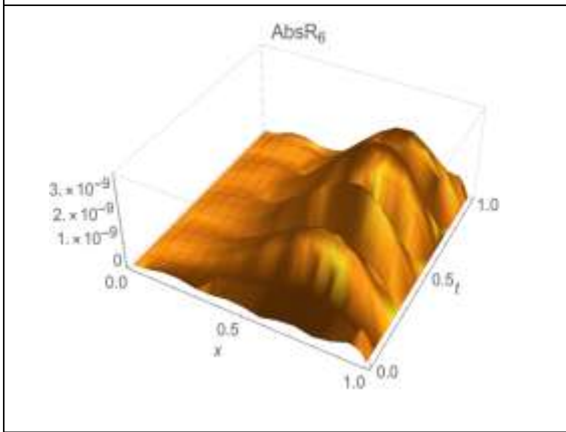


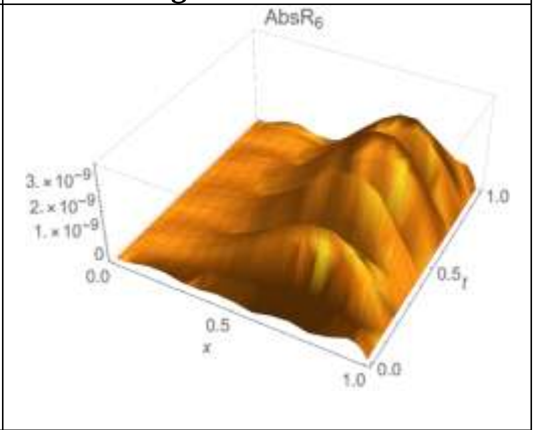
Figure 3: The absolute error values obtained using the proposed techniques of the example (4) with order approximation $n = 4$.



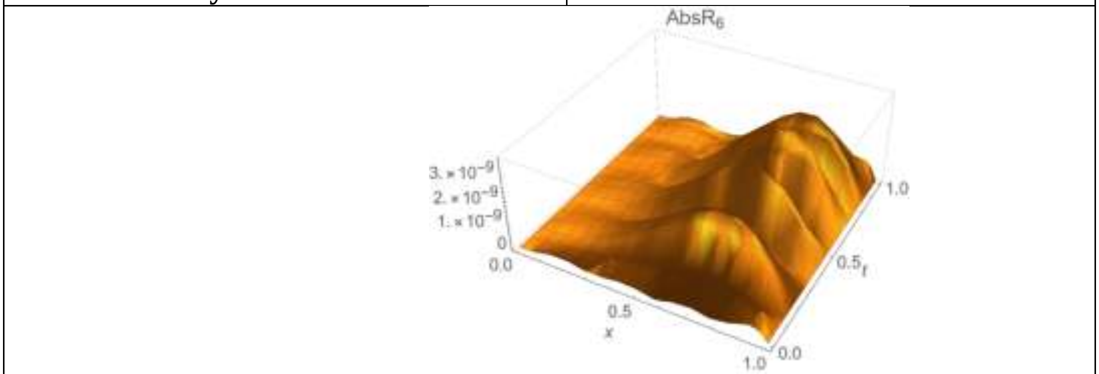


DECMs Bernstein **DECMs Legendre**





DECMs Chebyshev **DECMs Hermite**



DECMs Bernoulli

Figure 4: The absolute error values obtained using the proposed techniques of the example (4) with order approximation $n = 6$.

5. Conclusions

This paper introduces and implements a new class of computational techniques (DECMs) for solving parabolic partial differential equations based on suitable orthogonal polynomials such as the Bernstein, the Legendre, the Chebyshev, the Hermite, and the Bernoulli polynomials. In this work, we develop and extend the ECM-described double power series expansion technique to get novel analytic approximate solutions to the problem. The time-dependent diffusion equations have been reduced to a linear algebraic system which is solved by Mathematica®12. The proposed procedures are straightforward, and it is demonstrated with examples that the methods can produce exact solutions when the solutions are expressed as polynomials. Otherwise, highly accurate solutions are obtained with small approximation orders for some nonlocal problems. Furthermore, the results demonstrate that the proposed approaches improve ECM in terms of accuracy and error rate.

References

[1] M. A. AL-Jawary and O. M. Salih, "Reliable iterative methods for 1D Swift–Hohenberg equation," *Arab Journal of Basic and Applied Sciences*, vol. 27, no. 1, pp. 56-66, 2020.

[2] F. Toutounian and E. Tohidi, "A new Bernoulli matrix method for solving second order linear partial differential equations with the convergence analysis," *Applied Mathematics and Computation*, vol. 223, pp. 298-310, 2013.

[3] P. Knabner and L. Angermann, "*Numerical methods for elliptic and parabolic partial differential equations: With contributions by Andreas Rupp*," Springer Nature, 2021.

[4] R. W. Huisen, S. H. Abd Almjeed and A. S. Mohammed, "A reliable iterative transform method for solving an epidemic model," *Iraqi Journal of Science*, vol. 62, no. 12, pp. 4839-4846, 2021.

[5] O. M. Salih and M. A. AL-Jawary, "Reliable iterative methods for solving 1D, 2D and 3D Fisher's equation," *IIUM Engineering Journal*, vol. 22, no. 1, pp. 138-166, 2021.

[6] E. M. Namah, "Variational Approximate Solutions of Fractional Delay Differential Equations with Integral Transform," *Iraqi Journal of Science*, vol. 62, no. 10, pp. 3679-3689, 2021.

[7] S. M. Abed and M. A. AL-Jawary, "Efficient Iterative Methods for Solving the SIR Epidemic Model," *Iraqi Journal of Science*, vol. 62, no. 2, pp. 613-622, 2021.

[8] G. J. Mohammed and F. S. Fadhel, "Extend differential transform methods for solving differential equations with multiple delay," *Ibn AL-Haitham Journal for Pure and Applied Science*, vol. 24, no. 3, pp. 1-10, 2017.

[9] M. Tatari and M. Dehghan, "On the solution of the non-local parabolic partial differential equations via radial basis functions," *Applied Mathematical Modelling*, vol. 33, no. 3, pp. 1729-1738, 2009.

[10] A. Canada, P. Drabek and A. Fonda, "Handbook of differential equations: ordinary differential equations," North-Holland, 2004.

[11] J. R. Cannon, "The solution of the heat equation subject to the specification of energy," *Quarterly of Applied Mathematics*, vol. 21, no. 2, pp. 155-160, 1963.

[12] W. A. Day, "Parabolic equations and thermodynamics," *Quarterly of applied mathematics*, vol.50, no. 3, pp. 523-533, 1992.

[13] A. Bouziani, "On a class of parabolic equations with a nonlocal boundary condition," *Bulletins de l'Académie Royale de Belgique*, vol. 10, no. 1, pp. 61-77, 1999.

[14] M. Turkyilmazoglu, "Parabolic partial differential equations with nonlocal initial and boundary values," *International Journal of Computational Methods*, vol. 12, no. 05, id. 1550024, 2015.

[15] M. Dehghan, "Application of the Adomian decomposition method for two-dimensional parabolic equation subject to nonstandard boundary specifications," *Applied mathematics and computation*, vol. 157, no. 2, pp. 549-560, 2004.

[16] M. Dehghan, "Finite difference procedures for solving a problem arising in modeling and design of certain optoelectronic devices," *Mathematics and Computers in Simulation*, vol. 71, no. 1, pp. 16-30, 2006.

[17] E. Tohidi and A. Kılıçman, "An efficient spectral approximation for solving several types of parabolic PDEs with nonlocal boundary conditions," *Mathematical Problems in Engineering*, id. 369029, 2014.

[18] A. Golbabai and M. Javidi, "A numerical solution for non-classical parabolic problem based on Chebyshev spectral collocation method," *Applied mathematics and computation*, vol. 190, no. 1, pp. 179-185, 2007.

[19] T. Allahviranloo and H. Sahihi, "Reproducing kernel method to solve parabolic partial differential equations with nonlocal conditions," *Numerical Methods for Partial Differential Equations*, vol. 36, no. 6, pp. 1758-1772, 2020.

[20] S. A. Yousefi, M. Behroozifar and M. Dehghan, "The operational matrices of Bernstein polynomials for solving the parabolic equation subject to specification of the mass," *Journal of computational and applied mathematics*, vol. 235, no. 17, pp. 5272-5283, 2011.

[21] M. A. AL-Jawary and G. H. Ibraheem, "Two meshless methods for solving nonlinear ordinary differential equations in engineering and applied sciences," *Nonlinear Engineering*, vol. 9, no. 1, pp. 244-255, 2020.

[22] R. T. Farouki, T. N. Goodman and T. Sauer, "Construction of orthogonal bases for polynomials in Bernstein form on triangular and simplex domains," *Computer Aided Geometric Design*, vol. 20, no. 4, pp. 209-230, 2003.
[23] S. A. Yousefi and M. Behroozifar, "Operational matrices of Bernstein polynomials and their applications," *International Journal of Systems Science*, vol. 41, no. 6, pp. 709-716, 2010.
[24] E. Hesameddini and M. Shahbazi, "Two-dimensional shifted Legendre polynomials operational matrix method for solving the two-dimensional integral equations of fractional order," *Applied Mathematics and Computation*, vol. 322, pp. 40-54, 2018.
[25] M. Dehghan and M. Tatari, "Solution of a parabolic equation with a time-dependent coefficient and an extra measurement using the decomposition procedure of Adomian," *Physica Scripta*, vol. 72, no. 6, pp. 425-431, 2005.
[26] S. Zhu and P. Satravaha, "Solving nonlinear time-dependent diffusion equations with the dual reciprocity method in Laplace space," *Engineering analysis with boundary elements*, vol. 18, no. 1, pp. 19-27, 1996.
[27] Y. P. Shih and J. Y. Han, "Double Walsh series solution of first-order partial differential equations," *International Journal of Systems Science*, vol. 9, no. 5, pp. 569-578, 1978.
[28] R. D. Russell and L. F. Shampine, "A collocation method for boundary value problems," *Numerische Mathematik*, vol. 19, no. 1, pp. 1-28, 1972.
[29] A. Bouziani, N. Merazga and S. Benamira, "Galerkin method applied to a parabolic evolution problem with nonlocal boundary conditions," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 69, no. 5-6, pp. 1515-1524, 2008.
[30] J. Matos, M. J. Rodrigues and P. B. Vasconcelos, "New implementation of the Tau method for PDEs," *Journal of computational and applied mathematics*, vol. 164, pp. 555-567, 2004.
[31] E. D. Eason, "A review of least-squares methods for solving partial differential equations," *International journal for numerical methods in engineering*, vol. 10, no. 5, pp. 1021-1046, 1976.
[32] E. Tohidi and M. Shirazian, "Numerical solution of linear HPDEs via Bernoulli operational matrix of differentiation and comparison with Taylor matrix method," *Mathematical Sciences Letters*, vol. 1, no. 1, pp. 61-70, 2012.
[33] M. Turkyilmazoglu, "Effective computation of exact and analytic approximate solutions to singular nonlinear equations of Lane–Emden–Fowler type," *Applied Mathematical Modelling*, vol. 37, no. 14-15, pp. 7539-7548, 2013.
[34] M. Turkyilmazoglu, "An effective approach for numerical solutions of high-order Fredholm integro-differential equations," *Applied Mathematics and Computation*, vol. 227, no. 15, pp. 384-398, 2014.
[35] M. Turkyilmazoglu, "High-order nonlinear Volterra–Fredholm-Hammerstein integro-differential equations and their effective computation," *Applied Mathematics and Computation*, vol. 247, pp. 410-416, 2014.
[36] M. Turkyilmazoglu, "Effective computation of solutions for nonlinear heat transfer problems in fins," *Journal of Heat Transfer*, vol. 136, no. 9, id. 091901, 2014.
[37] M. Turkyilmazoglu, "Solution of initial and boundary value problems by an effective accurate method," *International Journal of Computational Methods*, vol. 14, no. 06, id. 1750069, 2017.
[38] M. Dehghan, "The one-dimensional heat equation subject to a boundary integral specification," *Chaos, Solitons & Fractals*, vol. 32, no. 2, pp. 661-675, 2007.
[39] M. Dehghan, "A computational study of the one-dimensional parabolic equation subject to nonclassical boundary specifications," *Numerical Methods for Partial Differential Equations: An International Journal*, vol. 22, no. 1, pp. 220-257, 2006.
[40] X. Li and B. Wu, "New algorithm for nonclassical parabolic problems based on the reproducing kernel method," *Mathematical Sciences*, vol. 7, no. 1, pp. 1-5, 2013.
[41] A. Saadatmandi and M. Razzaghi, "A Tau method approach for the diffusion equation with nonlocal boundary conditions," *International Journal of Computer Mathematics*, vol. 81, no. 11, pp. 1427-1432, 2004.
[42] M. Dehghan, "Efficient techniques for the second-order parabolic equation subject to nonlocal specifications," *Applied Numerical Mathematics*, vol. 52, no. 1, pp. 39-62, 2005.

[43] E. Tohidi and F. Toutounian, "Numerical solution of time-dependent diffusion equations with nonlocal boundary conditions via a fast matrix approach," *Journal of the Egyptian Mathematical Society*, vol. 24, no. 1, pp. 86-91, 2016.
[44] Q. Guan, M. Gunzburger and X. Zhang, "Collocation method for one dimensional nonlocal diffusion equations," *Numerical Methods for Partial Differential Equations*, vol. 38, no. 6, pp. 1618-1635, 2022.
[45] C. S. Liu and J. R. Chang, "Solving a nonlinear heat equation with nonlocal boundary conditions by a method of nonlocal boundary shape functions," *Numerical Heat Transfer, Part B: Fundamentals*, vol. 80, no.1-2, pp. 1-13, 2021.
[46] E. Hesameddini and M. Riahi, "Hybrid Legendre Block-Pulse functions method for solving partial differential equations with non-local integral boundary conditions," *Journal of Information and Optimization Sciences*, vol. 40, no. 7, pp. 1391-1403, 2019.
[47] N. Kadkhoda, "A numerical approach for solving variable order differential equations using Bernstein polynomials," *Alexandria Engineering Journal*, vol. 59, no. 5, pp. 3041-3047, 2020.
[48] G. H. Ibraheem and M. A. AL-Jawary, "The operational matrix of Legendre polynomials for solving nonlinear thin film flow problems," *Alexandria Engineering Journal*, vol. 59, no. 5, pp. 4027-4033, 2020.
[49] S. Singh, V. K. Patel and V. K. Singh, "Operational matrix approach for the solution of partial integro-differential equation," *Applied Mathematics and Computation*, vol. 283, pp. 195-207, 2016.
[50] E. Tohidi, "Application of Chebyshev collocation method for solving two classes of non-classical parabolic PDEs," *Ain Shams Engineering Journal*, vol. 6, no.1, pp. 373-379, 2015.
[51] A. Secer, N. Ozdemir and M. Bayram, "A Hermite polynomial approach for solving the SIR model of epidemics," *Mathematics*, vol. 6, no. 12, pp. 305, 2018.
[52] M. Gülsu, H. Yalman, Y. Öztürk and M. Sezer, "A new Hermite collocation method for solving differential difference equations," *Applications and Applied Mathematics: An International Journal*, vol. 6, no. 1, pp. 116-129, 2011.
[53] H. D. Azodi and M. R. Yaghouti, "Bernoulli polynomials collocation for weakly singular Volterra integro-differential equations of fractional order," *Filomat*, vol. 32, no. 10, pp. 3623-3635, 2018.
[54] S. Singh, V. K. Patel, V. K. Singh and E. Tohidi, "Application of Bernoulli matrix method for solving two-dimensional hyperbolic telegraph equations with Dirichlet boundary conditions," *Computers and Mathematics with Applications*, vol. 75, no. 7, pp. 2280-2294, 2018.