Supplement Extending Modules

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Abstract

In this note we consider a generalization of the notion of extending modules namely supplement extending modules. And study the relation between extending and supplement extending modules. And some properties of supplement extending. And we proved the direct summand of supplement extending module is supplement extending, and the converse is true when the module is distributive. Also we study when the direct sum of supplement extending modules is supplement extending.

Keywords: Closed submodule; Supplement submodule;Extending modules.

مقاسات التوسع المكملة

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الخلاصة

تم في هذا البحث أعطاء تعميم لمفهوم مقاسات التوسع بأسم مقاسات التوسع المكملة. ودرسنا الخواص الرئيسية لمقاسات التوسع المكملة. ودرسنا العلاقة بين مقاسات التوسع ومقاسات التوسع المكملة. و برهنا أن كل مركبة جمع مباشر لمقاسات التوسع تكون مقاس توسع،وبرهنا أن الجمع المباشر لمقاسات التوسع المكملة يكون مقاس توسع مكمل بشرط أن يكون المقاس قابل للتوزيع.

Introduction

Throughout this paper R will be a commutative ring with identity and all modules will be unitary left R–modules. A proper submodule N of an R–module M is called an essential in M if for every nonzero submodule K of M then $N \cap K \neq 0$. Equivalently, N is essential in M if and only if every nonzero element of M has a nonzero multiple in N [1]. A submodule N of M is called small in M denoted by $N \ll M$ whenever for some submodule L of M, N+L=M implies L=M [2].A submodule N of M is called closed in M if it has no proper essential extension in M [1]. A module M is called an extending if every submodule of M is essential in a direct summand. Equivalently, M is extending if and only if every closed submodule in M is a direct summand [3].

A submodule N of M is called supplement submodue in M if there exists a submodule K of M such that N+K=M and N is minimal with this property . Equivalently, if N+K=M and $N \cap K \ll N$. It is clear that every direct summand is supplement submodule [2].

In this paper, we replace the condition of extending modules which is every submodule is essential in a direct summand by the condition that every submodule is essential in a supplement submodule Equivalently, every closed submodule in M is supplement submodule and we call the module that satisfy this condition by supplement extending module.

This paper is structured in three sections, in the first section we introduce some general properties of supplement submodule that we need in section two and three.

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In section two we give the definition of supplement extending module and show that any direct summand of it is supplement extending.

In section three we give a sufficient condition under with M⊕N is supplement extending where M and N are supplement extending.

1. Some basic properties of closed and supplement submodules.

In this section we collect some well-known facts of closed submodule and we show other properties of supplement submodule.

Proposition 1.1 [1]: Let M be an R-module and A be a submodule of M. If B is a complement of A, then A⊕B is essential in M.

Recall that the definition of relative complement such that let A be a submodule of an R-module C. A relative complement for A in C is any submodule B of C which is maximal with respect to the property A∩B =0 see [1].

Proposition 1.2 [1]: If M= A⊕B is an R-module,then A is closed in M .

Proposition 1.3 [1]: Let A be a submodule of an R-module M. Then the following statements are equivalent:

1. A is a closed submodule in M.

2. If $A \subseteq B$, and B is essential in M, then $\frac{B}{A}$ is essential in$\frac{M}{A}$.

3. A is a relative complement for some submodule B of M.

Proposition1.4: Let A, B and C be submodules of an R-module M with B⊆A, then:

1. There exists a closed submodule H in M such that C is essential in H [1].

2. If B is closed in A, and A is closed in M, then B is closed in M [1].

Lemma[4] 1.5: Let M be an R-module.If K is a supplement submodule in M, then K is a supplement submodule in every U⊆M which K⊆U.

Proposition1.6: Let M be an R–module. If A is supplement submodule in M then $\frac{A}{N}$ is supplement submodule in$\frac{M}{N}$, where N is submodule of A.

Proof: Since A is supplement submodule in M then there exists a submodule K of M such that A+K =M and A∩K≪A. Now, we have $\frac{A}{N}+\frac{K+N}{N}=\frac{M}{N}$ to show $(\frac{A}{N} \cap \frac{K+N}{N})\ll \frac{A}{N}$. $\frac{A}{N} \cap \frac{K+N}{N}=\frac{A\cap(K+N)}{N}=\frac{(A\cap K)+N}{N}$ by [modular law]. Let $\frac{A\cap K+N}{N}+\frac{L}{N}=\frac{A}{N}$ where $L \subseteq A$ and $N \subseteq L$. such that $\frac{A\cap K+N+L}{N}=\frac{A}{N}$. Hence A∩K+N+L=A, but A∩K≪A then N+L=A but N⊆L then L=A and hence$\frac{L}{N}=\frac{A}{N}$. Then $(\frac{A}{N} \cap \frac{K+N}{N})\ll \frac{A}{N}$.

Lemma1.7 [4]: Let M be an R-module and V be a supplement submodule in M.Then every supplement submodule in V is a supplement submodule in M.

Lemma1.8 [5]: If A and B are supplements of K and L in R–modules M and N respectively, then A⊕B is supplement of K⊕L in M⊕N.

2. Supplement extending modules:

In this section, we introduce the concept of supplement extending modules and we discuss some of the basic properties of these modules and other related concepts.

Definition2.1: An R–module M is called supplement extending , if every submodule of M is essential in a supplement submodule in M . A ring R is supplement extending if R is supplement extending when considered as an R–module .

Proposition2.2: Let M be an R–module , then M is supplement extending if and only if every closed submodule in M is a supplement submodule in M .

Proof: Let A be a closed submodule in M. Since M is supplement extending module , then there exists a supplement submodule B in M such that A is essential in B. But A is closed, then A=B.

Conversely, Let A be a submodule of M. So, by Prop. 1.4 there exists a closed submodule B in M such that A is essential in B, so by assumption, B is supplement submodule in M, and hence M is supplement extending module .

Remarks and Examples2.3:

1. It is clear every extending module is supplement extending then Z, Q and M=$Z_2 \oplus Z_4$ as Z-module are supplement extending.

2. Every semi simple R- module is supplement extending.

3. A Z-module M=$Z_4 \oplus Z_4$ is not supplement extending since $\{(\bar{0}, \bar{2})\}$ is closed submodule in M which is not supplement submodule.

4. 4.Not every supplement extending module is extending as the following example. Consider the module $M=Z_8\oplus Z_2$ as a Z-module.The closed submodule in M are $\quad$ A= $\{(\bar{1},\bar{0})\}$, B= $\{(\bar{0},\bar{1})\}$, C= $\{(\bar{1},\bar{1})\}$, D= $\{(\bar{2},\bar{1})\}$ and E= $\{(\bar{4},\bar{1})\}$, and the summand submodules of M are A, B, C, E, F= $\{(\bar{0},\bar{0})\}$ and G=M. And hence A, B, C, E, F and G are supplement submodules in M.It is enough to check that D satisfy the definition of supplement, the only submodule A of M satisfy D+A=M to show D∩A≪D. D∩A=$\{(\bar{4},\bar{0})\}$=H. Let L be a submodule of D such that H+L=D then L=D. M is supplement extending. But M is not extending, Since D is closed but not summand.

Proposition2.4: Let M be an R–module such that M is supplement extending. Then every closed submodule in M is supplement extending.

Proof: Let N be closed submodule in M and let A be a closed submodule in N. then by Prop.1.4 A is closed submodule in M, but M is supplement extending. So, A is supplement submodule in M. By Prop. 1.5, A is supplement submodule in N. Hence N is supplement extending.

$\quad$ Since every direct summand is closed submodule by Prop. 1.2 then we get the following:

Corollary2.5: Every direct summand of supplement extending module is supplement extending.

Proposition2.6: Every uniform R–module is supplement extending module.

$\quad$ Now, we called an R–module M is supplement simple if the only supplement submodule in M are M and 0.

Remark2.7: Let M be supplemented simple R–module, if M is supplement extending then M is uniform.

Proof: Let A be a nonzero submodule of M and let K be a relative complement of K in M. So, by Prop. 1.1, A⊕K is essential in M, but by Prop. 1.3, then K is closed in M, but M is supplement extending. So, K is supplement submodule in M. Since M is supplemented simple and K≠M, then K=0. Then A is essential in M.

Proposition2.8: Let M be a supplement extending module and N be a closed submodule in M, then $\frac{M}{N}$ is supplement extending.

Proof: Let $\frac{K}{N}$ be a submodule of $\frac{M}{N}$, where K is submodule of M. Since M is supplement extending module. So, there exists a supplement submodule A in M such that K is essential in A. Since N⊆K and N is closed in M, then by Prop. 1.3 $\frac{K}{N}$ is essential in $\frac{A}{N}$ but A is supplement in M. So, by Prop. 1.6. $\frac{A}{N}$ is supplement in $\frac{M}{N}$.

$\quad$ The following theorem gives a characterization for supplement extending modules.

Theorem2.9: For any R–module M, the following statements are equivalent:

1. M is supplement extending.
2. Each closed submodule in M is supplement submodule in M.
3. If A is direct summand of injective hull E(M) of M, then A∩M is supplement submodule in M.

Proof: 1→2, clear from Prop. 2.2.

2→3. Let A be a direct summand of E(M),i.e. E(M)=A⊕B, where B is submodule of E(M), to show that A∩M is closed in M, let A∩M is essential in H, where H is submodule of M and h∈H. So, h=a+b, where a∈A and b∈B. Suppose that h∉A thus b≠0, but M is essential in E(M). So, there exists r∈R such that 0≠rb∈M. Now, rh=ra+rb and hence ra=(rh+rb)∈(M∩A)⊆H.

Thus, rb=(rh-ra)∈(B∩H). Since A∩M is essential in H, then 0=((A∩M)∩B) is essential in (H∩B), and hence H∩B=0. Thus rb=0 which is a contradiction.

Thus A∩M is closed in M. So, by (2) A∩M is supplement in M.

3→1. Let A be a submodule of M and let B be relative complement of A in M, then by Prop. 1.1 (A⊕B) is essential in M. But M is essential in E(M), therefore A⊕B is essential in E(M), thus E(A)⊕E(B)=E(A⊕B)=E(M). Since E (A) is summand of E(M), then E(A)∩M is supplement submodule in M, but A is essential in E(A) and M is essential in M. So, by [1] A =A∩M is essential in E (A) ∩M, which is supplement in M and hence M is supplement extending.

$\quad$ Let M be an R-module. Recall that Z(M)={x∈M: ann(x) is essential in R} is called singular submodule of M, where ann(x)={r∈R : rx=0} If Z(M)=M, then M is singular module. If Z(M)=0, then M is a nonsingular module, see[1].

$\quad$ Before We give out next result, We need the following lemma.

Lemma2.10[6]: Let f:M→N be an epimorphism of modules and L be a closed submodule in N. Suppose that N is nonsingular, then H=f^{-1}(L) is closed submodule in M.

Proposition2.11: Let M be a supplement extending module, then any nonsingular image of M is supplement extending.

Proof: Let f: M→N is an epimorphism and let L be a closed submodule in N, then by Lemma 2,10.H=f^{1}(L) is a closed in M, but M is supplement extending. Then there exists a submodule K of M such that H+K=M and (H∩K)≪H. Then N = f(M) = f(H+K) = f(H)+f(K) = L+ f(K) and by[2,Lemma 3.1.10] f(H∩K)=f(H)∩f(K), since ker$f$⊆H from proof of lemma 2.10 L∩f(K)≪f(H)=L. Hence N is supplement extending.

Let M be an R-module. Recall that M is called a multiplication R-module if for each submodule N of M, there exists an ideal I of R such that N=IM. See [7].

Proposition2.12: Let M be a finitely generated faithful multiplication R –module. Then R is supplement extending if and only if M is supplement extending.

Proof: Let N be a closed submodule in M, since M is faithful multiplication R–module. So, N=[N:M]M where [N:M]={r∈R:rM⊆N} be an ideal of R but N is closed submodule in M. So, by [9, Prop. 3.31, Ch 3], [N: M] is closed ideal in R, but R is supplement extending when considered as an R–module. So, [N: M] is supplement in R such that there exists an ideal J of R such that [N: M] +J=R and [N: M]∩J≪[N:M]. Now, M=RM= ([N: M] +J) M= [N: M] M+JM=N+JM. Now, to show that ([N: M] M∩JM)≪ [N: M] M. Let ([N: M] M∩JM) +KM= [N: M] M since M is multiplication module. ([N: M]∩J)+K]M=[N:M]M but M is finitely generated faithful multiplication R–module. So, by [8, Theorem 3.1], ([N:M]∩J)+K]=[N:M] then K=[N:M] and clear that KM=[N:M]M. Hence ([N: M] M∩JM)≪ [N: M] M. So, N is supplement in M. Hence M is supplement extending.

Conversely**,** Let I be a closed ideal in R. To show IM is closed in M. Since M is multiplication R–module, then IM=[IM:M]M where [IM:M]={r∈R:rM⊆IM}=I. But M is finitely generated faithful multiplication R –module. So, by [8, Theorem 3.1], I= [IM: M] but [IM: M] is closed in R, then by [9, Prop. 3.31, Ch 3] [IM: M] M is closed submodule in M, but M is supplement extending module. So, [IM: M] M is supplement submodule in M such that there exists a submodule KM of M such that [IM: M] M+KM=M and ([IM: M] M∩KM)≪ [IM: M] M. Where K be an ideal in R and because M is multiplication module, then [IM: M] M+KM=M. Now, ([IM: M] +K) M=M=RM but M is finitely generated faithful multiplication module. Then by [8, theorem 3.1], [IM: M] +K =R. Now, to show that ([IM: M]∩K)≪ [IM:M]. Let ([IM:M]∩K)+L=([IM:M], clear that ((([IM:M]∩K)+L)M=([IM:M]M, then ((([IM:M]M∩KM)+LM =([IM:M]M , and hence LM=[IM:M]M, but M is finitely generated faithful multiplication module then by [8, Theorem 3.1], L=[IM:M] and ([IM:M]∩K)≪ [IM:M] , and hence [IM:M] is supplement ideal in R. So, R is supplement extending.

Now, recall that an R-module M is called lifting module provided that, for any submodule N of M, there exists a direct summand L of M such that L⊆N and $\frac{N}{L}$ is small in $\frac{M}{L}$ [10].

Proposition 2.13 [11]: Let M be an R-module. The following statements are equivalent:

1. M is lifting.

2. M is amply supplemented and every supplement submodule of M is a direct summand of M.

Now, by Prop.2.13 since every supplement submodule is a direct summand, we have the following corollary.

Corollary2.14: Let M be a lifting R-module then the following are equivalent:

1. M is extending.

2. M is supplement extending.

3. The direct sum of supplement extending modules.

In this section we show that by example that a direct sum of supplement extending may not be supplement extending module. And we give a sufficient condition under which M⊕N is supplement extending where M and N are supplement extending modules.

Example3.1: Let M=Z_4⊕Z_4 as Z-module, Z_4 is supplement extending but M is not (see 2.3).

Proposition3.2: Let M=M_1⊕M_2 where M, M_1 and M_2 are R–modules and M be a distributive module, then M is supplement extending if and only if each M_i is supplement extending (i=1,2).

Proof: If M is supplement extending then each M_i is supplement extending (i=1, 2) by Prop. 2.4. Conversely, let L be a closed submodule in M. To proof L∩M_i is closed in M_i (i=1, 2).

Since M is distributive module, then we have L= $((L \cap M_1) \oplus (L \cap M_2))$. Hence $L \cap M_1$ is closed in M_1 and $L \cap M_2$ is closed in M_2. But M_1 and M_2 are supplement extending modules, then there exists a submodule K_1 of M_1 such that $K_1 + (L \cap M_1) = M_1$ and $K_1 \cap L \cap M_1 = (K_1 \cap L) \ll (L \cap M_1)$. Now, $M = M_1 \oplus M_2 = (K_1 + (L \cap M_1)) \oplus (K_2 + (L \cap M_2)) = (K_1 \oplus K_2) + L$. Then $K_1 \oplus K_2 + L$ and $(K_1 \oplus K_2) \cap L = ((K_1 \cap L) \oplus (K_2 \cap L)) \ll ((L \cap M_1) \oplus (L \cap M_2))$. $((K_1 \oplus K_2) \cap L) \ll M_1 \oplus M_2 = M$, and hence M is supplement extending.

Recall that a submodule N of an R–module M is called fully invariant if for every endomorphism f: M→M, $f(N) \subseteq N$. see [12].

Proposition3.3: Let $M = \oplus_i M_i$ (for each $i \in I$) be an R–module, where each M_i is submodule of M. If M is supplement extending then each M_i is supplement extending ($i \in I$). The converse is true if each closed submodule in M is fully invariant.

Proof: Suppose that M is supplement extending. Since M_i is direct summand of M for each $i \in I$, then M_i is supplement extending for each $i \in I$ by Prop. 2.4. The converse, let S be a closed submodule in M and by [13, Prop. 3.7]. Since $S \cap M_i$ is summand of S, then $S \cap M_i$ is closed in S, but S is closed in M therefore $S \cap M_i$ is closed in M by Prop. 1.4. But $S \cap M_i \subseteq M_i$ then $S \cap M_i$ is closed in M_i but M_i is supplement extending module for each $i \in I$. Thus, $\oplus_i (S \cap M_i) = S$ is supplement submodule in M by [5, lemma 2.2], and hence M is supplement extending module.

Proposition3.4: Let M and N be supplement extending modules such that annM+annN=R then M⊕N is Supplement extending.

Proof: Let A be a closed submodule in M⊕N. Since annM+annN=R, then by the same way of the proof [12, Prop. 4.2, ch1], A=C⊕D where C and D are submodule of M and N respectively. Since A≠0, then either C≠0 or D≠0. If C≠0 and D=0, then A is submodule of M, but M is supplement extending by Prop. 2.2 A is supplement submodule in M such that there exists a submodule B of M such that A+B=M and $A \cap B \ll A$. And M is submodule of M⊕N then by Lemma 1.7, A is supplement submodule in M⊕N and hence M⊕N is supplement extending as same as when A=0. Now, let C≠0 or D≠0, then A=C⊕D, clear that C and D are closed in M and N respectively, but M and N are supplement extending modules. So, C and D are supplements in M and N respectively, then by Prop 1.8 A=C⊕D is supplement in M⊕N, and hence M⊕N is Supplement extending.

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