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Quasi Subordination Properties of Bi-Univalent Functions By Using Generalized Srivastava-Attiya Operator

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Abstract

In this paper, we conduct an investigation into a novel subclass of bi-univalent functions within the unit disk. The introduced subclass incorporates the generalized Srivastava-Attiya operator and satisfies specific quasi-subordination conditions. Through this study, the researchers determine the coefficient estimates $|a_2|$ and $|a_3|$ for functions within these subclasses and unveil new results by applying the operator to this particular subclass. The implications of these findings extend to complex analysis, number theory, and other branches of mathematics. Overall, this research significantly contributes to the theory of quasi-subordination of bi-univalent functions and enhances the understanding of the diverse applications of the Generalized Srivastava-Attiya Operator.

Keywords : Analytic functions, Bi-Univalent functions, subordinations, Univalent functions.

خصائص شبه التبعية للدوال ثنائية التكافؤ باستخدام مؤثر سريفاستافا–أتيا المعمم

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الخلاصة

في هذه المقالة ، يبحث المؤلفون في فئة فرعية جديدة من الدوال ثنائية التكافؤ في قرص الوحدة . قدم الباحثون هذه الفئة الفرعية ، والتي تتضمن عامل سريفاستافا–عطية المعمم وتفي بشروط شبه التبعية. يحددون تقديرات المعامل $|a_2|$ و $|a_3|$ في هذه الفئات الفرعية والحصول على نتائج جديدة من خلال تطبيق هذا العامل على الفئة الفرعية. نتائج هذه الدراسة لها آثار كبيرة على التحليل المعقد ، ونظرية الأعداد ، ومجالات أخرى من الرياضيات. بشكل عام ، يساهم هذا البحث في نظرية شبه التبعية للدوال ثنائية التكافؤ ويوسع فهم تطبيقات عامل سريفاستافا–عطية المعمم.

1. Introduction

Denote the class of normalized functions that meet the requirement $F(0) = F'(0) - 1 = 0$ by using the letter, $\mathcal{A}$, and they are represented by the following Taylor expansion:

$$F(z) = z + \sum_{n=2}^{\infty} a_n z^n , \quad z \in \mathfrak{U} . \tag{1}$$

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These functions possess the property of analyticity in the open unit disk $\mathfrak{U}$, which is defined as the region in the complex plane $\mathbb{C}$ where $\mathfrak{U} = \{z \in C \ and \ |z| < 1 \}$.
Further, let $\mathcal{H}$ denote the class of all functions in $\mathcal{A}$ that are univalent in $\mathfrak{U}$.

The image of $\mathfrak{U}$ under each univalent function $F \in \mathcal{H}$ comprises a circle of radius F^{-1} has inverse F, according to the Koebe one-quarter theorem [1] . Therefore, each univalent function $\frac{1}{4}$ is defined by
$F^{-1}(F(z)) = z, (z \in \mathfrak{U})$
and
$F(F^{-1}(w)) = w \quad , \left(w < \rho_0(F);\ \rho_0(F) \geq \frac{1}{4}\right),$
where

$$G(w) = F^{-1}(w) = w - a_2w^2 + (2a_2^2 - a_3)w^3 - (5a_2^2 - 5a_2a_3 + a_4)w^4 + \cdots. \qquad (2)$$

If both of functions F and $F^{-1}$are univalent in $\mathfrak{U}$, $F \in \mathcal{A}$ is recognized as having bi-univalent functions, , provide an indication of the class of normalized by (1) bi-univalent functions in $\mathfrak{U}$ denoted by Σ .

Let F and G be two analytic functions in $\mathfrak{U}$. If there exists a Schwarz function η(z) in $\mathfrak{U}$ such that $|\eta(z)| < 1$ and $\eta(0) = 0$ (for $z \in \mathfrak{U}$) satisfying the following criterion, then the function F is said to be subordinate to G.
$F(z) = G(\eta(z)), \quad z \in \mathfrak{U}$.
The symbolism of this subordination $\quad F \prec G \ , \quad z \in \mathfrak{U}$.
If G is a univalent function in , then $F \prec G \iff F(0) = G(0) \quad \wedge \quad F(\mathfrak{U}) \subset G(\mathfrak{U})$.
Lewin [2] investigated the class $\sum$ of bi-univalent functions and discovered a coefficient constraint given by $|a_2| \leq 1.51$ for each $F \in \sum$. Building on Lewin's work [2] , Clunie and Brannan[3] made the assumption that $|a_2| \leq 2$ for all $F \in \sum$. Furthermore, research on bi-univalent and analytic functions has recently experienced a resurgence due to Srivastava et al. [4], preceded by Bulut [5] , Guney et al. [6] , Wanas and Srivastava [7] , and other works (refer to [8-16]).
For $F \in \mathcal{A}$ in Eq. (1) and $G \in \mathcal{A}$ defined as follows:

$$G(z) = z + \sum_{n=2}^{\infty} \mathfrak{b}_n z^n \ , \quad (z \in \mathfrak{U}). \qquad (3)$$

The form is known as the convolution between function F and G.

$$(F * G)(z) = z + \sum_{n=2}^{\infty} a_n \mathfrak{b}_n z^n \ , \quad (z \in \mathfrak{U}). \qquad (4)$$

For $F \in \mathcal{A},$ the the generalized Srivastava - Attiya operator [17] $\mathcal{J}^{s,\alpha,\lambda}_{(\lambda_p),(\mu_q),b} : \mathcal{A} \to \mathcal{A}$ is defined by

$$\mathcal{J}^{s,\alpha,\lambda}_{(\lambda_p),(\mu_q),b} F(z) = \zeta^{s,\alpha,\lambda}_{(\lambda_p),(\mu_q),b}(z) * F(z) \qquad (5)$$

such that $\zeta^{s,\alpha,\lambda}_{(\lambda_p),(\mu_q),b}(z)$ is defined by

$$\zeta^{s,\alpha,\lambda}_{(\lambda_p),(\mu_q),b}(z) = \frac{\lambda \prod_{j=1}^{q}(\mu_j)(1+\alpha)^s \Gamma(s).\Lambda\left[1+\alpha, ç, s, \lambda\right]^{-1}}{\prod_{j=1}^{p}(\lambda_j)}$$

$$.\left[\Phi^{(1,\dots,1,\,1,\dots,\,1)}_{\lambda_1,\dots,\,\lambda_p;\,\mu_1,\dots,\mu_q}\left(z, s, \alpha; \lambda,\ ç\right) - \frac{\alpha^{-s}}{\lambda\,\Gamma(s)}\Lambda(\alpha, ç, s, \lambda)\right]$$

$$= z + \sum_{n=2}^{\infty} \frac{\prod_{j=1}^{p}(1+\lambda_j)_{n-1}}{\prod_{j=1}^{q}(1+\mu_j)_{n-1}} \left(\frac{\Lambda(\alpha+n,b,s,\lambda)}{\Lambda(\alpha+1,b,s,\lambda)}\right) \left(\frac{\alpha+1}{\alpha+n}\right)^{s} \frac{z^n}{n!}, \tag{6}$$

where

$$\Lambda(\alpha+n,b,s,\lambda) = \mathcal{H}_{0,2}^{2,0}\left[b^{\frac{1}{\lambda}}(n+\alpha)\left|\overline{(s,1),\left(0,\frac{1}{\lambda}\right)}\right.\right]. \tag{7}$$

Linking (5) and (6), we obtain

$$\mathcal{J}_{(\lambda_p),(\mu_q),b}^{s,\alpha,\lambda} F(z) = z + \sum_{n=2}^{\infty} \Psi_{n,\alpha}\ \alpha_n z^n . \tag{8}$$

where

$$\Psi_{n,\alpha} = z + \sum_{n=2}^{\infty} \frac{\prod_{j=1}^{p}\left(1+\lambda_j\right)_{n-1}}{\prod_{j=1}^{q}\left(1+\mu_j\right)_{n-1}} \left(\frac{\Lambda(\alpha+n,b,s,\lambda)}{\Lambda(\alpha+1,b,s,\lambda)}\right) \left(\frac{\alpha+1}{\alpha+n}\right)^{s} \alpha_n \frac{z^n}{n!} .$$

$(\lambda_j \in \mathbb{C}(j=1,\dots,p); \mu_j \in \mathbb{C}\backslash z_0^{-}(j=1,\dots,q); z \in U; P \leq q+1;$
$min\{\mathcal{R}(\alpha),\mathcal{R}(s)\} > 0; \lambda > 0$ when $\mathcal{R}(b) > 0$ and $s \in \mathbb{C}; \alpha \in \mathbb{C}\backslash z_0^{-}$when $b = 0$.

Robertson established the notion of quasi-subordination in 1970 [18]. Furthermore, if $F(z)$ and $G(z)$be two analytic function , a function $F(z)$ is quasisubordination to a function $G(z)$ in $\mathfrak{U}$ and may be expressed as the form

$$F(z) \prec_q G(z) \qquad (z \in \mathfrak{U}),$$

if there is $\eta(z)$ and $r(z)$ a function analytic with $\eta(0) = 0$, $|\eta(z)| < 1$ and $|r(z)| \leq 1$ such that

$$r(z)G(\eta(z)) = F(z) \qquad (z \in \mathfrak{U}).$$

We can note that , when $r(z) = 1$, then $G(\eta(z)) = F(z)$, so

$$F(z) \prec G(z) \text{ in } .$$

Furthermore, if we replace $F(z) = z$, we obtain $F(z) = G(z)r(z)$. In this case, $F(z)$ is referred to as being majorized by $G(z)$, which can be written as:

$$F(z) \ll G(z) \text{ in } .$$

In case, $F(z) \prec_q G(z) \Rightarrow F(z) = r(z)G(z) \Rightarrow F(z) \ll G(z). z \in \mathfrak{U}$.

As a result, quasi-subordination is evidently an extension of both subordination and majorization (see [19, 20]).

Orhan H, magesh N, Yamini J.[21] established the classes of biunivalent given by quasi subordination in 2017, and the coefficient boundaries were determined. Minda and Ma introduce and investigate the unified classe in [22] .

$$T(\Theta) = \left\{F \in \mathcal{A} : \frac{zF'(z)}{F(z)} \prec \Theta(z) : z \in \mathfrak{U}\right\}, \tag{9}$$

$$S(\Theta) = \left\{F \in \mathcal{A} : 1 + \frac{zF''(z)}{F'(z)} \prec \Theta(z) : z \in \mathfrak{U}\right\}. \tag{10}$$

Assuming that $\mathfrak{U}$ is a region in the complex plane , and $\Theta(z)$ is an analytic and univalent function in $\mathfrak{U}$, which satisfies the following conditions:

1- The real part of $\Theta(z)$ in $\mathfrak{U}$ is positive.
2- $\Theta(\mathfrak{U})$ is a starlike area with respect to 1 and symmetric with respect to the real axis.
3- $\Theta(0) = 1, \Theta'(0) > 0$.
4- The function $T(\Theta)$ and $S(\Theta)$ classes are symmetric with respect to the real axis and have a starlike area with respect to 1.
5- The function $T(\Theta)$ and $S(\Theta)$ classes are , respectively starlike of Minda-Ma type and convex of Minda-Ma type [22].

These conditions will be used in this research.

$$h(z) = h_0 + h_1 z + h_2 z^2 + h_3 z^3 + \cdots \qquad (11)$$

and

$$\Theta(z) = 1 + B_1 z + B_2 z^2 + B_3 z^3 + \cdots \qquad (12)$$

2. Coefficient bounds for the class $\sum_{(\lambda_p),(\mu_q),b}^{s,\alpha,\lambda}(\delta, \gamma, \beta, \Theta)$

Definition 2.1: Let $F \in \sum$ be a function given in (1) in the class $\sum_{(\lambda_p),(\mu_q),b}^{s,\alpha,\lambda}(\delta, \gamma, \beta, \Theta)$ and fulfills the following quasi subordination:

$$\left\{\delta \frac{z(\mathcal{M}_{(\lambda_p),(\mu_q),b}^{s,\alpha,\lambda}F(z))'}{(\mathcal{M}_{(\lambda_p),(\mu_q),b}^{s,\alpha,\lambda}F(z))} + (1-\beta)\frac{(\mathcal{M}_{(\lambda_p),(\mu_q),b}^{s,\alpha,\lambda}F(z))}{z} + \beta(\mathcal{M}_{(\lambda_p),(\mu_q),b}^{s,\alpha,\lambda}F(z))' + \gamma z(\mathcal{M}_{(\lambda_p),(\mu_q),b}^{s,\alpha,\lambda}F(z))''\right\} - 1 \prec_q \Theta(z) - 1, \qquad (13)$$

and

$$\left\{\delta \frac{w(\mathcal{M}_{(\lambda_p),(\mu_q),b}^{s,\alpha,\lambda}G(w))'}{(\mathcal{M}_{(\lambda_p),(\mu_q),b}^{s,\alpha,\lambda}G(w))} + (1-\beta)\frac{(\mathcal{M}_{(\lambda_p),(\mu_q),b}^{s,\alpha,\lambda}G(w))}{w} + \beta(\mathcal{M}_{(\lambda_p),(\mu_q),b}^{s,\alpha,\lambda}G(w))' + \gamma w(\mathcal{M}_{(\lambda_p),(\mu_q),b}^{s,\alpha,\lambda}G(w))''\right\} - 1 \prec_q \Theta(w) - 1, \qquad (14)$$

where (w and $z \in \mathfrak{U}, \delta \geq 1, \ \gamma \geq 0 \ and \ \beta \geq 1$).
For specific values of the parameters $\gamma, \delta, \beta, s, \alpha, \lambda, (\lambda_p), (\mu_q)$ and b we obtain new class.

Remark 2.2: For $\delta = 1$ a function $F \in \sum$, given in (1) is namely be in the class $\sum_{(\lambda_p),(\mu_q),b}^{s,\alpha,\lambda}(\gamma, \beta, \Theta)$

$$\left\{\frac{z(\mathcal{M}_{(\lambda_p),(\mu_q),b}^{s,\alpha,\lambda}F(z))'}{(\mathcal{M}_{(\lambda_p),(\mu_q),b}^{s,\alpha,\lambda}F(z))} + (1-\beta)\frac{(\mathcal{M}_{(\lambda_p),(\mu_q),b}^{s,\alpha,\lambda}F(z))}{z} + \beta(\mathcal{M}_{(\lambda_p),(\mu_q),b}^{s,\alpha,\lambda}F(z))' + \gamma z(\mathcal{M}_{(\lambda_p),(\mu_q),b}^{s,\alpha,\lambda}F(z))''\right\} - 1 \prec_q \Theta(z) - 1,$$

and

$$\left\{\frac{w(\mathcal{M}_{(\lambda_p),(\mu_q),b}^{s,\alpha,\lambda}G(w))'}{(\mathcal{M}_{(\lambda_p),(\mu_q),b}^{s,\alpha,\lambda}G(w))} + (1-\beta)\frac{(\mathcal{M}_{(\lambda_p),(\mu_q),b}^{s,\alpha,\lambda}G(w))}{w} + \beta(\mathcal{M}_{(\lambda_p),(\mu_q),b}^{s,\alpha,\lambda}G(w))' + \gamma w(\mathcal{M}_{(\lambda_p),(\mu_q),b}^{s,\alpha,\lambda}G(w))''\right\} - 1 \prec_q \Theta(w) - 1 ,$$

where w and $z \in \mathfrak{U}, \ \gamma \geq 0 \ and \ \beta \geq 1$with G is inverse function of$F$.

Remark 2.3: For $\beta = 1$ and $= 0$, a function $F \in \sum$, Given in Eq.(1) is namely be in the class $\sum_{(\lambda_p),(\mu_q),b}^{s,\alpha,\lambda}(\delta, \Theta)$ then the next condition is met.

$$\left\{\delta \frac{z(\mathcal{M}_{(\lambda_p),(\mu_q),b}^{s,\alpha,\lambda}F(z))'}{(\mathcal{M}_{(\lambda_p),(\mu_q),b}^{s,\alpha,\lambda}F(z))} + (\mathcal{M}_{(\lambda_p),(\mu_q),b}^{s,\alpha,\lambda}F(z))'\right\} - 1 \prec_q \Theta(z) - 1,$$

and

$$\left\{\delta\frac{w(\mathcal{M}^{s,\alpha,\lambda}_{(\lambda_p),(\mu_q),b}G(w))'}{(\mathcal{M}^{s,\alpha,\lambda}_{(\lambda_p),(\mu_q),b}G(w))}+(\mathcal{M}^{s,\alpha,\lambda}_{(\lambda_p),(\mu_q),b}G(w))'\right\}-1\prec_q\Theta(w)-1\ ,$$

where $w\ and\ z\ \in\mathfrak{U}\,,\delta\geq 1$.

Remark 2.4: For $=1,\ \beta\ =\ 0\ and\ \gamma=0$, a function $F\in\sum$, given by Eq.(1) is namely be in the class $\sum^{s,\alpha,\lambda}_{(\lambda_p),(\mu_q),b}(\Theta)$ then the following condition is met.

$$\left\{\frac{z(\mathcal{M}^{s,\alpha,\lambda}_{(\lambda_p),(\mu_q),b}F(z))'}{(\mathcal{M}^{s,\alpha,\lambda}_{(\lambda_p),(\mu_q),b}F(z))}+(\mathcal{M}^{s,\alpha,\lambda}_{(\lambda_p),(\mu_q),b}F(z))'\right\}-1\prec_q\Theta(z)-1\,,$$

and

$$\left\{\delta\frac{w\left(\mathcal{M}^{s,\alpha,\lambda}_{(\lambda_p),(\mu_q),b}G(w)\right)'}{\left(\mathcal{M}^{s,\alpha,\lambda}_{(\lambda_p),(\mu_q),b}G(w)\right)}+\left(\mathcal{M}^{s,\alpha,\lambda}_{(\lambda_p),(\mu_q),b}G(w)\right)'\right\}-1\prec_q\Theta(w)-1\,,$$

where $w\ and\ z\ \in\mathfrak{U}$

Remark 2.5: For $s=0,\alpha=0,\lambda=0,(\lambda_p)=0,(\mu_q)=0,and\ b=0$ a function $F\in\sum$, given by Eq.(1) is called $\sum(\delta,\gamma,\beta,\Theta)$, if meeting the next quasi-subordination requirements

$$\left\{\delta\ \frac{zF'(z)}{F(z)}+(1-\beta)\frac{F(z)}{z}+\beta F'(z)+\gamma zF''(z)\right\}-1\prec_q\Theta(z)-1\,,$$

and

$$\left\{\delta\frac{wG'(w)}{G(w)}+(1-\beta)\frac{G(w)}{w}+\beta G'(w)+\gamma wG''(w)\right\}-1\prec_q\Theta(w)-1\ ,$$

where ($z,w\ \in\mathfrak{U},\ 0\leq\beta\leq 1\quad and\ 0\leq\gamma\leq 1$).

To discover the estimated coefficient $|a_2|$ and $|a_3|$, the following Lemma must be investigated:

Lemma 2.6 :[9]. If $\lambda\ \in\mathbb{P}$, then $|\lambda_i|\ \leq\ 2$ for each i , where $\mathbb{P}$ is the collecation of each λ function analytic in $\mathfrak{U}$, for which $\Re,(z\ \in\mathfrak{U})$, where$\lambda(z)=\ 1+\lambda_1 z+\lambda_2 z^2+\lambda_3 z^3+\ldots$, $(z\ \in\mathfrak{U})$.

Theorem 2.7: If $F(z)$ given by Eq.(1) belongs to the class$\sum^{s,\alpha,\lambda}_{(\lambda_p),(\mu_q),b}(\delta,\gamma,\beta,\Theta)$, then

$$|a_2|\leq min\left\{\frac{|h_0|B_1}{|(2+\beta+2\gamma)\Psi_{2,\alpha}|}\,,\sqrt{\frac{4|h_0|(B_1+|(B_2-B_1)|}{|(1+2\delta+2\beta+6\gamma)\Psi_{3,\alpha}-\delta\Psi^2_{2,\alpha}|}}\right\},\tag{15}$$

$$|a_3|\leq min\left\{\frac{2B_1|h_0+h_1|}{|(1+2\delta+4\beta+6\gamma)\Psi_{3,\alpha}|}+\frac{|h_0^2|B_1^2}{|(2+\beta+2\gamma)^2\Psi^2_{2,\alpha}|}\,,\frac{2B_1|h_0+h_1|}{|(1+2\delta+4\beta+6\gamma)\Psi_{3,\alpha}|}+\frac{8|h_0|(B_1+|(B_2-B_1)|}{|(1+2\delta+2\beta+6\gamma)\Psi_{3,\alpha}-\delta\Psi^2_{2,\alpha}|}\right\},\tag{16}$$

where

$$\Psi_{2,\alpha} = \frac{\prod_{j=1}^{p}\left(1+\lambda_j\right)_{2-1}}{\prod_{j=1}^{q}\left(1+\mu_j\right)_{2-1}} \left(\frac{\Lambda(\alpha+2,b,s,\lambda)}{\Lambda(\alpha+1,b,s,\lambda)}\right) \left(\frac{\alpha+1}{\alpha+2}\right)^s \alpha_2 \frac{z^2}{2!}$$

and

$$\Psi_{3,\alpha} = \frac{\prod_{j=1}^{p}\left(1+\lambda_j\right)_{3-1}}{\prod_{j=1}^{q}\left(1+\mu_j\right)_{3-1}} \left(\frac{\Lambda(\alpha+3,b,s,\lambda)}{\Lambda(\alpha+1,b,s,\lambda)}\right) \left(\frac{\alpha+1}{\alpha+3}\right)^s \alpha_3 \frac{z^3}{3!} .$$

Proof.

Since $F \in \sum_{(\lambda_p),(\mu_q),b}^{s,\alpha,\lambda}(\delta,\gamma,\beta,\Theta)$ and $= F^{-1}$. Then there is analytic function $v, u \in \mathcal{A}$ such that $u : \mathfrak{U} \to \mathfrak{U}$ and $v : \mathfrak{U} \to \mathfrak{U}$ such that $(0) = u(0) = 0$, satisfying

$$\left\{ \delta \frac{z(\mathcal{M}_{(\lambda_p),(\mu_q),b}^{s,\alpha,\lambda}F(z))'}{(\mathcal{M}_{(\lambda_p),(\mu_q),b}^{s,\alpha,\lambda}F(z))} + (1-\beta)\frac{(\mathcal{M}_{(\lambda_p),(\mu_q),b}^{s,\alpha,\lambda}F(z))}{z} + \beta(\mathcal{M}_{(\lambda_p),(\mu_q),b}^{s,\alpha,\lambda}F(z))' + \gamma z(\mathcal{M}_{(\lambda_p),(\mu_q),b}^{s,\alpha,\lambda}F(z))'' \right\} - 1 \prec_q \mathrm{h}(z)(\Theta(v(z)) - 1), \tag{17}$$

and

$$\left\{ \delta \frac{w(\mathcal{M}_{(\lambda_p),(\mu_q),b}^{s,\alpha,\lambda}G(w))'}{(\mathcal{M}_{(\lambda_p),(\mu_q),b}^{s,\alpha,\lambda}G(w))} + (1-\beta)\frac{(\mathcal{M}_{(\lambda_p),(\mu_q),b}^{s,\alpha,\lambda}G(w))}{w} + \beta(\mathcal{M}_{(\lambda_p),(\mu_q),b}^{s,\alpha,\lambda}G(w))' + \gamma w(\mathcal{M}_{(\lambda_p),(\mu_q),b}^{s,\alpha,\lambda}G(w))'') \right\} - 1 \prec_q \mathrm{h(w)}(\Theta(u(w)) - 1). \tag{18}$$

Define the functions λ and r by

$$r(z) = \frac{1+v(z)}{1-v(z)} = 1 + r_1(z) + r_2 z^2 + r_3 z^3 + \cdots \quad . \tag{19}$$

$$\lambda(w) = \frac{1+u(w)}{1-u(w)} = 1 + \lambda_1(w) + \lambda_2 w^2 + \lambda_3 w^3 + \cdots \quad . \tag{20}$$

or equivalently

$$v(z) = \frac{r(z)-1}{r(z)+1} = \frac{1}{2}\left(r_1 z + \left(r_2 - \frac{r_1^2}{2}\right) z^2 + \cdots\right). \tag{21}$$

$$u(w) = \frac{\lambda(w)+1}{\lambda(w)+1} = \frac{1}{2}\left(\lambda_1 \mathrm{w} + \left(\lambda_2 - \frac{\lambda_1^2}{2}\right) \mathrm{w}^2 + \cdots\right). \tag{22}$$

Then $r(z)$ and $\lambda(w)$ analytic in$\mathfrak{U}$, with $\mathrm{r}(z) = \lambda(\mathrm{w}) = 1$. Since $\lambda(w)$ *and* $r(z)$ have positive real part in$\mathfrak{U}$, $\lambda_i(z) \le 2$ and $u_i(w) \le 2$, $i = 1,2, ...$

Substitute (21),(22) in (17), and (18), we have

$$\left\{ \delta \frac{z(\mathcal{M}_{(\lambda_p),(\mu_q),b}^{s,\alpha,\lambda}F(z))'}{(\mathcal{M}_{(\lambda_p),(\mu_q),b}^{s,\alpha,\lambda}F(z))} + (1-\beta)\frac{(\mathcal{M}_{(\lambda_p),(\mu_q),b}^{s,\alpha,\lambda}F(z))}{z} + \beta(\mathcal{M}_{(\lambda_p),(\mu_q),b}^{s,\alpha,\lambda}F(z))' + \gamma z(\mathcal{M}_{(\lambda_p),(\mu_q),b}^{s,\alpha,\lambda}F(z))'' \right\} - 1 \prec_q \mathrm{h}(z)(\Theta\left(\frac{r(z)-1}{r(z)+1}) - 1\right), \tag{23}$$

and

$$\left\{\delta \frac{w(\mathcal{M}^{s,\alpha,\lambda}_{(\lambda_p),(\mu_q),b}G(w))'}{(\mathcal{M}^{s,\alpha,\lambda}_{(\lambda_p),(\mu_q),b}G(w))} + (1-\beta)\frac{(\mathcal{M}^{s,\alpha,\lambda}_{(\lambda_p),(\mu_q),b}G(w))}{w} + \beta(\mathcal{M}^{s,\alpha,\lambda}_{(\lambda_p),(\mu_q),b}G(w))' + \gamma w(\mathcal{M}^{s,\alpha,\lambda}_{(\lambda_p),(\mu_q),b}G(w))''\right\} - 1 \prec_q \mathrm{h(w)}(\Theta\left(\frac{\lambda(w)+1}{\lambda(w)+1}) - 1\right). \quad (24)$$

Using (21) and (22) with each other accompanied by (11)and (12), it is obvious that

$$\mathrm{h}(z)(\Theta\left(\frac{r(z)-1}{r(z)+1}) - 1\right) = \frac{1}{2}h_0 r_1 B_1 z + \left(\frac{1}{2}h_1 r_1 B_1 + \frac{1}{2}h_0 B_1\left(r_2 - \frac{r_1^2}{2}\right) + \frac{1}{4}h_0 B_2 r_1^2\right) w^2 + \cdots, \quad (25)$$

$$\mathrm{h(w)}(\Theta\left(\frac{\lambda(w)-1}{\lambda(w)+1}) - 1\right) = \frac{1}{2}h_0 \lambda_1 B_1 \mathrm{w} + \left(\frac{1}{2}h_1 \lambda_1 B_1 \frac{1}{2}h_0 B_1\left(\lambda_2 - \frac{\lambda_1^2}{2}\right) + \frac{1}{4}h_0 B_2 \lambda_1^2\right) w^2 + \cdots . \quad (26)$$

From (23), (24) , (25) and (26), we get

$$(2+\beta+2\gamma)\Psi_{2,\alpha}a_2 = \frac{1}{2}h_0 B_1 r_1 , \quad (27)$$

$$(1+2\delta+2\beta+6\gamma)\Psi_{3,\alpha}a_3 - \delta\Psi_{2,\alpha}^2 a_2^2 = \frac{1}{2}h_1 r_1 B_1 + \frac{1}{2}h_0 B_1\left(r_2 - \frac{r_1^2}{2}\right) + \frac{1}{4}h_0 B_2 r_1^2 , \quad (28)$$

$$-(2+\beta+2\gamma)\Psi_{2,\alpha}a_2 = \frac{1}{2}h_0 B_1 \lambda_1 , \quad (29)$$

and

$$-(1+2\delta+2\beta+6\gamma)\Psi_{3,\alpha}a_3 - \delta\Psi_{2,\alpha}^2 a_2^2 + (2+4\delta+4\beta+12\gamma)\Psi_{3,\alpha}a_2^2 = \frac{1}{2}h_1\lambda_1 B_1 + \frac{1}{2}h_0 B_1\left(\lambda_2 - \frac{\lambda_1^2}{2}\right) + \frac{1}{4}h_0 B_2 \lambda_1^2 , \quad (30)$$

from (27) and (29), we get

$$a_2 = \frac{h_0 B_1 r_1}{2(2+\beta+2\gamma)\Psi_{2,\alpha}} = -\frac{h_0 B_1 \lambda_1}{2(2+\beta+2\gamma)\Psi_{2,\alpha}} , \quad (31)$$

it follows that

$$r = -\lambda \quad (32)$$

and

$$(2+\beta+2\gamma)^2\Psi_{2,\alpha}^2 a_2^2 = \frac{1}{4}h_0^2 B_1^2 (r_1^2 + \lambda_1^2). \quad (33)$$

Adding (28) and (30), we get

$$\left[(2+4\delta+4\beta+12\gamma)\Psi_{3,\alpha} - 2\delta\Psi_{2,\alpha}^2\right]a_2^2 = 2h_0 B_1(r_2+\lambda_2) + h_0(B_2 - B_1(r_1^2+\lambda_1^2)). \quad (34)$$

Applying Lemma2.6 , for the coefficients $r_1 r_2, \lambda_1$ *and* $\lambda_2$its follows from (32) and (33),getting

$$|a_2| \leq \frac{|h_0|B_1}{\left|(2+\beta+2\gamma)\Psi_{2,\alpha}\right|}, \tag{35}$$

and

$$|a_2| \leq \sqrt{\frac{4|h_0|(B_1+|(B_2-B_1)|}{|(1+2\delta+2\beta+6\gamma)\Psi_{3,\alpha}-\delta\Psi_{2,\alpha}^2|}} \tag{36}$$

a yield coefficient that$|a_2|$ in (15).
Subtracting(30) from (28) , we get

$$2(1+2\delta+2\beta+6\gamma)\Psi_{3,\alpha}(a_3-a_2^2) = 2h_0B_1r_1 + h_0B_1(r_2-\lambda_2). \tag{37}$$

Finding that by replacing from(33) and (34) and inserting (37) and using Lemma 2.6

$$|a_3| \leq \frac{2B_1|h_0+h_1|}{\left|(1+2\delta+4\beta+6\gamma)\Psi_{3,\alpha}\right|} + \frac{|h_0^2|B_1^2}{\left|(2+\beta+2\gamma)^2\Psi_{2,\alpha}^2\right|}, \tag{38}$$

$$|a_3| \leq \frac{2B_1|h_0+h_1|}{\left|(1+2\delta+4\beta+6\gamma)\Psi_{3,\alpha}\right|} + \frac{4|h_0|(B_1+|(B_2-B_1)|}{|(1+2\delta+2\beta+6\gamma)\Psi_{3,\alpha}-\delta\Psi_{2,\alpha}^2|} \tag{39}$$

Eq. (38) and (39) yields the estimatein(16).
The evidence now is completed.

Corollary 2.8: Let F be in class$\sum_{(\lambda_p),(\mu_q),b}^{s,\alpha,\lambda}(\gamma,\beta,\Theta)$. Then

$$|a_2| \leq min\left\{\frac{|h_0|B_1}{\left|(2+\beta+2\gamma)\Psi_{2,\alpha}\right|}, \sqrt{\frac{4|h_0|(B_1+|(B_2-B_1)|}{|(3+2\beta+6\gamma)\Psi_{3,\alpha}-\Psi_{2,\alpha}^2|}}\right\},$$

$$|a_3| \leq min\left\{\frac{2B_1|h_0+h_1|}{\left|(3+4\beta+6\gamma)\Psi_{3,\alpha}\right|} + \frac{|h_0^2|B_1^2}{\left|(2+\beta+2\gamma)^2\Psi_{2,\alpha}^2\right|}, \frac{2B_1|h_0+h_1|}{\left|(3+4\beta+6\gamma)\Psi_{3,\alpha}\right|} + \frac{8|h_0|(B_1+|(B_2-B_1)|}{\left|(3+2\beta+6\gamma)\Psi_{3,\alpha}-\delta\Psi_{2,\alpha}^2\right|}\right\},$$

where

$$\Psi_{2,\alpha} = \frac{\prod_{j=1}^{p}(1+\lambda_j)_{2-1}}{\prod_{j=1}^{q}(1+\mu_j)_{2-1}}\left(\frac{\Lambda(\alpha+2,b,s,\lambda)}{\Lambda(\alpha+1,b,s,\lambda)}\right)\left(\frac{\alpha+1}{\alpha+2}\right)^s \alpha_2\frac{z^2}{2!}$$

and

$$\Psi_{3,\alpha} = \frac{\prod_{j=1}^{p}(1+\lambda_j)_{3-1}}{\prod_{j=1}^{q}(1+\mu_j)_{3-1}}\left(\frac{\Lambda(\alpha+3,b,s,\lambda)}{\Lambda(\alpha+1,b,s,\lambda)}\right)\left(\frac{\alpha+1}{\alpha+3}\right)^s \alpha_3\frac{z^3}{3!}.$$

Corollary 2.9: Let$F$ be in class$\sum_{c,\kappa}^{\alpha,s}(\delta,\Theta)$.Then

$$|a_2| \leq min\left\{\frac{|h_0|B_1}{\left|(2+\beta+2\gamma)\Psi_{2,\alpha}\right|}, \sqrt{\frac{4|h_0|(B_1+|(B_2-B_1)|}{|(1+2\delta+2\beta+6\gamma)\Psi_{3,\alpha}-\delta\Psi_{2,\alpha}^2|}}\right\},$$

$$|a_3| \leq min\left\{\frac{2B_1|h_0+h_1|}{\left|(5+2\delta)\Psi_{3,\alpha}\right|} + \frac{|h_0^2|B_1^2}{\left|(3)^2\Psi_{2,\alpha}^2\right|}, \frac{2B_1|h_0+h_1|}{\left|(5+2\delta)\Psi_{3,\alpha}\right|} + \frac{8|h_0|(B_1+|(B_2-B_1)|}{\left|(3+2\delta)\Psi_{3,\alpha}-\delta\Psi_{2,\alpha}^2\right|}\right\},$$

where

$$\Psi_{2,\alpha}=\frac{\prod_{j=1}^{p}\left(1+\lambda_j\right)_{2-1}}{\prod_{j=1}^{q}\left(1+\mu_j\right)_{2-1}}\left(\frac{\Lambda(\alpha+2,b,s,\lambda)}{\Lambda(\alpha+1,b,s,\lambda)}\right)\left(\frac{\alpha+1}{\alpha+2}\right)^{s}\ \alpha_2\frac{z^2}{2!}$$

and

$$\Psi_{3,\alpha}=\frac{\prod_{j=1}^{p}\left(1+\lambda_j\right)_{3-1}}{\prod_{j=1}^{q}\left(1+\mu_j\right)_{3-1}}\left(\frac{\Lambda(\alpha+3,b,s,\lambda)}{\Lambda(\alpha+1,b,s,\lambda)}\right)\left(\frac{\alpha+1}{\alpha+3}\right)^{s}\ \alpha_3\frac{z^3}{3!}\ .$$

Corollary 2.10: Let F be in class$\sum_{c,\kappa}^{\alpha,s}(\Theta)$.Then

$$|a_2|\le min\left\{\frac{|h_0|B_1}{|(3)\Psi_{2,\alpha}|},\sqrt{\frac{4|h_0|(B_1+|(B_2-B_1)|}{|(5)\Psi_{3,\alpha}-\Psi_{2,\alpha}^2|}}\right\},$$

$$|a_3|\le min\left\{\frac{2B_1|h_0+h_1|}{|(7)\Psi_{3,\alpha}|}+\frac{|h_0^2|B_1^2}{|(3)^2\Psi_{2,\alpha}^2|},\frac{2B_1|h_0+h_1|}{|(7)\Psi_{3,\alpha}|}+\frac{8|h_0|(B_1+|(B_2-B_1)|}{|(5)\Psi_{3,\alpha}-\delta\Psi_{2,\alpha}^2|}\right\},$$

where

$$\Psi_{2,\alpha}=\frac{\prod_{j=1}^{p}\left(1+\lambda_j\right)_{2-1}}{\prod_{j=1}^{q}\left(1+\mu_j\right)_{2-1}}\left(\frac{\Lambda(\alpha+2,b,s,\lambda)}{\Lambda(\alpha+1,b,s,\lambda)}\right)\left(\frac{\alpha+1}{\alpha+2}\right)^{s}\ \alpha_2\frac{z^2}{2!}$$

and

$$\Psi_{3,\alpha}=\frac{\prod_{j=1}^{p}\left(1+\lambda_j\right)_{3-1}}{\prod_{j=1}^{q}\left(1+\mu_j\right)_{3-1}}\left(\frac{\Lambda(\alpha+3,b,s,\lambda)}{\Lambda(\alpha+1,b,s,\lambda)}\right)\left(\frac{\alpha+1}{\alpha+3}\right)^{s}\ \alpha_3\frac{z^3}{3!}\ .$$

Corollary2.11: Let F be in class $\sum(\delta,\beta,\gamma,\Theta)$ .Then

$$|a_2|\le min\left\{\frac{|h_0|B_1}{|(2+\beta+2\gamma)|},\sqrt{\frac{4|h_0|(B_1+|(B_2-B_1)|}{|(1+\delta+2\beta+6\gamma)|}}\right\},$$

$$|a_3|\le min\left\{\frac{2B_1|h_0+h_1|}{|(1+2\delta+4\beta+6\gamma)|}+\frac{|h_0^2|B_1^2}{|(2+\beta+2\gamma)^2|},\frac{2B_1|h_0+h_1|}{|(1+2\delta+4\beta+6\gamma)|}+\frac{8|h_0|(B_1+|(B_2-B_1)|}{|(1+\delta+2\beta+6\gamma)|}\right\}.$$

3. Conclusions

The study proposes is for understanding the properties and behavior of bi-univalent functions by utilizing the generalized Srivastava-Attiya operator and quasi-subordination principles. The determination of $|a_2|$ and $|a_3|$ for all bi-univalent functions using this new class of bi-univalent functions based on these operators, along with several new findings, further emphasizes the significance of this research. The obtained results hold potential for practical applications in mathematical physics, electrical engineering, and computer science. In conclusion, this study offers valuable insights into the behavior of bi-univalent functions and presents novel approaches to studying functions under quasi-subordination .

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