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The Optimal Harvesting Strategy of A Discretized Stage-Structured Prey-Predator Fractional Model with Crowley-Martin Functional Response

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Abstract:

This paper focuses on studying the dynamical behavior of a discrete-time stage structured prey-predator fractional model with Crowley-Martin type functional response. This study starts with the construction of a discretization of the fractional stage structured prey-predator model by using piecewise constant arguments process then we determine and set conditions that are required to achieve the local stability of all the equilibrium points of the considered system. The results of the study show that the discrete–fractional system may have one or two non-negative interior equilibrium points. One of them becomes locally asymptotically stable under certain conditions. Also, we present and extend the considered system to the optimal control problem to get maximum harvest profit strategy. For that, the Pontryagin's maximal principal to get the optimal solution is applied. Numerical simulation is given to verify the analytical results.

Keywords: Difference-equations, Stage structured model, Crowley-Martin, Optimal harvesting.

استراتيجيةالحصاد الأمثل لنموذج كسري متقطع نوع فريسة ومفترس للفئات العمرية مع استجابة كراولي – مارتن الوظيفية

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الخلاصة:

يركز هذا البحث على دراسة السلوك الديناميكي لنموذج متقطع كسري منظم من الفريسة والمفترس مع استجابة وظيفية من نوع Crowley–Martin. تبدأ هذه الدراسة ببناء النموذج الاحيائي (الفريسة–المفترس) منظم المرحلة الكسرية باستخدام عملية التحويل الثابتة، كذلك تم تحديد ووضع الشروط المطلوبة لتحقيق الاستقرار المحلي لجميع نقاط التوازن للنظام المدروس. أظهرت نتائج الدراسة أن النظام الجزئي المنفصل قد يحتوي على نقطتي توازن داخليين غير سالبين. تصبح أحداها مستقرَّة محليًا في ظل ظروف معينة. كذلك تم توسيع النظام المقترح الى مسالة التحكم المثلى للحصول على إستراتيجية مثلى لتعظيم الحصاد.تم تطبيق مبدأ Pontryagin الأعظم للحصول على الحل الأمثل. واخيرا تم إعطاء محاكاة عددية لتاكيد النتائج التحليلية وكذلك حل مسالة السيطرةالمثلى.

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1. Introduction:

The Biological mathematical modelling concerns with the interaction among two or more populations. The relationship of prey–predator species is well known and widely employed in the past and it becomes prominent and remarkable works nowadays. Scientists and many researchers have modified and analysed the prey–predator models by applying and employing many factors, namely time delay, disease, harvesting, functional response, scavenger etc., in their works, see, [1]–[6]. The dynamics of diffusive prey predator model was studied with different functional responses and stage-structured for the prey in [7]–[12]. One of the most efficient methods of obtaining precise information on the behaviour of the biological system is the difference equations due to have no overlap between successive generations and for which births occur in regular [13]and[14]. A discrete-time and fractional predator-prey system have widely investigated and studied in the last decades, see [12], [15]–[20] and references therein. Fractional order derivative becomes one of the central interdisciplinary subjects in physical, biological, sciences and engineering. In [21]–[23], the authors discussed the harvesting for a prey-predator model with stage-structured for the prey and investigated the stability concepts using different mathematical techniques. In [24], the authors studied and investigated a fractional-order prey-predator system with with Crowley-Martin functional response and constant harvesting rate. Their system is given as follows:

$$\left.\begin{array}{r} D_t^q(x_1(t)) = rx_2(t) - (1+b)x_1(t) \\ D_t^q(x_2(t)) = d_1x_1(t) - (d_2+h)x_2(t) - \frac{a_1x_2(t)x_3(t)}{(1+x_2(t))(1+x_3(t))} \\ D_t^q(x_3(t)) = \frac{a_2x_2(t)x_3(t)}{(1+x_2(t))(1+x_3(t))} - d_3x_3(t) \end{array}\right]. \quad (1)$$

Where $x_1(t)$, $x_2(t)$ and $x_3(t)$ are the density of immature prey species, mature prey species positive constants. The description and interpretation of them can be found in [24]. $D_t^q f(t)$ stands for Caputo fractional order derivatives.

The main objective of this paper is to study the dynamical behavior of the discretization of model (1) with Crowley -Martian functional response and we extend the consider model to the optimal harvesting, for more details about the using optimal control theory to get the optimal harvesting amount, see [25]–[28].

2. Discrete Fractional-Order Model:

In this section, the discretization process that is found in [29]–[31] is applied on the system (1) and the process is given as follows:

Let $x_{1,0}(t) = x_{1,0}$, $x_{2,0}(t) = x_{2,0}$ and $x_{3,0}(t) = x_{3,0}$ be the initial conditions for system (1), then the corresponding discrete time model with piecewise constant argument is:

$$\left.\begin{array}{r} D_t^q(x_1(t)) = rx_2(\left[\frac{t}{s}\right]s) - (1+b)x_1(\left[\frac{t}{s}\right]s) \\ D_t^q(x_2(t)) = d_1x_1(\left[\frac{t}{s}\right]s) - (d_2+h)x_2(\left[\frac{t}{s}\right]s) - \frac{a_1x_2(\left[\frac{t}{s}\right]s)x_3(\left[\frac{t}{s}\right]s)}{(1+x_2(\left[\frac{t}{s}\right]s))(1+x_3(\left[\frac{t}{s}\right]s))} \\ D_t^q(x_3(t)) = \frac{a_2x_2(\left[\frac{t}{s}\right]s)x_3(\left[\frac{t}{s}\right]s)}{(1+x_2(\left[\frac{t}{s}\right]s))(1+x_3(\left[\frac{t}{s}\right]s))} - d_3x_3(\left[\frac{t}{s}\right]s) \end{array}\right],$$

First, let $t \in [0, s) as \frac{t}{s} \in [0,1)$ then we get:

$$\left.\begin{array}{c} D_t^q(x_1(t)) = rx_{2,0} - (1+b)x_{1,0} \\ D_t^q(x_2(t)) = d_1x_{1,0} - (d_2+h)x_{2,0} - \dfrac{a_1x_{2,0}x_{3,0}}{(1+x_{2,0})(1+x_{3,0})} \\ D_t^q(x_3(t)) = \dfrac{a_2x_{2,0}x_{3,0}}{(1+x_{2,0})(1+x_{3,0})} - d_3x_{3,0} \end{array}\right]$$

And the solution of the new system is:

$$x_{1,1}(t) = x_{1,0} + J^q\left[rx_{2,0} - (1+b)x_{1,0}\right] = x_{1,0} + \frac{t^q}{q\Gamma(q)}\left[x_{2,0} - (1+b)x_{1,0}\right]$$

$$x_{2,1}(t) = x_{2,0} + J^q\left[d_1x_{1,0} - (d_2+h)x_{2,0} - \frac{a_1x_{2,0}x_{3,0}}{(1+x_{2,0})(1+x_{3,0})}\right]$$

$$= x_{2,0} + \frac{t^q}{q\Gamma(q)}\left[d_1x_{1,0} - (d_2+h)x_{2,0} - \frac{a_1x_{2,0}x_{3,0}}{(1+x_{2,0})(1+x_{3,0})}\right]$$

$$x_{3,1}(t) = x_{3,0} + J^q\left[\frac{a_2x_{2,0}x_{3,0}}{(1+x_{2,0})(1+x_{3,0})} - d_3x_{3,0}\right] = x_{3,0} + \frac{t^q}{q\Gamma(q)}\left[\frac{a_2x_{2,0}x_{3,0}}{(1+x_{2,0})(1+x_{3,0})} - d_3x_{3,0}\right].$$

Therefore, the discrete time model becomes as follows:

$$\left.\begin{array}{c} x_{1,n+1} = x_{1,n} + \frac{s^q}{q\Gamma(q)}\left[rx_{2,n} - (1+b)x_{1,n}\right] \\ x_{2,n+1} = x_{2,n} + \frac{s^q}{q\Gamma(q)}\left[d_1x_{1,n} - (d_2+h)x_{2,n} - \frac{a_1x_{2,n}x_{3,n}}{(1+x_{2,n})(1+x_{3,n})}\right] \\ x_{3,n+1} = x_{3,n} + \frac{s^q}{q\Gamma(q)}\left[\frac{a_2x_{2,n}x_{3,n}}{(1+x_{2,n})(1+x_{3,n})} - d_3x_{3,n}\right] \end{array}\right]. \quad (2)$$

Where $q \in (0,1) and\ s > 0$.

To get the fixed points of the discrete time system (2), we can solve the following algebraic system:

$$\left.\begin{array}{c} x_1 = x_1 + \frac{s^q}{q\Gamma(q)}\left[rx_2 - (1+b)x_1\right] \\ x_2 = x_2 + \frac{s^q}{q\Gamma(q)}\left[d_1x_1 - (d_2+h)x_2 - \frac{a_1x_2x_3}{(1+x_2)(1+x_3)}\right] \\ x_3 = x_3 + \frac{s^q}{q\Gamma(q)}\left[\frac{a_2x_2x_3}{(1+x_2)(1+x_3)} - d_3x_3\right] \end{array}\right].$$

It is clearly that the system (2) has at least three equilibrium points these points are:

1- The trivial fixed point $e_0 = (0,0,0)$ always exists.

2- The free predator fixed point $e_1 = (x_1{}^*, x_2{}^*, 0)$ exists if $x_2{}^* = \left(\frac{1+b}{r}\right)x_1{}^*$ and $r = \frac{d_2+h+bd_2+hb}{d_1}$. Therefore, the system (2) may have two positive equilibrium points.

3- The interior fixed point$e_2 = (x_1^*, x_2^*, x_3^*)$ exists if $\frac{d1r}{1+b} > d_2 + h$ and $\frac{a_1x_3^*}{(1+x_3^*)(\frac{d1r}{1+b} - d_2 - h)} > 1$,where $x_2^* = \frac{a_1x_3^*}{(1+x_3^*)(\frac{d1r}{1+b} - d_2 - h)} - 1$,$x_1^*$= $(\frac{r}{1+b})x_2^*$, $and\ x_3^*$is a positive root of the following equation:

$$x_3^{*2} + ax_3^* + b_1 = 0 \text{ , where } a = \frac{a_1(d_3) - a_1a_2 + a_2(\frac{d1r}{1+b} - d_2 - h)}{a_1(d_3)} \quad \text{and} \quad b_1 = \frac{a_2(\frac{d1r}{1+b} - d_2 - h)}{a_1(d_3)}.$$

3. Stability Analysis:

In this section, the properties of the local stability for the discretized fractional–order model (2) are determined. The stability of the system (2) is investigated based on the values of the roots (eigenvalues) of characteristic polynomial of variation matrix $J(x_1, x_2, x_3)$ evaluated at fixed point of system (2), so the variation matrix $J(x_1, x_2, x_3)$ of the system (2) at any point (x_1, x_2, x_3) is given by:

$$J(x_1, x_2, x_3) = \begin{bmatrix} 1 - W(1+b) & Wr & 0 \\ Wd_1 & 1 - W\left(d_2 + h + \frac{a_1 x_3}{(1+x_3)(1+x_2)^2}\right) & -W\left(\frac{a_1 x_2}{(1+x_2)(1+x_3)^2}\right) \\ 0 & W\left(\frac{a_2 x_3}{(1+x_3)(1+x_2)^2}\right) & 1 + W\left(\frac{a_2 x_2}{(1+x_2)(1+x_3)^2} - d_3\right) \end{bmatrix} \quad (3)$$

Where $W = \frac{s^q}{q\Gamma(q)}$.

Lemma (3.1): A fixed point $e^* = (x_1, x_2, x_3)$ of the system (2) is said to be a sink (locally asymptotically) stable if $|\xi_i| < 1$ for all i =1,2,3, source or unstable if $|\xi_i| > 1$ for all i=1,2,3 and non-hyperbolic fixed point if $|\xi_i| = 1$ for at least one i =1,2,3. Where ξ_i are the eigenvalues of characteristic polynomial of the Jacobian matrix evaluated at the fixed point e^*. The following theorem is helpful for the investigated of the nature dynamical behavior of system (2) at the fixed points.

Theorem(3.2): [13], [31]

1- Let $F(x) = x^2 + px + q$ be a polynomial of degree two. If $F(1) > 0, F(-1) > 0$ and $q < 1$. Then the roots of $F(x)$ inside the unit circle.

2- Let $F(x) = x^3 + p_1 x^2 + p_2 x + p_3$ be a polynomial of degree three. The roots of F inside the unit circle if and only if the following conditions hold:

i. $F(1) > 0$

ii. $F(-1) < 0$

iii. $|p_2 - p_3 p_1| < 1 - {p_3}^2$

Proof :See [13,31]

Theorem (3.3): The trivial fixed point e_0 of system (2) is:

i. A sink point if $(d_2 + h + bd_2 + bh) > rd_1$, $W \in (0, N_1)$, and $N_2 < r < N_3$, otherwise it is unstable point.

ii. A non-hyperbolic point if $W = N_1$ or $r = N_2$ or $r = N_3$.

Where $N_1 = \frac{2}{d_3}, N_2 = \frac{W(d_2+h+bd_2+bh)-(1+b+d_2+h)}{Wd_1}$, and $N_3 = \frac{1}{d_1}[(d_2 + h + bd_2 + bh + h) - \frac{2}{W}(1 + b + d_2 + h) + \frac{4}{W^2}]$.

Proof:

The Jacobian matrix at e_0 is given by:

$$J(e_0) = \begin{bmatrix} 1 - W(1+b) & Wr & 0 \\ Wd_1 & 1 - W(d_2 + h) & 0 \\ 0 & 0 & 1 - Wd_3 \end{bmatrix},$$

The characteristics polynomial equation is:

$F(\lambda) = (1 - Wd_3 - \lambda)(F_*(\lambda)) = 0$, where $F_*(\lambda) = \lambda^2 + p\lambda + q$

, $p = W(1 + b + d_2 + h) - 2$ and $q = 1 + W^2(d_2 + h + bd_2 + bh - rd_1) - W(1 + b + d_2 + h)$.

It is clear that the first eigenvalue $\lambda_1 = 1 - W\, d_3$ and the other eigenvalues are $\lambda_{2,3}$ which are the roots of $F_*(\lambda)$. Now if $W \in (0, N_1)$, then $0 < W < \frac{2}{d_3} \Leftrightarrow -2 < -Wd_3 < 0 \Leftrightarrow -1 < 1 - Wd_3 < 1$ then $|\lambda_1| < 1$.

If $d_2 + h + bd_2 + bh > rd_1$, then $W^2(d_2 + h + bd_2 + bh - rd_1) > 0$ this gives $1 + W(1 + b + d_2 + h) - 2 - W(1 + b + d_2 + h) + 1 + W^2(d_2 + h + bd_2 + bh - rd_1) > 0$. Therefore, $F_*(1) > 0$. If $r < N_3$, then $r < \frac{1}{d_1}[(d_2 + h + bd_2 + bh + h) - \frac{2}{W}(1 + b + d_2 + h) + \frac{4}{W^2}]$. Hence, $W^2 rd_1 < W^2(d_2 + h + bd_2 + bh - rd_1) - 2W(1 + b + d_2 + h) + 4$. Therefore, $F_*(-1) > 0$.

If $r > N_2$ then $r > \frac{W(d_2+h+bd_2+bh)-(1+b+d_2+h)}{Wd_1}$. Hence, $rWd_1 > W(d_2 + h + bd_2 + bh) - (1 + b + d_2 + h)$. Thus, $W^2(d_2 + h + bd_2 + bh - rd_1) - W(1 + b + d_2 + h) < 0$ .Therefore, $q < 1$. According to Theorem (3.2) part 1, the trivial fixed point e_0 is a sink point. . Otherwise e_0 is unstable. If $W = N_1$ or $r = N_2$ or $r = N_3$, e_0 is a non-hyperbolic point.

Theorem (3.4): The free predator fixed point e_1 is always a non-hyperbolic point.

Proof:

The Jacobian matrix at e_1 is given by:

$$J\,(e_1) = \begin{bmatrix} 1 - W(1+b) & W(\frac{d_2+h+d_2b+hb}{d_1}) & 0 \\ Wd_1 & 1 - W(d_2+h) & -W(\frac{a_1(\frac{1+b}{r})x_1{}^*}{(1+\left(\frac{1+b}{r}\right)x_1{}^*)}) \\ 0 & 0 & 1 + W(\frac{a_2\left(\frac{1+b}{r}\right)x_1{}^*}{\left(1+\left(\frac{1+b}{r}\right)x_1{}^*\right)} - d_3) \end{bmatrix},$$

The characteristics equation is then:

$$G(\lambda) = \left(1 + W\left(\frac{a_2\left(\frac{1+b}{r}\right)x_1{}^*}{\left(1+\left(\frac{1+b}{r}\right)x_1{}^*\right)} - d_3\right) - \lambda\right)(G_*(\lambda)) = 0\,.$$

Where $G_*(\lambda) = \lambda^2 + \varphi_1\lambda + \varphi_2$,

$\varphi_1 = W(1 + b + d_2 + h) - 2$, and $\varphi_2 = 1 - W(1 + b + d_2 + h)$.

We can easily see that $G_*(1) = 0$. Therefore, the $\lambda = 1$ is an eigenvalue of $J\,(e_1)$ and the free predator fixed point e_1 is a non-hyperbolic point.

To discuss the local stability of the positive point e_2 of the system (2), we have the next theorem.

Theorem (3.5): The positive fixed point e_2 is a locally stable if the following conditions hold:

1 − $K_1K_3K_4 + rd_1K_5 > K_1K_2K_5$,

2− $Z_1 < Z_2$,

3 − $|\rho_2 - \rho_3\rho_1| < 1 - \rho_3{}^2$.

Where, $Z_1 = 4W(K_1 + K_2) + 2W^2(K_2K_5 + rd_1 + K_1K_5) + W^3(K_1K_3K_4 + rd_1K_5)$, $Z_2 = 8 + 4WK_5 + 2W^2(K_1K_2 + K_3K_4) + W^3K_1K_2K_5$, $\rho_1 = W(K_1 + K_2 - K_5) - 3$, $\rho_2 = 2W(K_5 - K_2 - K_1) + W^2\big(K_3K_4 - K_2K_5 - rd_1 + K_1(K_2 - K_5)\big) + 3$ and $\rho_3 = W(K_1 + K_2 - K_5) + W^2(K_2K_5 - K_3K_4 - K_1(K_2 - K_5) + rd_1) + W^3(K_1K_3K_4 - K_1K_2K_5 + rd_1K_5) - 1$.

Proof:

The Jacobian matrix of system (3) at e_2 is:

$J\,(e_2) = \begin{bmatrix} 1 - WK_1 & Wr & 0 \\ Wd_1 & 1 - WK_2 & -WK_3 \\ 0 & WK_4 & 1 + WK_5 \end{bmatrix}$,

$K_1 = (1 + b),\ K_2 = (d_2 + h + \frac{a_1 x_3{}^*}{(1+x_3{}^*)(1+x_2{}^*)^2})$,

$K_3 = \left(\frac{a_1 x_2{}^*}{(1+x_2{}^*)(1+x_3{}^*)^2}\right)$,$K_4 = \left(\frac{a_2 x_3{}^*}{(1+x_3{}^*)(1+x_2{}^*)^2}\right)$, and

$K_5 = \left(\frac{a_2 x_2{}^*}{(1+x_2{}^*)(1+x_3{}^*)^2} - d_3\right)$.

Therefore, the characteristic polynomial equation of the Jacobian matrix is:

$P(\lambda) = \lambda^3 + \rho_1\lambda^2 + \rho_2\,\lambda + \rho_3 = 0$.

Now, by simple computation, we can get that if $rd_1K_5 + K_1K_3K_4 > K_1K_2K_5$, then $(1) > 0$.

We assume that $Z_1 < Z_2$ that is $4W(K_1 + K_2) + 2W^2(K_2K_5 + rd_1 + K_1K_5 + W^3(K_1K_3K_4 + rd_1K_5) < 8 + 4WK_5 + 2W^2K_1K_2 + 2W^2K_3K_4 + W^3K_1K_2K_5$, then with some computation steps, we can get $P(-1) < 0$. Finally, if condition (3) holds, we can apply Theorem 3.2 part (2), and we obtain that the fixed point e_2 is locally stable.

4. Optimal Harvesting:

In this section, we focus on examining how to maximize the profits from harvesting mature prey species which can be defined by:

$$J(h_k) = max \sum_{k=0}^{T-1}\left(c_1 h_k x_{2,k} - c_2 {h_k}^2\right) . \qquad (4)$$

Where c_1 represents the price for harvesting, $c_2{h_k}^2$ is the total cost and k represents the step of time. In this problem , the aim is to maximize (4) subject to the state- equations:

$$\left.\begin{aligned} x_{1,k+1} &= x_{1,k} + \frac{s^q}{q\Gamma(q)}\left[r x_{2,k} - (1+b)x_{1,k}\right] \\ x_{2,k+1} &= x_{2,k} + \frac{s^q}{q\Gamma(q)}\left[d_1 x_{1,k} - (d_2 + h_k)x_{2,k} - \frac{a_1 x_{2,k} x_{3,k}}{(1+x_{2,k})(1+x_{3,k})}\right] \\ x_{3,k+1} &= x_{3,k} + \frac{s^q}{q\Gamma(q)}\left[\frac{a_2 x_{2,k} x_{3,k}}{(1+x_{2,k})(1+x_{3,k})} - d_3\right] \end{aligned}\right]. \qquad (5)$$

The control harvesting variable $h_k$is subjected to the constrains $0 \le h_k \le h_{max}, k = 1,2,3 \ldots T$.

Theorem (4.1): Suppose that the optimal harvesting is given by h_k with the optimal state solutions $x_{1,k}, x_{2,k}$ and $x_{3,k}$, then the adjoint functions $\mu_{1,k}, \mu_{2,k}$ and $\mu_{3,k}$ exist and satisfy the following:

$$\mu_{1,k} = \mu_{1,k+1}\left(1 - \frac{s^q}{q\Gamma(q)}[1+b]\right) + \mu_{2,k+1}\left(\frac{s^q}{q\Gamma(q)} d_1\right),$$

$$\mu_{2,k} = c_1 h_k + \mu_{1,k+1}\left(\frac{s^q}{q\Gamma(q)} r\right) + \mu_{2,k+1}\left(1 - \frac{s^q}{q\Gamma(q)}\left[(d_2 + h_k) + \frac{a_1 x_{3,k}}{\left(1 + x_{2,k}\right)^2\left(1 + x_{3,k}\right)}\right]\right) + \mu_{3,k+1}\left(\frac{s^q}{q\Gamma(q)}\left[\frac{a_2 x_{3k}}{\left(1 + x_{2,k}\right)^2\left(1 + x_{3,k}\right)}\right]\right),$$

$$\mu_{3,k} = \mu_{3,k+1}\left(1 + \frac{s^q}{q\Gamma(q)}\left[\frac{a_2 x_{2,k}}{\left(1 + x_{2,k}\right)\left(1 + x_{3,k}\right)^2} - d_3\right]\right) - \mu_{2,k+1}\left(\frac{s^q}{q\Gamma(q)}\left[\frac{a_1 x_{2,k}}{\left(1 + x_{2,k}\right)\left(1 + x_{3,k}\right)^2}\right]\right).$$

With $\mu_{1,T} = \mu_{2,T} = \mu_{3,T} = 0$. In addition, the optimal solution is given by

$$h_k^* = \begin{cases} 0 & \frac{\left(c_1 - \mu_{2,k+1}\frac{s^q}{q\Gamma(q)}\right)x_{2,k}}{2c_2} < 0 \\ \frac{\left(c_1 - \mu_{2,k+1}\frac{s^q}{q\Gamma(q)}\right)x_{2,k}}{2c_2} & 0 < \frac{\left(c_1 - \mu_{2,k+1}\frac{s^q}{q\Gamma(q)}\right)x_{2,k}}{2c_2} \le h_{max} \\ h_{max} & h_{max} < \frac{\left(c_1 - \mu_{2,k+1}\frac{s^q}{q\Gamma(q)}\right)x_{2,k}}{2c_2} \end{cases}$$

Proof: The Hamiltonian function of the problem is given by:
$H_k = c_1 h_k x_{2,k} - c_2 {h_k}^2 + \mu_{1,k+1}\left(x_{1,k} + \frac{s^q}{q\Gamma(q)}\left[r x_{2,k} - (1+b)x_{1,k}\right]\right) + \mu_{2,k+1}\left(x_{2,k} + \frac{s^q}{q\Gamma(q)}\left[d_1 x_{1,k} - (d_2 + h)x_{2,k} - \frac{a_1 x_{2,k} x_{3,k}}{(1+x_{2,k})(1+x_{3,k})}\right]\right) + \mu_{3,k+1}\left(x_{3,k} + \frac{s^q}{q\Gamma(q)}\left[\frac{a_2 x_{2,k} x_{3,k}}{(1+x_{2,k})(1+x_{3,k})} - d_3 x_{3k}\right]\right)$. Then by applying Pontryagin's maximum principle [21], [32], the necessary conditions for the optimal control strategy are:

$\mu_{1,k} = \frac{\partial H_k}{\partial x_{1,k}} = \mu_{1,k+1}\left(1 - \frac{s^q}{q\Gamma(q)}[1+b]\right) + \mu_{2,k+1}\left(\frac{s^q}{q\Gamma(q)} d_1\right)$,

$$\mu_{2,k} = \frac{\partial H_k}{\partial x_{2,k}} = c_1 h_k + \mu_{1,k+1}\left(\frac{s^q}{q\Gamma(q)} r\right) + \mu_{2,k+1}\left(1 - \frac{s^q}{q\Gamma(q)}\left[(d_2 + h_k) + \frac{a_1 x_{3,k}}{\left(1+x_{2,k}\right)^2\left(1+x_{3,k}\right)}\right]\right) + \mu_{3,k+1}\left(\frac{s^q}{q\Gamma(q)}\left[\frac{a_2 x_{3k}}{\left(1+x_{2,k}\right)^2\left(1+x_{3,k}\right)}\right]\right),$$

$\mu_{3,k} = \frac{\partial H_k}{\partial x_{3,k}} = \mu_{3,k+1}\left(1 + \frac{s^q}{q\Gamma(q)}\left[\frac{a_2 x_{2,k}}{(1+x_{2,k})(1+x_{3,k})^2} - d_3\right]\right) - \mu_{2,k+1}\left(\frac{s^q}{q\Gamma(q)}\left[\frac{a_1 x_{2,k}}{(1+x_{2,k})(1+x_{3,k})^2}\right]\right)$,
with the optimality condition
$0 = \frac{\partial H_k}{h_k} = c_1 x_{2,k} - 2c_2 h_k - \mu_{2,k+1}\left(\frac{s^q}{q\Gamma(q)} x_{2,k}\right)$, we get the characterization of the optimal solution.

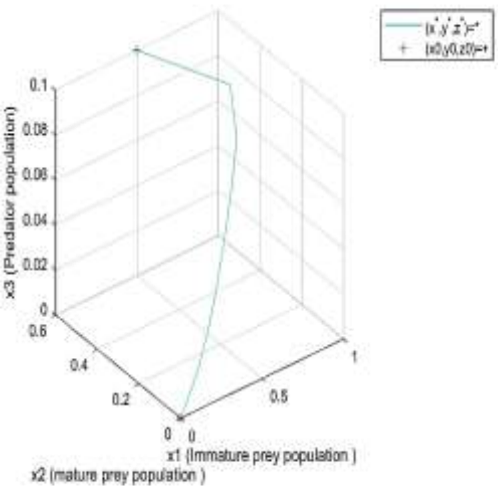
5. Numerical Simulation:

In this section, we give a numerical simulation that confirms our analytical results, especially the conditions of Theorems (3.3) and (3.5) in the section (3). Moreover, we solve the optimal control problem by using iterative method that can be found in [1,14,20,25,33] to get the optimal harvesting strategy on the mature prey species.

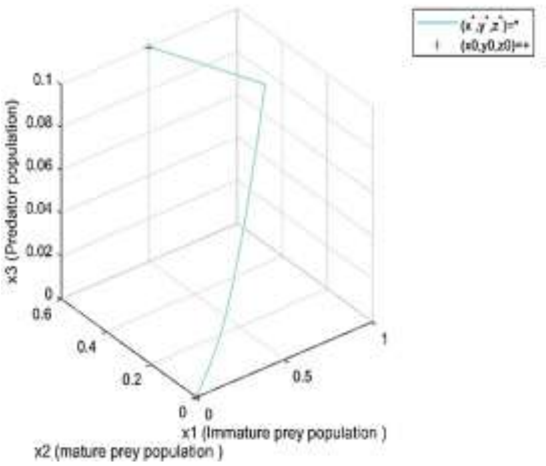
Example 1. In Table 1, we consider a set of parameters that confirm our analytical results for the equilibrium points e_0 and e_2, respectively. Figures 1-2 show the stability at different values of fractional order q of the equilibrium point e_0 and e_2, respectively.

Table 1: The parameter values at the equilibrium point $e_0$ande_2

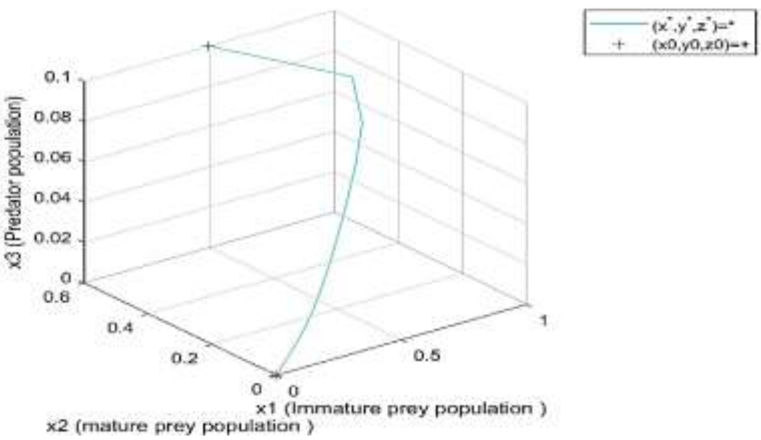
Parameter's value	At the fixed point e_0	At the fixed point e_2
d_1	0.1	0.16
b	0.1	0.88
h	0.03	0.18
d_3	0.5	0.2
r	2	7.78
a_2	0.5	0.73
d_2	0.3	0.42
a_1	0.6	0.79
$x_1(0)$	0.5	2.4
$x_2(0)$	0.6	0.5
$x_3(0)$	0.1	0.09
s	0.65	0.5
T	200	4000



(A)q=0.8

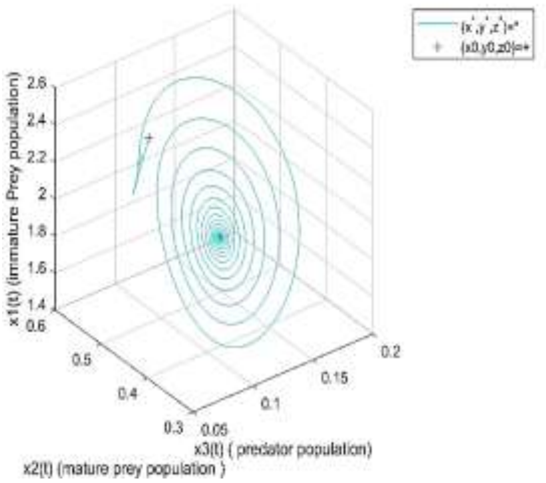


(B)q=0.9

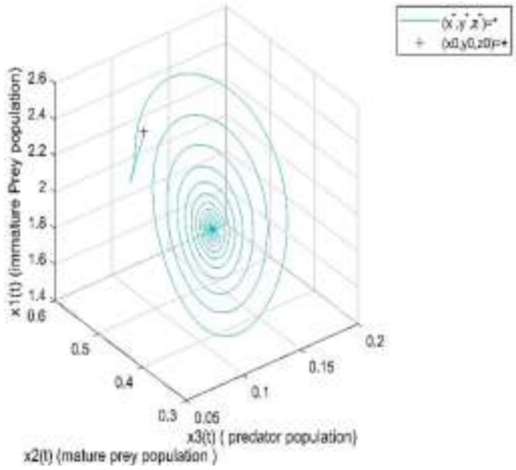


(C) q=0.98

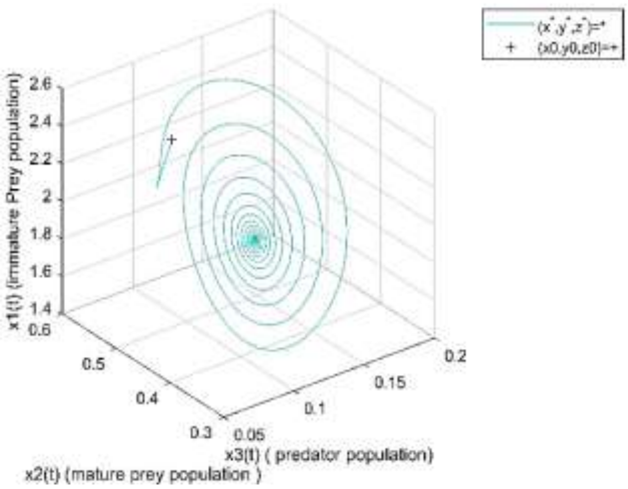
Figure 1: This figure represents the local stability of the trivial equilibrium point e_2 for system (2) for different values of q, namely: A) q=0.8,B)q=0.9and C)q=0.98with $s = 0.65$ and$T = 200$



(A) q=0.8



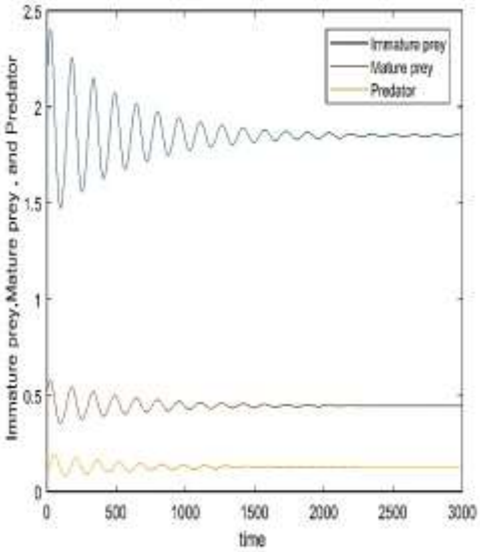
(B) q=0.9

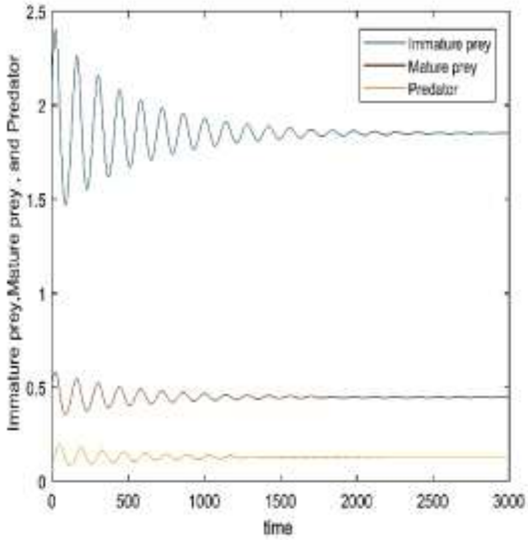


(C) q=0.98

Figure 2: This figure represents the local stability of the positive equilibrium point e_2 for system (2) for different values of q, namely: A)q= 0.8,B)q=0.9,and C)=0.98with $s = 0.5$ and$T = 4000$.

Also, in Figure (3),the dynamical of a discrete system (2) with time *T*, for q=0.8 and 0.98 at the positive equilibrium point e_2 is illustrated.





(A) q=0.8 (B) q=0.98

Figure 3: The dynamical of immature, mature prey, and predator populations of system (2) with time for q=0.8 and 0.98 at the equilibrium point $e_2$in(A) and (B) the blue line represents the immature prey populations, pink line represents the, mature prey populations and orange line represents the predator populations.

Example 2. We choose the values of parameters which are given in Table 2, with these values of parameters fractional order q =0.9, $T = 6000$, and s=0.05. For the control problems (5) and (6) with the initial conditions, and by using the iterative method which is founded in references [23], [32], [33]we obtain the total optimal harvesting $J = 0.5871$. So, Figures 5-6 represent the effect of optimal harvesting on the mature prey species and the optimal control variables a function of time illustrates in Figure 7.

Table 2: The parameter values at optimal harvesting

Parameter's	values
d_1	0.145
b	0.88
h	0.065
d_3	0.2
r	7.78
a_2	0.7
d_2	0.6
a_1	0.75
$x_1(0)$	2
$x_2(0)$	0.419
$x_3(0)$	0.0002
c_1	0.01
c_2	0.01

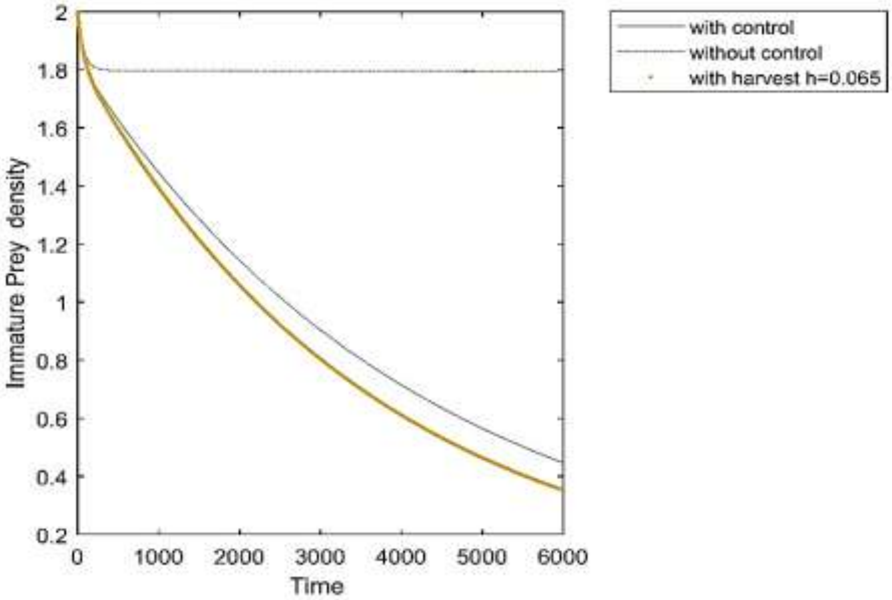


Figure 4: The trajectory with, without and constant harvesting 0.065 of the immature prey populations for the discrete system (2). The blue line represents the immature prey populations with control problem, the orange line represents the immature prey populations without harvesting, and pink line represents the immature prey populations with a constant harvesting 0.065.

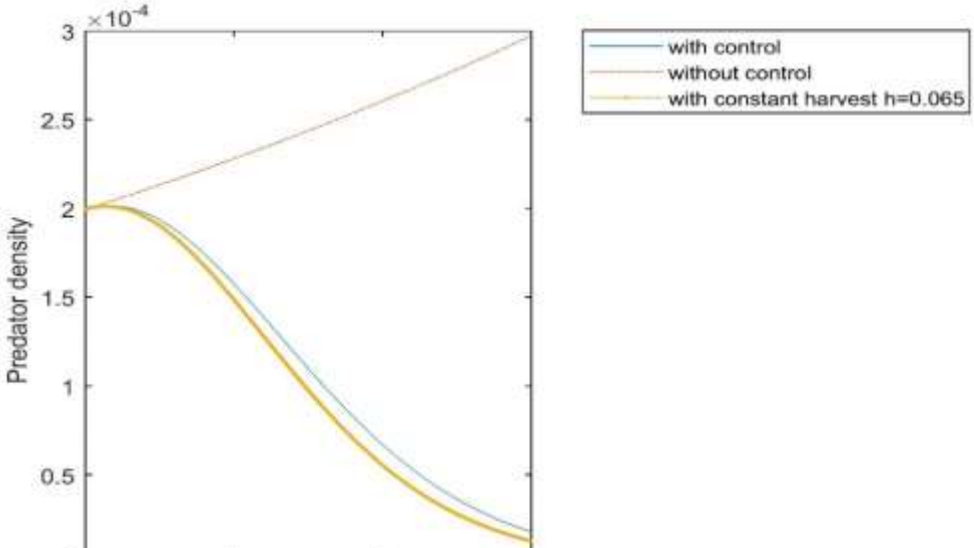


Figure 5: The trajectory with, without and constant harvesting 0.065 of the mature prey populations for the discrete system (2). The blue line represents the mature prey populations with control problem. the orange line represents the mature prey populations without harvesting, and pink line represents the prey populations with a constant harvesting 0.065.

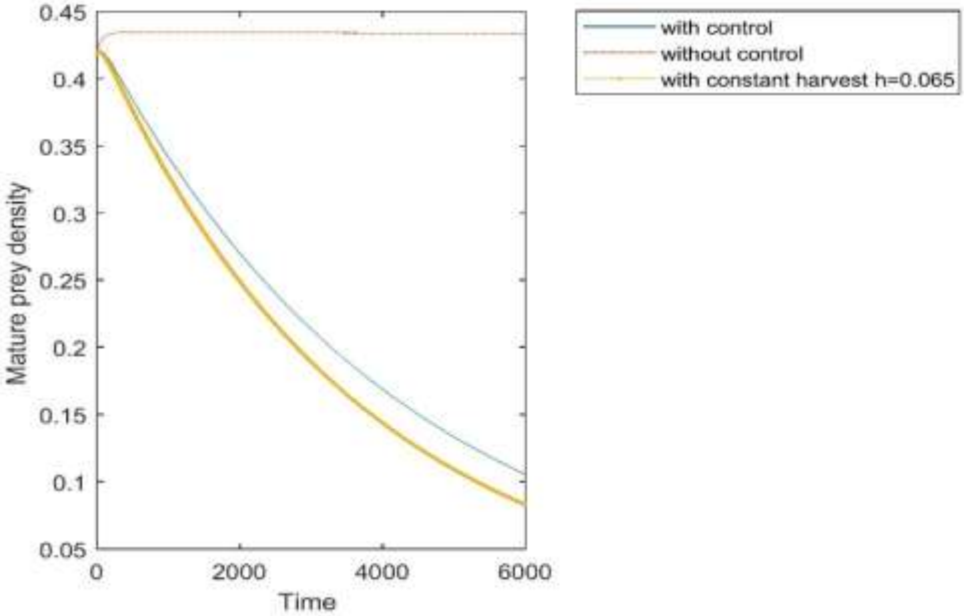


Figure 6: This figure represents the plotting of trajectory with, without and constant harvesting 0.065 of the predator populations for the discrete system (2) .The blue line represents the predator populations with control problem , the orange line represents predator populations without harvesting, and pink line represents the predator populations with a constant harvesting 0.065.

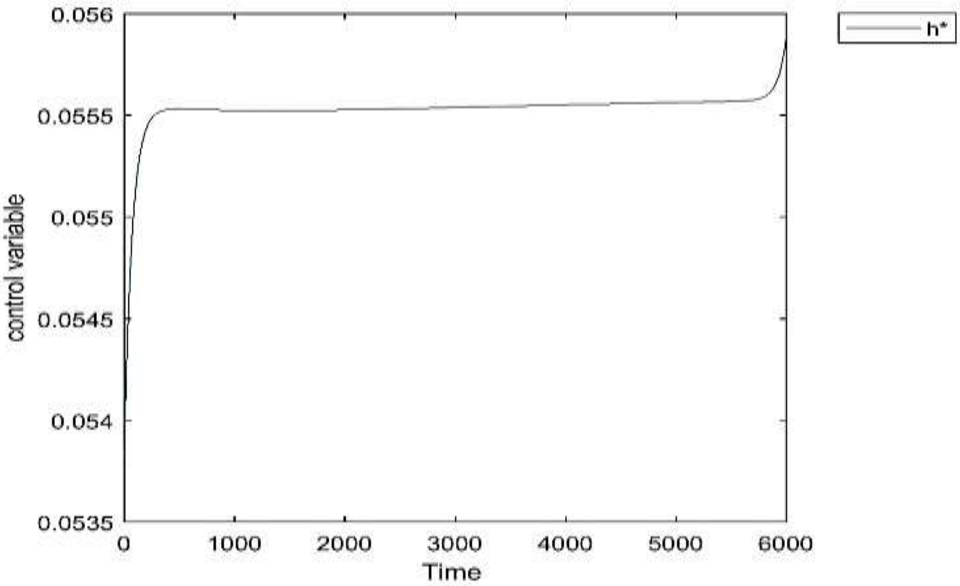


Figure 7: This figure represents the plotting of the effect of optimal harvesting for the discrete system (2) as a function of time.

6. Conclusions:

In this paper, the discrete system is extended to the optimal harvesting on mature prey species. In addition, the considered has at least three local stability at all three fixed points of the discrete system was presented according to Theorems (3.3) and (3.5). It found that Both step size s and fractional order q and the relation among the parameters in the system (2) have been influence on the properties of stability of the system. Also, the necessary conditions for the optimal control strategies were derived so, the ecological interpretation for the local stability of the interior equilibrium point system (2) was given that the ecosystem life under control for time. Finally, random values of the system's parameters have been selected to illustrate the stability of all three fixed points and optimal harvesting. These numerical values exactly meet the theoretical results.

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