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Supplement Continuous and Quasi-Supplement Continuous Modules

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Abstract
In this research, we introduce the concepts of P_i-module where (i=1,2,3), as proper generalizations of C_i-conditions, where (i=1,2,3). The relationships between these concepts and previous ones were explained, and a comprehensive description of each was provided. Additionally, by relying on P_i-module, new concepts were developed, such as supplement continuous and quasi-supplement continuous, which represent generalizations of continuous and quasi-continuous. Illustrative examples were provided, and the possibility of their inheritance was discussed.

Keywords: Extending Modules, Supplement Extending Modules, P_1-Modules, P_2-Modules, P_3-module, supplement continuous Modules, quasi-supplement continuous Modules.

المقاسات التكميلية المستمرة وشبه التكميلية المستمرة

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الخلاصة
في هذا البحث، نقدم مفاهيم المقاسات من نمط P_i- حيث (i=1,2,3)، والتي تمثل تعميمات فعلية لشروط C_i − حيث (i=1,2,3)، وتم شرح العلاقات بينها وبين المفاهيم السابقة وتم تقديم توصيف شامل لكل منها.بالإضافة الى ذلك ومن خلال الاعتماد على المقاسات من نمط $\boldsymbol{P_i}$ حيث (i=1,2,3)، استطعنا تقديم مفاهيم جديدة وهي المقاسات المكملة المستمرة والمقاسات شبة التكميلية المستمرة، والتي تمثل تعميمات فعلية حول المقاسات المستمرة والمقاسات شبة المستمرة كما تم تقديم أمثلة توضيحية ومناقشة إمكانية نقل هذه المفاهيم بطريقة وراثية.

1. Introduction

Throughout this paper all rings have an identity and the modules are unitary. A submodule S of an R-module H is said to be essential in H and denote by ($S \leq_e H$) if $S \cap D \neq (0)$, $\forall\, (0) \neq D \leq$ H,[1], p. 15]. A module H is said to be uniform, if $S \leq_e$ H, $\forall\, S \leq$ H [2], p. 37]. A submodule S of an R-module H is closed in H (denoted by $S \leq_c$ H), if $S \leq_e D \leq$ H then $S = D$, [1], p. 18]. A module H is called extending module if every submodule of H is essential in a direct summand of H [3]. There are many researchers who have made many generalizations about extending modules see [4] and [5]. The submodule S is called a supplement submodule of N in H if S+N = H and S∩ N ≪ S, [6]. H is called a supplement extending module if every submodule of H is essential in a supplement submodule in H [7]. Also, H is said to be supplement simple if (0) and H are the only supplement in H [8].

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R. Wisbauer mentioned and defined a local module H to be a module that has a proper submodule which contains all other proper submodules of H. Equivalently, H is called local if it is hollow and has a unique maximal submodule [[8], p.52]. An R-module H is said to be lifting if for every submodule S of H there exists a submodule A of S such that H = A$\oplus$ D and S$\cap$ D$\ll$H, where D submodule of H [9]. It can be proving that every local module is lifting. An R-module H is said to quasi-injective if it is injective with respect to itself [10]. An R-module H is said to be continuous, if it satisfies the conditions (C1): Every submodule of H is essential in a direct summand of H; and (C2): Every submodule of H which is isomorphic to a direct summand of H is a direct summand of H. Also, an R-module H is quasi-continuous if it satisfies the conditions (C1) and (C3): If two direct summands of H have zero intersection, then their sum is a direct summand of H. There are a lot of researchers who dealt with these concepts and made many studies about them, for example[3], [11] and [12].

2. P_1-module, P_2-module and P_3-module

Definition 2.1: An R-module H is said to be P_1-module if every submodule of H is essential in a supplement in H. Actually, this definition is the same definition of supplement extending modules which introduced by M. Tawfeek [8].
Definition 2.2: An R-module H is said to be P_2-module if each submodule of H which is isomorphic to a direct summand of H is a supplement in H.
Definition 2.3: An R-module H is said to be P_3-module if two direct summands of H have zero intersection, then their sum is a supplement in H.
Proposition 2.4: Every P_2-module is P_3-module.
Proof: Suppose that $H_1$and H_2 are summands of P_2-module H such that $H_1 \cap H_2$ = 0, we must show that $H_1 \oplus H_2$is supplement submodule in H. Now, since $H_1 \leq_{\oplus} H$, so there exists a submodule H_1^* of H such that H = $H_1 \oplus H_1^*$. Also take π to be projection map from H ontoH_1^* and $\pi|_{H_2}$be the restriction map of π on H_2, also by [[12], Proposition 2.2 we get $H_1 \oplus H_2$= $H_1 \oplus \pi H_2$and $\pi|_{H_2}$ isomorphic to H_1^*, but H is P_2-module. So, $\pi(H_2)$supplement in H. Now, since $\pi(H_2) \leq H_1^*$ implies $H_1^* = \pi(H_2) + H_1^*$,and since we have H =$H_1 \oplus H_1^*$ then H= $H_1 \oplus (\pi(H_2) + H_1^*) = (H_1 \oplus \pi(H_2)) + H_1^*$. Now since $H_1 \cap H_2 = 0$ and $H_1 \cap H_1^* = 0$, then we get $(H_1 \oplus H_2) \cap H_1^*$ =0 it flows that $H = (H_1 \oplus H_2) \oplus H_1^*$. that means $H_1 \oplus H_2$is direct summand of H and by the fact that every direct summand is supplement then we get $H_1 \oplus H_2$is supplement in H. Therefore, H is P_3-module.

Remarks and examples 2.5:
1. If an R-module H has (C_1) condition then it is P_1-module, since every summand is supplement. But the converse argument is not generally true for example H=$Z_8 \oplus Z_2$ it is P_1-module but not (C_1) condition.
2. If an R-module H has (C_2) condition then it is P_2-module.
Proof: Let S $\leq$H such that S$\cong$D where D$\leq_{\oplus}$ H. Since H has (C_2) then S$\leq_{\oplus}$ H and by every summand is supplement, we get that S is supplement submodule in H.
3. Clearly if an R-module H has (C_3) condition then it is P_3-module.
4. The converse of Proposition 2.4, does not generally hold, for example, Z as Z- module is P_3-module but not P_2-module. Since, 2Z$\cong$Z and Z is a direct summand of Z. But 2Z is not supplement of Z.
5. Clearly every semisimple module is P_i-module, $(i = 1,2,3)$. but the converse argument does not generally hold. For example, let H=$Z_8 \oplus Z_2$ as Z-module is P_1-module, Q as Z-module is P_2-module and, Z as Z-module is P_3-module, but they are not semisimple
6. Every uniform is P_i-module, $(i = 1,3)$.

Proof: Since every uniform module satisfies(C_1) and(C_3) conditions, by [[12] Proposition 2.5. P.20] and by (1) and (3), we get the required. But the converse argument is not generally true. For example, H=$Z_6 \oplus Z_3$ by (5) it is P_1-module and P_3-module but not uniform.

Now we will set some sufficient conditions to make the relation between the concepts are equivalent

Proposition 2.6: Let H be a lifting R-module, then H is (P_i)-module iff H has (Ci) condition.$(i = 1,2,3)$.

Proof:⟹) Case one: it is clear by [7].

Case two: Let S ≤H such that S≅D where $D \leq_{\oplus}$ H. Since H is P_2-module then S is supplement in H, since H is lifting, we get that $S \leq_{\oplus}$ H.

Case three: By using same technique in case two we complete the proof in Case three

⟸)By Remarks and examples 2.5, (1), (2) and (3).

Corollary 2.7: Let H be a local R-module. Then H is (P_i)-module iff H has (Ci) condition.$(i = 1,2,3)$.

Proof: Since every local is lifting module by [[8], Remark 2.2.7], then by Proposition 2.6 the result is verified.

Proposition 2.8: Let H be a semisimple R-module, then H is (P_i)-module iff H has (Ci) condition, where $(i = 1,2,3)$.

Proof: Since every submodule of H is direct summand and supplement then the proof hold.

The next results are partial answer about the question: when do P_i-modules, $(i = 1,2,3)$ have properties inherited by a submodule?

Proposition 2.9: Any direct summand of a P_1-module is again a P_1-module.

Proof: By [7].

Proposition 2.10: Any direct summand of a P_2 -module is again a P_2-module.

Proof: Let H be a P_2-module and S $\leq_{\oplus}$ H. Now let D and L are two isomorphic submodules of S such that D $\leq_{\oplus}$ S, then by [[13], Lemma 1.1.14], we get D $\leq_{\oplus}$ H, and since $D \cong L$ and $D \leq_{\oplus} H$, then by P_2 -module properties we get L is supplement submodule in H, hence, by [[9], p.235], we get L is supplement submodule in S. Therefore, S is P_2-module.

Proposition 2.11: Any direct summand of a P_3-module is again a P_3-module.

Proof: Let H be P_3-module and let S $\leq_{\oplus}$ H, and D $\leq_{\oplus}$ S and L $\leq_{\oplus}$ S such that $D \cap L = 0$. Now by [[13], Lemma 1.1.14], we get $D \leq_{\oplus}$ Hand $L \leq_{\oplus}$ H. Since H is P_3-module, thus $D \oplus L$ is supplement submodule in H. Now, with the same previous technique demonstrated in Proposition 2.10, the proof is done.

Proposition 2.12: If $H = H_1 \oplus H_2$, where H_1 and H_2 are submodules of H where H is P_3-module, and g is Homomorphism from H_1into $H_2$with ker g $\leq_{\oplus}$ H_1, then Im g $\leq_{\text{sup}} H_2$.

Proof: Let g: $H_1 \longrightarrow H_2$ be homomorphism

Case one: If g is monomorphism, then Ker g=0 implies Ker g $\leq_{\oplus} H_1$. Now to prove that Im g is supplement inH_2. For this let S={$h_1 + g(h_1)$: $h_1 \in H_1$}. We claim that $H = S \oplus H_2$, if $m \in H$, then $m = h_1 + h_2$ where $h_1 \in H_1$ and $h_2 \in H_2$. Therefor $m = h_1 + g(h_1) - g(h_1) + h_2 \in S + H_2$, and so $H = S + H_2$. Now, we show that $S \cap H_2 = \{0\}$, let $m \in S \cap H_2$ where $m = h_1 + g(h_1)$, for some $h_1 \in H_1$, hence $h_1 = m - g(h_1) \in H_1 \cap H_2 = \{0\}$, thus $m = 0$. Then $H = S \oplus H_2$ and S $\leq_{\oplus}$ H. Moreover, to show that $S \cap H_1 = \{0\}$, if m∈ $S \cap H_1$, then $m = h_1 + g(h_1)$ for some $h_1 \in H_1$. Consequently $g(h_1) = m - h_1 \in H_1 \cap H_2 = \{0\}$. So $g(h_1) = 0$, then g(0) =0. Since g is monomorphism, we have $h_1 = 0$ hence $m = 0$. Then $S \leq_{\oplus}$ H and $H_1 \leq_{\oplus}$ H and by H is P_3-module we get $S \oplus H_1$ is supplement in H. Finally, to show that $H_1 \oplus S = H_1 \oplus \text{Img}$, for $m \in$ Img then $m = g(h_1)$, where $h_1 \in H_1$and so, $m = h_1 - h_1 + g(h_1) \in S + H_1$,and, hence $H_1 \oplus S = H_1 \oplus \text{Img}$, and since $S \oplus H_1$is supplement in H, Img is supplement in H. Therefore, Img is supplement in H_2

Case two: If g is any homomorphism, with Ker g $\leq_{\oplus}$ H_1, to prove Img is supplement in H_2. Now, since Ker g $\leq_{\oplus}$ H_1, then there exists $D \leq H_1$, such that $H_1 = \ker g \oplus D$, then H = $H_1 \oplus H_2 = \ker g \oplus D \oplus H_2$, and the restriction map $g|_D: D \rightarrow H_2$ is monomorphism. By any direct summand of P_3-module is again P_3-module Proposition 2.11, and by case one we get Im g =Im $g|_D \leq_{sup} H_2$.

Corollary 2.13: If $H = H_1 \oplus H_2$, such that H_1 and H_2 submodule of H where H is P_3-module, and g is monomorphism from H_1 into H_2, then Im g $\leq_{sup} H_2$.

Proof: We deduce the proof from the above proposition.

Definition 2.14: An R-module H is called P_2 -module relative to S if any submodule $S_1 \leq S$ with $S_1 \cong D$ where $D \leq_{\oplus}$ H implies $S_1 \leq_{sup}$ S.

Proposition 2.15: Let $H = H_1 \oplus H_2$, such that H_1 and H_2 are submodules in H where H is P_3-module, then H_1 is a P_2-module relative to H_2 and H_2 is a P_2 -module relative to H_1.

Proof: Let $S \leq_{\oplus} H_1$ and $D \leq H_2$ such that $S \cong D$. suppose $\pi : H_1 \rightarrow S$ be the natural projection and g is an isomorphism from S into D. Hence $ig\pi$ is a homomorphism from H_1 into H_2, where i is inclusion from D into H_2. Clearly $\ker(ig\pi) = \ker(\pi) \leq_{\oplus} H_1$ by Proposition 2.12, Img(ig π) = D is supplement in H_2. Therefore, H_1 is a P_2-module relative to H_2. With the same technique we complete the proof that H_2 is a P_2-module relative to H_1.

The next corollary is a direct result of Proposition 2.15.

Corollary 2.16:

1. If $H \oplus H$ is a P_3-module, then H is a P_2-module.

2. $H = \oplus H_i, (i = 1,2,..n)$ is a P_3-module iff every H_i is a P_2-module.

Proposition 2.17: The conditions are equivalent for right R-module H.

1. H is a P_3-module.

2. If $S \leq_{\oplus} H$, $D \leq_{\oplus} H$ and $S \cap D \leq_{\oplus} H$, then $S + D \leq_{sup} H$.

Proof: (1) ⇒ (2) By hypothesis $S \cap D \leq_{\oplus} H$, $H = (S \cap D) \oplus K$, where $K \leq H$. Obviously, $S = (S \cap D) \oplus (S \cap K)$ and $D = (S \cap D) \oplus (D \cap K)$. By $S \leq_{\oplus} H$ and $(S \cap D) \leq_{\oplus} S$, we get $(S \cap K) \leq_{\oplus} H$ and by $D \leq_{\oplus} H$ and $(D \cap K) \leq_{\oplus} D$, we get $(D \cap K) \leq_{\oplus} H$. And $(S \cap K) \cap (D \cap K) = (S \cap D) \cap K = 0$, since H is P_3-module, so we get $(S \cap K) \oplus (D \cap K)$ supplement in H, and let $B = (S \cap K) \oplus (D \cap K)$ supplement in H. in fact B is satisfy summand condition, then we get $B \leq_{\oplus} H$, and $(S \cap D) \cap B = 0$, since H is P_3-module so we get $(S \cap D) \oplus B$ supplement in H. Now, $S + D = [(S \cap D) \oplus (S \cap K)] + [(S \cap D) \oplus (D \cap K)] = (S \cap D) \oplus (S \cap K) \oplus (D \cap K) = (S \cap D) \oplus B$ supplement in H.

(2) ⇒ (1) Clear.

3. Supplement continuous and quasi supplement continuous module

Definition 3.1: An R-module H is said to be supplement continuous, if it satisfies the condition of P_1-module and P_2-module.

Definition 3.2: An R-module H is said to be quasi-supplement continuous if it satisfies the condition of P_1-module and P_3-module.

Remarks and examples 3.3:

1. Every supplement continuous module is quasi- supplement continuous module, by Definition 3.1, and Proposition 2.4. But the converse argument is not generally true. For example, Z as Z-module is P_1-module and P_3-module. But not P_2-module, since $2Z \cong Z$ and $Z \leq_{\oplus}$ Z. But 2Z is not supplement of Z.
2. Every (quasi-)continuous is (quasi-)supplement continuous, by Remarks and examples 2.5, (1), (2), and (3).
3. Obviously, every uniform module H is quasi- supplement continuous, by Remarks and examples 2.5, (6).

4. Every quasi-injective module is (quasi-)supplement continuous. By [12]; Proposition 2.1] and by Remark and example 2.5, (1), (2) and (3). But the converse argument is not generally true. For example, Z as Z-module by (3) is quasi-supplement continuous but it is not quasi-injective

The Next is an illustrative diagram of the relationships between previous concepts:

$$\begin{array}{ccccccc} \text{Quasi-injective} & \Longrightarrow & \text{continuous} & \Longrightarrow & \text{quasi-supplement continuous} & \Longrightarrow & C_1 \\ \Downarrow & & \Downarrow & & \Downarrow & & \Downarrow \\ \text{supplement continuous} & & \Longrightarrow & & \text{quasi-supplement continuous} & \Longrightarrow & P_1\text{-module} \end{array}$$

The next lemma is a direct result of Proposition 2.11:

Lemma 3.4: Let $H=\bigoplus H_i,(i = 1,2,..n)$ if H is a quasi-supplement continuous then H_i is supplement continuous.

Proof: By Corollary 2.16, (2), we get H_i is P_2-module and by [7], we obtain H_i is P_1-module. Therefore, H_i is supplement continuous.

Proposition 3.5: Let H be lifting R-module, then H is (quasi-) continuous if and only if H is (quasi-)supplement continuous.

Proof: $\Longrightarrow$)By Remark and examples 3.3, (2).

$\Longleftarrow$)By Proposition 2.6.

A module H is said to be automorphism-invariant if f(H) ⊆ H for any automorphism f of the injective hull of H, [14] . A module H is referred to be pseudo-injective if for any submodule S in H, every monomorphism S → H can be extended to an endomorphism of the module H, [15] . It was shown that a module H is automorphism invariant iff it is pseudo-injective. In [14]. Also, every pseudo-injective module as well as every automorphism invariant module is a C_2[16]. If R is a prime ring, then all nonsingular automorphism-invariant right R-modules are quasi-injective. [[17]; Theorem 5]. Recall that a ring R is prime if aRb = 0 implies that a = 0 or b = 0.

Proposition 3.6: Let H be (local-)lifting pseudo-injective then the next concepts are equivalent:

1. H is continuous;
2. H is quasi- continuous;
3. H is extending;
4. H is supplement -continuous;
5. H is quasi- supplement continuous;
6. H is supplement extending.

Proof: (1) $\Longrightarrow$) (2) $\Longrightarrow$)(3) Clearly.

(3) $\Longrightarrow$)(4) Since H is extending then H is P_1-module and by [15], we get that H satisfy (C_2) condition, also by Corollaries 2.6 and 2.7, H is P_2-module, which implies H is supplement-continuous

(4) $\Longrightarrow$) (5) $\Longrightarrow$)(6) Clearly.

(6) $\Longrightarrow$) (1) Since H is supplement extending module then by [7], we get that H satisfies (C_1)condition, also by [15], then H is satisfied (C_2)condition, therefore, H is continuous.

Corollary 3.7: Let H be (local-)lifting automorphism invariant module then the next concepts are equivalent:

1. H is continuous;
2. H is quasi- continuous;
3. H is extending;
4. H is supplement -continuous;
5. H is quasi- supplement continuous;

6. H is supplement extending.

Proof: By [16], we deduce the proof from the above proposition.

Proposition 3.8: If H is an R-module such that H ⊕ H is P_3-module., then the next concepts are equivalents;
1. H is supplement continuous;
2. H is quasi-supplement continuous;
3. H is supplement extending.
Proof: (1)⟹) (2)⟹) (3) Clearly,
(3) ⟹) (1) Snice H is supplement extending, then H is P_1-module. Now, since H ⊕H is P_3-module, then by Corollary 2.16, (1), we get that H is a P_2-module. Therefore, H is supplement continuous.
Proposition 3.9: Let H be (local) lifting module, such that H ⊕ H is P_3-module then the next concepts are equivalent;
1. H is continuous;
2. H is quasi- continuous;
3. H is extending;
4. H is supplement -continuous;
5. H is quasi- supplement continuous;
6. H is supplement extending.

Proof: (1) ⟹) (2) ⟹)(3) Clearly.
(3) ⟹)(4) Since H is extending then H is P_1-module and by Corollary 2.16, (1), we get that H is a P_2-module, which implies H is supplement-continuous
(4) ⟹) (5) ⟹)(6) clearly
(6) ⟹)(1) Since H is supplement extending module then by [7], we get H is satisfied (C_1)condition, and by Corollary 2.16, (1), we get that H is P_2-module, and by Corollaries 2.6, and 2.7, then H is continuous.

Lemma 3.10: Let H be a supplement simple R-module. If H is a supplement extending, then it is uniform, [8].

Proposition 3.11: If a supplement simple module H is a P_1-module, then H is quasi-supplement continuous.

Proof: By Lemma 3.10 and by Remark and example 3.3, (3), H is quasi-supplement continuous.
From Propositions 2.9, 2.10, and 2.11, we answer in the next proposition the about question: when does a (quasi)-supplement continuous module have properties inherited by a submodule?

Proposition 3.12: If S direct summand of H (quasi)-supplement continuous module then S is (quasi)-supplement continuous modules.

4. Conclusions

Through this paper, we reached the next conclusions: any P_2-module is P_3-module but the converse argument does not generally hold. Additionally, we prove that each (quasi-)continuous and quasi-injective modules are (quasi)-supplement continuous. Further, we show that P_1 −modules, P_2-modules P_3-modules, and (quasi)-supplement continuous module are inherited by direct summand. Finally, we introduced some conditions that make the above concepts equivalent.

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