Determination of Timewise-Source Coefficient in Time-Fractional Reaction-Diffusion Equation from First Order Heat Moment

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Received: 18/2/2023 Accepted: 1/4/2023 Published: 30/3/2024

Abstract
This article aims to determine the time-dependent heat coefficient together with the temperature solution for a type of semi-linear time-fractional inverse source problem by applying a method based on the finite difference scheme and Tikhonov regularization. An unconditionally stable implicit finite difference scheme is used as a direct (forward) solver. While by the MATLAB routine lsqnonlin from the optimization toolbox, the inverse problem is reformulated as nonlinear least square minimization and solved efficiently. Since the problem is generally incorrect or ill-posed that means any error inclusion in the input data will produce a large error in the output data. Therefore, the Tikhonov regularization technique is applied to obtain stable and accurate results. Finally, to demonstrate the accuracy and effectiveness of our scheme, two benchmark test problems have been considered, and its good working with different noise levels.

Keywords: Implicit finite difference scheme (IFDS), Tikhonov technique, Caputo fractional derivative, Time-fractional source inverse problem, Stability analysis.

1. Introduction
The use of fractional partial differential equations to model many systems and processes is still ongoing and their applications are very wide [1, 2]. They have many types, the most

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important is the diffusion equation of fractional order that was used for the first time in physics [3] to describe the diffusion in media using fractal geometry. However, sometimes, a part of the data may not be given. These data are source terms, diffusion coefficients, initial data, boundary data, and e.t. Therefore, these data must be determined by additional information, and this is called fractional inverse problems. Work and study of direct problems of PDE have become widespread. In another hand, work on inverse problems is very little and more recent, articles on this aspect are few and very limited, see [4 – 11] for some examples of work to solve inverse problems.

In this article, we consider the time-fractional inverse source problem to reconstruct the unknown time-source coefficient c(t) in the following time-fractional reaction–diffusion equation:

\[ D_α^t u(x, t) = u_{xx} - a(t)u(x, t) + c(t)F(x, t, u), \quad (x, t) \in (0,1) \times (0, T). \]  

(1)

The initial condition

\[ u(x, 0) = φ(x), \quad x \in [0,1], \]  

(2)

and nonlocal boundary conditions;

\[ u(0, t) = u(1, t); \quad u_x(t, t) = 0, \quad t \in [0, T], \]  

(3)

and integral over determination condition

\[ \int_0^1 xu(x, t)dx = e(t), \quad 0 \leq t \leq T, \]  

(4)

where \( F(x, t, u), φ(x) \) and \( e(t) \) are given functions, \( a(t) \) is given positive function, \( c(t) \) is unknown time-dependent coefficient. Mathematically, Eq. (1) is a parabolic fractional partial differential equation where \( u(x, t) \) represents the concentration of one substance, \( u_{xx} \) is a diffusion term, \( a(t)u(x, t) \) + \( c(t)F(x, t, u) \) represents the reaction term where \( F(x, t, u) \) is a non-linear source term and \( a(t) > 0 \) can be regarded as a control parameter. \( D_α^t u(x, t) \) is the Caputo time-fractional derivative of order \( 0 < α < 1 \), [12]:

\[ D_α^t u = \frac{1}{Γ(1-α)} \int_0^t (t-s)^{-α} \frac{∂u(x,s)}{∂s} \ ds, \quad 0 \leq t \leq T, \]

The unique solvability for the inverse problem (1)-(4) is established in [12] and reads under the following assumptions:

(A1) \( a \in C[0, T] \) is a positive function and \( M_a = \| a \|_{C[0, T]} \).

(A2) \( φ \in C^4[0, 1] \) such that \( φ(0) = φ(1), φ′(1) = 0, φ''(0) = φ''(1), φ'''(1) = 0. \)

(A3) Let the function \( F(x, t, u) \) be continuous with respect to all arguments in \( (0,1) \times (0, T) \times R \) and satisfies the following conditions:

1. \( F(., t, u) \in C^4[0, 1], t \in [0, T], \quad F(x, t, u)|_{x=0} = F(x, t, u)|_{x=1}, \)
2. \( F_x(x, t)|_{x=1} = 0, \quad F_{xx}(x, t)|_{x=0} = F_{xx}(x, t)|_{x=1} = F_{xxx}(x, t)|_{x=1} = 0. \)
3. There exists a nonnegative function \( b(x, t) \) such that for each \( u, ū \in R \) and \( (x, t) \in (0,1) \times (0, T) \),

\[ \left| \frac{∂^r}{∂x^r} F(x, t, u) - \frac{∂^r}{∂x^r} F(x, t, ū) \right| \leq b(x, t)|u - ū|, \quad r = 0, 1, 2 \]

where \( b \in L^2((0,1) \times (0, T)], \quad \max \| b(., t) \|_{L^2((0,1))} < \infty; \)
4. \( M_F = \max \left\{ \left\| \frac{∂^r}{∂x^r} F(., ., u) \right\|_{L^2((0,1) \times (0, T])}; \quad r = 0, 1, 2. \right\} \)
5. There exists a positive constant \( F_m \) such that \( | \int_0^1 xF(x, t, u) \ dx | > F_m \) for each \( t \in [0, T] \) and uniformly to \( u \in R. \)

(A4) \( E \in C([0, T]) \) and \( E(0) = \int_0^1 xφ(x) \ dx. \)

In our article, a stable numerical solution to problem (1) - (4) should be obtained by using implicit finite difference (IFD) with Tikhonov regularization [29]. First, we apply (IFD) to obtain the direct solution to problem (1)- (3). Next, we use the Tikhonov regularization to stabilize this problem.

The article consists of six sections: (IFDS) is given in section 2 to obtain the numerical solution of problems (1)-(3). Section 3 is developed to investigate the stability and convergence of the numerical procedure. The numerical approach to solving (TFSIP) of equations (1)-(4) is given in section 4. In section 5, some experiments are provided. The end section gives the conclusions for this work.

2. Fractional Finite Differences Scheme (FFDS)

In this section, we give a direct solution method to problem (1)-(3).

We start by letting \( M \) and \( N \) are two positive integers. Consider a regular grid in \( I = [0,1] \times [0,T] \) as:
\[
\Omega = \{(x_i = ih, t_j = jk), j = 0,1,2, ..., N; i = 0,1,2, ..., M\},
\]

The step length in space and time are \( h = 1/M \) and \( k = T/N \), respectively. In addition, suppose
\[ u_i^j = u(x_i, t_j), \quad a_j = a(t_j), \quad c_j = c(t_j), \quad F_i^j = F(x_i, t_j, u(x_i, t_j)) \]

\[ \phi_i = \varphi(x_i), \quad E_j^j = E(t_j) \quad \text{for} \quad j = 0, 1, 2, ..., N; \quad i = 0, 1, 2, ..., M, \]

\[ u_{xx}(x_i, t_j) \approx \frac{u_{i-1}^j - 2u_i^j + u_{i+1}^j}{h^2} \quad \text{for} \quad j = 0, 1, 2, ..., N; \quad i = 1, 2, ..., M \]

and

\[ u_x(x_i, t_j) \approx \frac{u_{i-1}^j - u_{i+1}^j}{2h} \]

The discrete form of the term \( D^\alpha_t u \) that is introduced in [4] is defined as the following:

\[ D^\alpha_t u(x_i, t_j) \approx q_{\alpha,k} \sum_{k=1}^{j} \omega_j^a \left( u_i^{j-k+1} - u_i^{j-k} \right) \]  \hspace{1cm} (5)

where \( q_{\alpha,k} = \frac{\Gamma(\alpha)k^{\alpha}}{(1-\alpha)\Gamma(1-\alpha)} \).

We must note that \( \omega_j^a \) satisfies the following fact:

\[ \omega_j^a = (j)^{1-\alpha} - (j - 1)^{1-\alpha} \quad j = 1, 2, ..., N. \]

From equation’s (2) and (3), we obtain the following discrete form of boundary

\[ u_0 = u_M^1, \quad u_{M+1}^j = u_{M-1}^j \quad j = 0, 1, 2, ..., N, \]  \hspace{1cm} (6)

and the discrete initial condition; so as:

\[ u_i^0 = \varphi_i, \quad i = 0, 1, 2, ..., M. \]  \hspace{1cm} (7)

By using equations (5), (6) and (7), we get the following formula for governing equation (1):

\[ q_{\alpha,k} \sum_{k=1}^{j} \omega_j^a \left( u_i^{j-k+1} - u_i^{j-k} \right) = \gamma \left[ u_{i-1}^j - 2u_i^j + u_{i+1}^j \right] - a_ju_i^j + c_jF_i^j \]

where \( \gamma = \frac{1}{h^2} \), \( i = 1, 2, ..., M - 1 \).

\[ q_{\alpha,k} \sum_{k=1}^{j} \omega_j^a \left( u_i^{j-k+1} - u_i^{j-k} \right) = [\gamma u_{i-1}^j - (2\gamma + a_j)u_i^j + \gamma u_{i+1}^j] + c_jF_i^j \]

Or

\[ q_{\alpha,k} \omega_1^a (u_i^1 - u_i^0) + q_{\alpha,k} \sum_{k=2}^{j} \omega_k^a \left( u_i^{j-k+1} - u_i^{j-k} \right) = [\gamma u_{i-1}^j - (2\gamma + a_j)u_i^j + \gamma u_{i+1}^j] + c_jF_i^j \]  \hspace{1cm} (8)

We must note that \( \omega_j^a \) satisfies the following fact:

\[ 1 = \omega_1^a > \omega_2^a > \omega_3^a > \cdots \to 0, \quad j = 1, 2, ..., N. \]

We apply equation (8) when \( i = 1, 2, \ldots, M - 1, M \), we have:

- At the first-time level (\( j = 1 \)),

\[ -\gamma u_{i-1}^j + (2\gamma + a_j + q_{\alpha,k} \omega_j^a)u_i^j - \gamma u_{i+1}^j = c_1F_i^j + q_{\alpha,k} \omega_j^a \varphi_i, \]  \hspace{1cm} (9)

and from (8) and (6) at \( i = M \), we have:

\[ -2\gamma u_{M-1}^j + (2\gamma + a_j + q_{\alpha,k} \omega_j^a)u_M^1 = c_1F_M^1 + q_{\alpha,k} \omega_j^a \varphi_M \]  \hspace{1cm} (9a)

- At \( j = 2, 3, ..., N \),

\[ -\gamma u_{i-1}^j + \left( 2\gamma + a_j + q_{\alpha,k} \omega_j^a \right)u_i^j - \gamma u_{i+1}^j = c_jF_i^j + q_{\alpha,k} \sum_{k=1}^{j-1} (\omega_k^a - \omega_{k+1}^a)u_i^{j-k} + q_{\alpha,k} \omega_j^a \varphi_i \]  \hspace{1cm} (10)

and from (8) and (6) at \( i = M \), we have:

\[ -2\gamma u_{M-1}^j + \left( 2\gamma + a_j + q_{\alpha,k} \omega_j^a \right)u_M^1 = c_jF_M^1 + q_{\alpha,k} \sum_{k=1}^{j-1} (\omega_k^a - \omega_{k+1}^a)u_i^{j-k} + q_{\alpha,k} \omega_j^a \varphi_M \]  \hspace{1cm} (10a)
Using equations (9), (9a), (10) and (10a), for \( j = 1, \ldots, N \) and \( i = 1, 2, \ldots, M \), it can be written in more compact form:

\[
AU^j = c_j F^j + q_{\alpha,k} U^0 ,
\]

\[
AU^j = c_j F^j + q_{\alpha,k} D_j^{-1} , j = 2, \ldots, N
\]

where

\[
F^j = [F_1^j, F_2^j, \ldots, F_N^j]^T , j = 1, 2, \ldots, N
\]

where \( u^0 \) is computed from equation (7). Also, we have \( U^j = [u_1^j, u_2^j, \ldots, u_M^j]^t \), \( D_j^{-1} = \sum_{k=1}^{M-1} (w_{k+1}^a - w_k^a) u_{j-k}^1 + w_k^a u_0^j \).

Finally, discretize integral condition (4) by using the trapezoidal rule as:

\[
e(t_j) = \frac{1}{2M} \left( x_0 u_0^j + x_M u_M^j + 2 \sum_{k=1}^{M-1} x_k u_k^j \right) , j = 0, 1, 2, \ldots, N.
\]

### 3. Stability and Convergence

Here, we use the Von Neumann method [30] to prove the stability of the scheme (8). Let the solution of the equations (9) and (10) with \( \phi(x) \) be \( U \) and \( \bar{U} \) is the solution to the perturbed data \( \bar{\phi}(x) \).

Define the error \( E = \bar{U} - U, E^j = \bar{U}^j - U^j = (e_0^j, e_1^j, \ldots, e_M^j)^t \) i.e. \( e_i^j = \bar{u}_i^j - u_i^j, j = 0, 1, 2, \ldots, N, i = 0, 1, 2, \ldots, M \).

**Theorem 1.** The FDM scheme (8) is unconditionally stable.

**Proof:**

Assume \( e_i^j = \xi_j e^{i \beta h} \), where \( \beta \) is a real spatial number [31] and \( \bar{i} = \sqrt{-1} \). From equation (9), we have

\[
-\gamma \xi_1 e^{i \beta (i-1)h} + (2\gamma + a_1 + q_{\alpha,k} w_1^a) \xi_1 e^{i \beta ih} - \gamma \xi_1 e^{i \beta (i+1)h} = \xi_0 e^{i \beta ih},
\]

where \( \bar{\alpha} = \max_{t \in [0,T]} |a(t)|, \) which can be reduced to

\[
-\gamma \xi_1 e^{-i \beta h} + (2\gamma + a_1 + q_{\alpha,k} w_1^a) \xi_1 - \gamma \xi_1 e^{i \beta h} = \xi_0 .
\]

Or,

\[
\xi_1 = \frac{\xi_0}{-\gamma e^{-i \beta h} + (2\gamma + a_1 + q_{\alpha,k} w_1^a) - \gamma e^{i \beta h}}
\]

which implies

\[
\xi_1 = \left( \frac{1}{2\gamma (1 - \cos \beta h) + a_1 + q_{\alpha,k} w_1^a} \right) \xi_0 . \tag{11}
\]

Since \( 2\gamma (1 - \cos \beta h) + a_1 + q_{\alpha,k} w_1^a \geq 1 \), it follows that \( \xi_1 \leq \xi_0 \).

Now, from equation (10) when \( j \geq 2 \) and substituting \( e_i^j = \xi_j e^{i \beta \bar{i} h} \), we have

\[
-\gamma \xi_j e^{-i \beta h} + (2\gamma + a_j + q_{\alpha,k} w_1^a) \xi_j - \gamma \xi_j e^{i \beta h} = q_{\alpha,k} \sum_{k=1}^{j-1} (w_k^a - w_{k+1}^a) \xi_{j-k} + q_{\alpha,k} w_1^a \xi_0 \]

\[
(-\gamma e^{-i \beta h} + (2\gamma + a_j + q_{\alpha,k} w_1^a) - \gamma e^{i \beta h}) \xi_j = q_{\alpha,k} \sum_{k=1}^{j-1} (w_k^a - w_{k+1}^a) \xi_{j-k} + q_{\alpha,k} w_j^a \xi_0
\]

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\[ \xi_j = \frac{q_{a,k} \sum_{k=1}^{j-1} (w_k^a - w_{k+1}^a) \xi_{j-k} + q_{a,k} \sum_{k=1}^{j-1} (w_k^a - w_{k+1}^a) \xi_0}{(2y(1-\cos \theta) + x^2) + q_{a,k} \sum_{k=1}^{j-1} (w_k^a - w_{k+1}^a) \xi_0}. \]

By induction, we have \( \xi_N \leq \xi_{N-1} \leq \cdots \leq \xi_1 \leq \xi_0 \). Thus \( |e_j^i| \leq |e_j^{i-1}| \), for all \( j \). This completes the proof of the unconditional stability of the scheme (8).

Now, we must prove the convergent. Consider \( \|E_j^i\|^2 = h \sum |e_j^i|^2 \). This Euclidean norm of the perturbation [31]. Therefore, the stability condition can be written as:

\[
\|E_j^i\|_2 \leq \|E_{j-1}^i\|_2, j = 1, 2, ..., N
\]

This relation implies that \( \|E_j^i\|_2 \leq \|E_0^i\|_2 \). Equation (8) can be rewritten as:

\[
AU_j^i = \mathcal{M} U_{j-1}^i,
\]

where \( \mathcal{M} \) is the difference operator defined as:

\[
(\mathcal{M} U_{j-1}^i)_i = \sum_{k=1}^{j-1} (w_k^a - w_{k+1}^a)(U_{j-k}^i)_i + w_j^a (U_0^i)_i
\]

From \( A E_j^i = \mathcal{M} E_{j-1}^i \), we have \( E_j^i = A^{-1} \mathcal{M} E_{j-1}^i \), by considering equation (15), gives

\[
\|A^{-1} \mathcal{M} E_{j-1}^i\|_2 \leq \|E_{j-1}^i\|_2.
\]

That is the operator \( A^{-1} \mathcal{M} \) is a non-expansive.

Now, take \( A = \mathcal{M} = R \), where \( R \) is the difference equation at mesh point \((i, k)\)th. So if \( R \) tends to zero \( (R \rightarrow 0) \), the scheme (8) is consistent [33,34].

Now, define \( e_j^i = u_j^i - U_j^i \), where \( U \) is the numerical solution and \( u \) is the exact solution, then we have [32].

\[
A e_j^i = \mathcal{M} e_{j-1}^i + R_j^i
\]

Therefore, from equation (17), we can write

\[
\|e_j^i\|_2 \leq \|A^{-1} \mathcal{M} e_{j-1}^i\|_2 + \|A^{-1}\|_2 \|R_j^i\|_2
\]

Because \( A^{-1} \mathcal{M} \) is non-expansive, then from equation (18) we have

\[
\|e_j^i\|_2 \leq \|e_{j-1}^i\|_2 + \|A^{-1}\|_2 \|R_j^i\|_2
\]

Thus, by induction, we have

\[
\|e_j^i\|_2 \leq \|A^{-1}\|_2 \sum_{k=1}^{j-1} \|R_k\|_2
\]

This inequality shows that if \( \|A^{-1}\|_2 \) for equations (9) – (10) is bounded, then, the error \( \|e_j^i\|_2 \rightarrow 0 \). This completes the proof of the convergence of the proposed method.

### 3.1 Numerical Example for Direct Problem

Here, an example is given for the direct problem, that is when the coefficient \( c(t) \) is known, to validate, stability, and accuracy for (FFDS).

**Example 1** Consider solving the problem (1)-(3) with the data:

\[
a(t) = 100e^{100t}, \quad t \in [0,1],
\]

\[
c(t) = e^t, \quad t \in [0,1],
\]

\[
\varphi(x) = 0, \quad x \in [0,1],
\]

\[
F(x, t, u) = \frac{1}{(2-a)^t - a u} - (2-12x^2) t + 100e^{100t} u, \quad (x, t) \in [0,1] \times [0,1].
\]
where \( u(x, t) = x^2(1 - x^2)t \) is the exact solution. Figure 1 shows the absolute error between the exact and numerical solutions of \( u(x, t) \) when \( \alpha \in \{0.25, 0.5, 0.75\}, \ M = N = 40 \). And, one can see from this 3D figure excellent agreement is obtained. Figure 2 shows the exact and numerical solutions for the thermal energy \( e(t) \).

\[ \begin{align*}
\text{Figure 1:} & \quad \text{The absolute error between the true and numerical solutions of } u(x, t) \text{ when } \alpha \in \{0.25, 0.5, 0.75\} \text{ for Example 1.} \\
\text{Figure 2:} & \quad \text{The required output } e(t), \text{ with } N = M = 40, \text{ for Example 1 with } \alpha = 0.5.
\end{align*} \]

4. Numerical Procedure for Time-Fractional Source Inverse Problem (TFSIP)

We aim to find the numerical solution for problem (1)-(4) which is described in Section 2. We want to find stable reconstructions for the unknown coefficient \( c(t) \) of the one-dimensional semi-linear time-fractional equation together with \( u(x, t) \) to fulfill equations (1)-(4).

Observe that from (1) and (4), we have:

\[
c(t) = \frac{D^\alpha e(t) + a(t)e(t)}{\int_0^1 xF(x,t,u)dx}.
\]

Therefore, by using \( t = 0 \) in the above equation,

\[
c(0) = \frac{D^\alpha e(0) + a(0)e(0)}{\int_0^1 xF(x,0,\varphi(x))dx},
\]

which is a constant initial guess. Now, we recast this problem as a nonlinear minimization problem. In the other words, we minimize the gap between measured data and computed solutions. The Tikhonov regularization functional can be found by imposing the condition (4):

\[
F(c) = \left\| \int_0^1 xu(x, t)dx - e(t) \right\|^2 + \beta\|c(t)\|^2,
\]

or in discretized form,
\[
F(c) = \sum_{j=1}^{N} \left( \int_{0}^{1} xu(x, t_j) dx - e(t_j) \right)^2 + \beta \sum_{j=1}^{N} c_j^2,
\]

where \( \beta > 0 \) represents a parameter of regularization. The `lsqnonlin` routine is used to achieve a minimum objective \( F \) function, for more details see [35]. `Lsqnonlin`’s routine starts from \( c(0) \) and tries to find a minimum of the scalar function of several variables, given the constraints, for more details see [4]. We take the parameters of the routine as:

- **Maximum number of iterations** = \( 10^2 \times (\text{number of variables}) \).
- **Solution and Objective function tolerances** = \( 10^{-15} \).

The fractional inverse problem (1)-(4) is tested subject to both noisy measurement and exact data (4). The noise-contaminated is simulated as:

\[
e^\varepsilon(t_j) = e(t_j) + \varepsilon_j, \quad j = 0, N
\]

where \( \varepsilon \) represents the Gaussian random vector with a mean equal to zero and standard deviation is given by:

\[
\sigma = p \times \max_{t \in [0, T]} |e(t)|,
\]

where \( p \) represents the percentage of noise. We anticipate the `normrnd` built in function to generate the random variables \( \varepsilon = (\varepsilon_j), j = 0, N \) as:

\[
\varepsilon = \text{normrnd}(0, \sigma, N).
\]

5. **Results and Discussions**

The Root Means Square Error (RMSE) has been used to check the accuracy of numerical results after employing the Tikhonov regularization technique, and its formula is given as follows:

\[
\text{RMSE}(c) = \left( \frac{1}{N} \sum_{j=1}^{N} (c_{\text{numerical}}(t_j) - c_{\text{exact}}(t_j))^2 \right)^{1/2},
\]

For simplicity, we fix \( T = 1 \) in all following numerical experiments.

**Example 5.1: (Smooth Coefficient)** Consider the fractional inverse problem (1)-(4) with input data in Example 1 of the direct problem with the \( e(t) = \frac{1}{12} t, t \in [0,1] \) and the coefficient \( c(t) \) is unknown. The initial guess is taken as \( c_0 = 1 \). Figure 3 shows the numerical solution of the time-dependent source function from first order heat moment (4) in comparison with the exact solution \( c(t) = e^t \) obtained by solving the inverse problem with the input data in Example 1 using the FFDS, which is described in Section 2, with \( M = N \in \{10, 20, 40\} \). In Figure 4, the counter of iterations required to reach the convergence of the functional (22) to a very low threshold value of \( O(10^{-15}) \) is plotted with \( M = N \in \{10, 20, 40\} \). From this figure, it can be observed a speed convergence was achieved in 5 iterations only to reach a very low value of order \( O(10^{-15}) \). However, the problem remains ill-posed and has to be regularized.
Figure 3: The numerical solutions for time-dependent source $c(t)$ and exact value ($c(t) = e^t$) for Example 5.1 with $p = 0$ and $\beta = 0$.

Figure 4: Objective function (22) for Example 5.1 when $\beta = 0$ and with $= 0\%$.

Next, we fix $N = M = 40$ and start our investigation with cases (i) no noise ($p = 0\%$) and (ii) noise ($p = 10\%$), included in the measurement data (4). Figure 5 explains the comparisons of numerical results with the exact solution for $c(t)$ with regularization ($\beta = 0$) and (a) no noise ($p = 0\%$) and (b) with noise $p \in \{3,10\}\%$. Subfigure (a) investigates the convergence of the numerical solution for $c(t)$, while we note the results in subfigure (b) are unstable and inaccurate and this is expected.
Figure 5: Reconstructed $c(t)$ for Example 5.1 with $\beta = 0$ and (a) $p = 0\%$ and (b) $p \in \{3,10\}\%$ noise.

The reconstructed $c(t)$ are presented in Figure 6 after applying the Tikhonov regularization with $p = 10\%$ and $\beta \in \{10^{-4},10^{-5},10^{-6}\}$. The speed converges minimization of the objective function (22), as a function of the number of iterations, for $\beta \in \{10^{-i}, i = 4,5,6\}$ shown in Figure 7. The 3D graph of absolute error between true solution and numerical solution for temperatures $u(x,t)$ is plotted in Figure 8 with (a) $p = 3\%$ and (b) $p = 10\%$ with (i, iv) $\beta = 10^{-4}$, (ii, v) $\beta = 10^{-5}$ and (iii, vi) $\beta = 10^{-6}$. Next, in Table 1, we compute the $RMSE$ errors (25) for $\beta \in \{0,10^i, i = 3,4,5,6\}$ and $p \in \{3,5,10\}\%$. Clearly, from Figures 5,6,8 and Table 1, it can be seen that there is good agreement and convergence between the numerical solutions of $c(t)$ and $u(x,t)$ with their corresponding exact solutions, where $p$ decreases from 10% to 3% and then to 0%.
Figure 6: The reconstructed $c(t)$ for Example 5.1. with $p = 10\%$ and different amounts for $\beta$.

Figure 7: Objective function (22), with different values for $\beta$ and $p = 10\%$, for Example 5.1
Figure 8: The absolute error between the exact and numerical solutions of \( u(x, t) \) for Example 5.1 with different values of \( p \) and \( \beta \).

Table 1: The RMSE value (25) for various amounts of noise \( p \in \{3, 5, 10\}\% \) and various regularization parameters \( \beta \in \{0, 10^{-4}, 10^{-5}, 10^{-6}\} \) for Example 5.1.

<table>
<thead>
<tr>
<th>( \beta )</th>
<th>( p = 3% )</th>
<th>( p = 5% )</th>
<th>( p = 10% )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( 0 )</td>
<td>0.4513</td>
<td>0.6272</td>
<td>1.8438</td>
</tr>
<tr>
<td>( 10^{-4} )</td>
<td>0.3869</td>
<td>0.5416</td>
<td>0.6748</td>
</tr>
<tr>
<td>( 10^{-5} )</td>
<td>0.2655</td>
<td>0.4589</td>
<td>0.3111</td>
</tr>
<tr>
<td>( 10^{-6} )</td>
<td>0.3819</td>
<td>0.6368</td>
<td>1.5274</td>
</tr>
</tbody>
</table>

Example 5.2: (Non-Smooth Coefficient)

In this example, the time-fractional inverse source problem that is represented by equations (1) - (4) is considered with the following data:

\[
\begin{align*}
    a(t) &= 100(e^{30t} + 1), \ t \in [0, 1], \\
    \varphi(x) &= \sin^4(2\pi x), \ x \in [0, 1], \\
    c(t) &= \frac{1}{2} + |t - 0.5|, \ t \in [0, 1], \\
    e(t) &= \left(\frac{3}{16}\right)(t + 1), \ t \in [0, 1],
\end{align*}
\]

and \( F(x, t, u) = \frac{u}{c(t)} \left( x^{2 \lambda - 1} + 16 \pi^2 \left( 1 - 3 \frac{\cos^2(2\pi x)}{\sin^2(2\pi x)} \right) + a(t) \right), (x, t) \in [0, 1] \times [0, 1], \)

where \( u(x, t) = (1 + t)\sin^4(2\pi x) \) is the exact solution to inverse problem. Figures 8-10 show the numerical and the exact solutions of the time-dependent source \( c(t) \) and the objective function (22). The results are obtained in the same way as Example 5.1. Figure 8 explains the comparisons of numerical results of the time-dependent source \( c(t) \) with exact non-smooth solution ( \( c(t) = \frac{1}{2} + |t - 0.5| \) ) with \( \beta = 0 \) (no regularization) and noise level \( p \in \{0, 1, 3, 5\}\% \). From the graph of this figure, we note that the numerical solution of \( c(t) \) converges when \( p = 0 \), while the results slightly deviated from the exact ones, as the noise percentage \( p \) increases from 1\% to 5\% and this is expected. In Figure 9, the numerical performance of functional (22) minimization is plotted. From this figure, it can be observed a speed convergence was achieved in 7 iterations only to reach a very low value of order \( O(10^{-14}) \). Therefore, problem (1)-(4) is ill-posed and has to be regularized. The converge minimization of the objective function (22), as a function of the number of iterations, for \( \beta \in \{10^{-2}, 10^{-3}, 10^{-4}\} \) shown in Figure 10. The associated results for \( c(t) \) are presented in Figure 11 after applying Tikhonov regularization.
Figure 8: Reconstructed $c(t)$ for Example 5.2 with $\beta = 0$ and different amounts for $p$.

Figure 9: Objective function (22) for Example 5.2, with $\beta = 0$ and when $p = 3\%$.

Figure 10: Objective function (22) for Example 5.2, with $p = 3\%$ and when different values for $\beta$. 
Figure 1: Reconstructed $c(t)$ with $p = 3\%$ and when different values for $\beta$ for Example 5.2.

The 3D graphs of the exact and numerical solutions for $u(x, t)$, and the absolute error are plotted in Figure 12 with (i) $p = 0\%$ and $\beta = 0$, (ii) $p = 3\%$ and $\beta = 10^{-2}$, (iii) $p = 3\%$ and $\beta = 10^{-3}$, and (iv) $p = 3\%$ and $\beta = 10^{-4}$. Other details about the number of function evaluations, number of iterations, the value of the objective function (Eq.22) and the rmse of $c(t)$ in (Eq.26) are given in Table 2. From Figures 11, 12 and Table 2, it can be seen that there is a good agreement between the numerical results of $c(t)$ and $u(x, t)$ and their analytical solutions for the exact data.
Figure 12: The exact solution, numerical solution for $u(x,t)$ and the absolute error, with (i) $p = 0\%$ and $\beta = 0$, (ii) $p = 3\%$ and $\beta = 10^{-2}$, (iii) $p = 3\%$ and $\beta = 10^{-3}$, and (iv) $p = 3\%$ and $\beta = 10^{-4}$ for Example 5.1.

Table 2: Number of iterations, number of function evaluations, value of the functional (22) and $\text{rmse}(c)$ for various amounts of noise and regularization, for Example 5.2.

<table>
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<tr>
<th>$\beta$</th>
<th>$p$</th>
<th>No. of iterations</th>
<th>No. of func. evaluations</th>
<th>Objective function value at final iteration (22)</th>
<th>$\text{rmse}(c)$</th>
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</thead>
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<tr>
<td>$0$</td>
<td>0%</td>
<td>6</td>
<td>294</td>
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<td>0.0002</td>
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<tr>
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<td></td>
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<tr>
<td></td>
<td>1%</td>
<td>7</td>
<td>336</td>
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<td>$10^{-3}$</td>
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<td>$10^{-4}$</td>
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6. Conclusions
In this article, a semi-linear time-fractional inverse source problem of determining temperature solution together with the time-dependent source has been investigated. The
fractional finite difference scheme (FFDS) used with the Tikhonov regularization technique for finding the stable solution of problem (1)-(4) has been utilized. Proved the stability and convergence of the proposed algorithm by the Von Neumann method (VNM). Finally, some test examples are given to validate this method.

Reference


