DOI: 10.24996/ijs.2019.60.9.16

ISSN: 0067-2904

Strongly and Semi Strongly E_h-b-Vex Functions: Applications to Optimization Problems

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Abstract

In this paper, we propose new types of non-convex functions called strongly E_h-b-vex functions and semi strongly E_h-b-vex functions. We study some properties of these proposed functions. As an application of these functions in optimization problems, we discuss some optimality properties of the generalized nonlinear optimization problem for which we use, as an objective function, strongly E_h-b-vex function and semi strongly E_h-b-vex function.

Keywords: E-convex functions, strongly E-convex set, strongly E_h-b-vex sets, strongly E_h-b-vex functions, semi strongly E_h-b-vex functions

الدوال المحدبة وشبه المحدبة بقوة من نوع $E_h- b-$vex وتطبيقاتها على مشاكل الأمثلية

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الخلاصه

في هذا البحث تم دراسة أنواع جديدة من الدوال غير المحدبة والمسماة الدوال المحدبة وشبه المحدبة بقوة من نوع $E_h- b-$vex . قمنا بدراسة بعض الخواص الجديدة لهذه الدوال. قمنا ايضاً بمناقشة بعض خواص الأمثلية لمشاكل الأمثلية غير الخطية المعممة من نوع $E_h- b-$vex .

1. Introduction

Convex analysis is considered as effective tool for dealing with problems in optimization and applied mathematics [1-4]. Several attempts are made to extend and generalize convex sets and functions into ones with less restrictive convexity assumptions. For instance, b-vex functions introduced by Bector and Singh [5] as a generalization of convex functions which met with convex functions in many properties. Another class of generalized convex functions introduced by Youness [6]. Youness introduced E-convex sets, E-convex functions, and E-convex programming by relaxing the definitions of convex sets, convex functions, and convex optimization problems and using the effect of a mapping E. After that, Chen [7] introduced semi E-convex, quasi semi E-convex, and pseudo semi E-convex functions, and studied some of their properties and relations with E-convex functions. Recently, Youness and Emam, [8, 9] extended E-convex sets, E-convex functions, and semi E-convex functions into strongly E-convex sets, strongly E-convex functions, and semi strongly E-convex functions, respectively. By combining E-convex and b-vex functions, Mishra et. al [10] introduced the class of E-b-vex and semi E-b-vex functions and study some of their properties and

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relations with E-convex functions. Very recently, Marian [11] introduced h-strongly E-convex functions as a combination of strongly E-convex functions and h-convex functions introduced by Házy [12]. For more results published on generalized convexity, (see, e.g. [13-19]). In this paper, we introduce two new classes of functions, namely, strongly E_h-b-vex and semi strongly E_h-b-vex functions. The new classes generalize E-b-vex, semi E-b-vex and h-strongly E-convex functions. We study some basic and optimality properties of these functions. The paper is organized as follows. In the next section we recall some preliminary concepts and some generalized convex functions introduced in the literature. For the sake of completeness, we define h- semi strongly E-convex function where h- strongly E-convex functions are already studied in [11]. In section 3, we define strongly E_h-b-vex and semi strongly E_h-b-vex functions which are new types of non-convex functions and study some of their general properties. Applications of strongly E_h-b-vex and semi strongly E_h-b-vex functions to non-linear optimization problems are studied in Section 4.

2. Preliminaries

Throughout the paper, we assume that $\mathbb{R}^n$ is the n-dimensional Euclidean space, $\mathbb{R}^+$ be a set of non-negative real numbers. For the sake of brevity, we adopt the following assumption

Assumption (A) Let K be a non-empty subset of $\mathbb{R}^n$. Assume that $f: \mathbb{R}^n \to \mathbb{R}$ is a real valued function, $E: \mathbb{R}^n \to \mathbb{R}^n$, and $b: \mathbb{R}^n \times \mathbb{R}^n \times [0,1] \to \mathbb{R}^+$ are given mappings where $\mu b(x, y, \mu) \in [0,1]$ for each $x, y \in \mathbb{R}^n$ and $\mu \in [0,1]$.

Let us recall some preliminaries and related concepts that will be needed to develop the new generalized functions. For the rest of the paper, $K, f,\ E$ and b are defined as in assumption (A) unless otherwise specified.

Definition 2.1 Let $\gamma \in \mathbb{R}$. Then, different types of γ-level sets associated with f and E are defined in the literature. Some of these sets are listed below.

1) $K_\gamma = \{k \in K: f(k) \leq \gamma\}$. [1]
2) $E - K_\gamma = \{k \in K: f(Ek) \leq \gamma\}$. [6]
3) $K_\gamma^E = \{E(k) \in E(K): f(k) \leq \gamma\}$. [16]

Definition 2.2 The epigraph of f associated with K_γ, for each $\gamma \in \mathbb{R}$, is denoted by $epi\ f$ and defined as $epi\ f = \{(k, \gamma) \in K \times \mathbb{R} : f(k) \leq \gamma\}$. [1]

Definition 2.3 The set K is said to be

1) convex if for every $k_1, k_2 \in K$, and for every $\mu \in [0,1]$, we have
$\mu k_1 + (1-\mu)k_2 \in K$. [1]
2) E-convex with respect to the operator E if for every $k_1, k_2 \in K$, and for every $\mu \in [0,1]$, we have $\mu E(k_1) + (1-\mu)E(k_2) \in K$. [6]
3) Strongly E-convex if for every $k_1, k_2 \in K$, and for every $\mu, \alpha \in [0,1]$, we have
$$\mu(\alpha k_1 + E(k_1)) + (1-\mu)(\alpha k_2 + E(k_2)) \in K. \quad [8]$$
4) E-b-vex if for every $k_1, k_2 \in K$, and for every $\mu \in [0,1]$, we have
$\mu b(k_1, k_2, \mu)E(k_1) + (1 - \mu b(k_1, k_2, \mu))E(k_2) \in K$. [10]

Definition 2.4 [9] Let $K \subseteq \mathbb{R}^n \times \mathbb{R}$, $I: \mathbb{R} \to \mathbb{R}$ be the identity function, and $E: \mathbb{R}^n \to \mathbb{R}^n$ is a given mapping then K is called strongly $E \times I$ convex if $(x, \beta), (y, \omega) \in K$ and $\mu, \alpha \in [0,1]$ then
$(\mu\big(\alpha x + E(x)\big) + (1-\mu)\big(\alpha y + E(y)\big), \mu(\alpha\beta + E(\beta)) + (1-\mu)(\alpha\omega + E(\omega)) \in K$.In other words,
$(\mu\big(\alpha x + E(x)\big) + (1-\mu)\big(\alpha y + E(y)\big), \mu(\alpha\beta + \beta) + (1-\mu)(\alpha\omega + \omega) \in K$.

Proposition 2.5 [8] If $K \subseteq \mathbb{R}^n$ is strongly E-convex set then $E(K) \subseteq K$.

Definition 2.6 [6] f is said to be E-convex function on K if and only if K is an E-convex set and for each $k_1, k_2 \in K$, and each $0 \leq \mu \leq 1$, we have
$f(\mu E(k_1) + (1-\mu)E(k_2)) \leq \mu f(E(k_1)) + (1-\mu) f(E(k_2)).$

Definition 2.7 [8] A function f is said to be strongly E-convex on K if and only if K is strongly E-convex set and for each $k_1, k_2 \in K$, and each $0 \leq \mu, \alpha \leq 1$, we have
$f(\mu\big(\alpha k_1 + E(k_1)\big) + (1-\mu)\big(\alpha k_2 + E(k_2)\big)) \leq \mu f(E(k_1)) + (1-\mu) f(E(k_2)).$

Remark 2.8 Every strongly E-convex function is E-convex function ($\alpha = 0$). The converse does not hold [8, Example 4].

Definition 2.9 [10] A function f is said to be E-b-vex on K if and only if K is E-b-vex set and for each $k_1, k_2 \in K$, and each $0 \leq \mu \leq 1$, we have

$f(\mu b(k_1,k_2,\mu)E(k_1)+(1-\mu b(k_1,k_2,\mu))E(k_2))\leq$
$\mu b(k_1,k_2,\mu)f(E(k_1))+(1-\mu b(k_1,k_2,\mu))f(E(k_2))$.

Mishra et al. [10] also introduced semi E-b-vex functions as follows.

Definition 2.10 A function f is referred to as semi E-b-vex on K if and only if K is E-b-vex set and for each $k_1,k_2\in K$, each $0\leq\mu\leq 1$, we have

$f(\mu b(k_1,k_2,\mu)E(k_1)+\left(1-\mu b(k_1,k_2,\mu)\right)E(k_2))\leq \mu b(k_1,k_2,\mu)f(k_1)+$
$(1-\mu b(k_1,k_2,\mu))f(k_2)$.

In what follow we recall the definition of h-convex introduced in [12]. Note that other versions of h-convex functions can be found in [20-21].

Definition 2.11 [12] Let $h:[0,1]\to\mathbb{R}$ be a function. Then f is said to be h-convex function if for each $k_1,k_2\in K$, and each $0\leq\mu\leq 1$ we have $f(\mu k_1+(1-\mu)k_2)\leq h(\mu)f(k_1)+h(1-\mu)f(k_2)$.

By making use of h-convex functions and strongly E-convex functions, Marian [11] introduced the h-strongly E-convex functions as follows.

Definition 2.12 Let $h:[0,1]\to\mathbb{R}$ be a function. Then f is said to be h-strongly E-convex function if for each $k_1,k_2\in K$, and each $0\leq\mu,\alpha\leq 1$ we have

$f(\mu(\alpha k_1+E(k_1))+(1-\mu)(\alpha k_2+E(k_2)))\leq h(\mu)f(E(k_1))+h(1-\mu)f(E(k_2))$.

Following the lines of Marian [11] and Youness and Emam [8], the definition of h- semi strongly E-convex function can be deduced.

Definition 2.13 Let $h:[0,1]\to\mathbb{R}$ be a function. Then f is said to be h-semi strongly E-convex function if for each $k_1,k_2\in K$, and each $0\leq\mu,\alpha\leq 1$ we have

$f(\mu(\alpha k_1+E(k_1))+(1-\mu)(\alpha k_2+E(k_2)))\leq h(\mu)f(k_1)+h(1-\mu)f(k_2)$.

Next, we show an example of h-semi strongly E-convex function that is not h-strongly E-convex.

Example 2.14 Let f and E are defined as in [Example 2.1, 8]. Namely, $f:\mathbb{R}\to\mathbb{R}$ such that

$$f(x)=\begin{cases}\sqrt{x} & x\geq 0\\ -\sqrt{-x} & x\leq 0\end{cases}$$

Let $E:\mathbb{R}\to\mathbb{R}$ such that $E(x)=-|x|$ and $h:[0,1]\to\mathbb{R}$ be defined as $h(\lambda)=2\lambda$. Direct computation shows that f is h-semi strongly E-convex function. However, f is not h-strongly E-convex function. Indeed, if we take $\mu=\frac{1}{2}$, $\alpha=1, x=2, y=5$. Then, $f\left(\mu\left(\alpha x+E(x)\right)+(1-\mu)\left(\alpha y+E(y)\right)\right)=f(0)=0$, and

$h(\mu)f(E(x))+h(1-\mu)f(E(y))$= -3.73. ■

Another example illustrates h-strongly E-convex function which is not h-semi strongly E-convex function is given below.

Example 2.15 Let f and E are defined as in [Example 2.3, 8] where $f:\mathbb{R}\to\mathbb{R}$ such that

$$f(x)=\begin{cases}(x-2)^2 & 0\leq x\leq 2\\ (x+2)^2 & -2\leq x\leq 0\end{cases}$$

Let $E:\mathbb{R}\to\mathbb{R}$ such that $E(x)=0$ and $h:[0,1]\to\mathbb{R}$ be defined as $h(\lambda)=2\lambda$.

Using the definition, we deduce that f is h-strongly E-convex function, but it is not h-semi strongly E-convex function. For $\mu=\alpha=0,\ x=0,\ y=1$.

Then, $f(\mu(\alpha x+E(x))+(1-\mu)(\alpha y+E(y)))=4$, and $h(\mu)f(x)+h(1-\mu)f(y)$= 2. ■

Remark 2.16 For simplicity in appearance, we omit in the proofs and calculations the parentheses from $E(x)$, and writing it instead as Ex whenever it seems convenient. We also discard the argument of the mapping b and express $b(x,y,\mu)$ as b.

3. Strongly E_h-b-vex and semi strongly E_h-b-vex functions

In this section, we introduce two generalized convex functions which are strongly E_h-b-vex and semi strongly E_h-b-vex functions where $h:[0,1]\to\mathbb{R}$ is a function defined as in Definition 2.10. The generalized functions extend the definitions of h-strongly E-convex functions, E-b-vex functions, and semi E-b-vex functions. As we mentioned earlier in the preliminary section, each of f,E,b, and K are defined as in assumption (A).

Definition 3.1 f is said to be strongly E_h-b-vex function if K is strongly E-convex and for each $k_1,k_2\in K$, and each $0\leq\mu,\alpha\leq 1$ we have

$f(\mu(\alpha k_1+Ek_1)+(1-\mu)(\alpha k_2+Ek_2))\leq h(\mu b)f(Ek_1)+h(1-\mu b)f(Ek_2)$.

Definition 3.2 f is said to be semi strongly E_h-b-vex function if K is strongly E-convex and for each $k_1, k_2 \in K$, and each $0 \leq \mu, \alpha \leq 1$ we have
$f(\mu(\alpha k_1 + E(k_1)) + (1-\mu)(\alpha k_2 + E(k_2))) \leq h(\mu b)f(k_1) + h(1-\mu b)f(k_2)$.

Note that strongly E_h-b-vex and semi strongly E_h-b-vex functions are considered as generalizations of h-strongly E-convex and h-semi strongly E-convex functions, respectively, in the following sense.

Remark 3.3 Every h-strongly (respectively, h-semi strongly) E-convex functions are strongly (respectively, semi strongly) E_h-b-vex functions. (Choose $b = 1$)

Proposition 3.4 Let $f, b, K,$ and E are defined as in assumption (A) such that the mapping $b = 1$ and let $h: [0,1] \to \mathbb{R}$ be a function such that $h(0) = 0$. Assume that f is semi strongly E_h-b-vex function on the strongly E-convex set K. Then $f(\alpha k + Ek) \leq h(1)f(k) \quad \forall k \in K, \alpha \in [0,1]$.

Proof Assume that f is semi strongly E_h-b-vex on the strongly E-convex set K, then for each $k_1, k_2 \in K$, and each $0 \leq \mu, \alpha \leq 1$ we have
$\mu(\alpha k_1 + Ek_1) + (1-\mu)(\alpha k_2 + Ek_2) \in K$ and
$f(\mu(\alpha k_1 + Ek_1) + (1-\mu)(\alpha k_2 + Ek_2)) \leq h(\mu b)f(k_1) + h(1-\mu b)f(k_2)$.
For $\mu = b = 1$, we get $f(\alpha k_1 + Ek_1) \leq h(1)f(k_1)$. ■

Proposition 3.5 Assume that f is strongly E_h-b-vex function on the strongly E-convex set K and $f(Ek) \leq f(k) \quad \forall k \in K$. Then f is semi strongly E_h-b-vex function on K.

Proof. From the assumptions, we have for each $k_1, k_2 \in K$, and each $0 \leq \mu, \alpha \leq 1$,
$\mu(\alpha k_1 + Ek_1) + (1-\mu)(\alpha k_2 + Ek_2) \in K$ and
$$f(\mu(\alpha k_1 + Ek_1) + (1-\mu)(\alpha k_2 + Ek_2)) \leq h(\mu b)f(Ek_1) + h(1-\mu b)f(Ek_2).$$
$$\leq h(\mu b)f(k_1) + h(1-\mu b)f(k_2),$$
Thus, f is semi strongly E_h-b-vex function on K as required. ■

Some properties that hold for strongly E_h-b-vex functions and semi strongly E_h-b-vex functions on K are given next.

Proposition 3.6 Let $f, g: \mathbb{R}^n \longrightarrow \mathbb{R}$ are two functions such that K is strongly E-convex set. If f and g are strongly (respectively, semi strongly) E_h-b-vex on K w.r.t the same mappings E and b, then $\alpha f + \beta g$ is strongly (respectively, semi strongly) E_h-b-vex on K , for all $\alpha, \beta \geq 0$.

Proof The proof follows directly using the definitions of strongly (respectively, semi strongly) E_h-b-vex functions. ■

Proposition 3.7 Let $f_i: \mathbb{R}^n \longrightarrow \mathbb{R}$ is bounded from above for each $i \in \Lambda$ and K is strongly E-convex set with the same mappings E and b and h is a positive function. Define, $f: \mathbb{R}^n \longrightarrow \mathbb{R}$ such that $f = \sup_{i\in\Lambda} f_i$. If f_i is strongly (respectively, semi strongly) E_h-b-vex function on K for each $i \in \Lambda$, then f is strongly (respectively, semi strongly) E_h-b-vex function on K.

Proof We prove the property for strongly E_h-b-vex functions and in a similar manner one can prove it for semi strongly E_h-b-vex functions. Assume that f_i is a strongly E_h-b-vex functions on K, $\forall i \in \Lambda$. Then, for each $k_1, k_2 \in K$ and $0 \leq \mu, \alpha \leq 1$ we have $\mu(\alpha k_1 + Ek_1) + (1-\mu)(\alpha k_2 + Ek_2) \in K$ and $\forall i \in \Lambda$
$f_i(\mu(\alpha k_1 + Ek_1) + (1-\mu)(\alpha k_2 + Ek_2)) \leq h(\mu b)f_i(Ek_1) + h(1-\mu b)f_i(Ek_2)$
Taking the supremum to the right side and then to the left side of the above inequality and using the assumptions on h and f_i, we get
$\sup_{i\in\Lambda} f_i(\mu(\alpha k_1 + Ek_1) + (1-\mu)(\alpha k_2 + Ek_2)) \leq \sup_{i\in\Lambda}\{h(\mu b)f_i(Ek_1) + h(1-\mu b)f_i(Ek_2)\}$, that is, $f(\mu(\alpha k_1 + Ek_1) + (1-\mu)(\alpha k_2 + Ek_2)) \leq h(\mu b)\sup_{i\in\Lambda} f_i(Ek_1) + h(1-\mu b)\sup_{i\in\Lambda} f_i(Ek_2) = h(\mu b)\, f(Ek_1) + h(1-\mu b)\, f(Ek_2)$.

Hence, f is strongly E_h-b-vex functions on K. ■

Proposition 3.8 Let f be a strongly (respectively, semi strongly) E_h-b-vex function on the strongly E-convex set K. Assume also that $G: \mathbb{R} \longrightarrow \mathbb{R}$ is non-decreasing sublinear function. Then $G o f$ is a strongly (respectively, semi strongly) E_h-b-vex function.

Proof. We show that the composition property satisfies for semi strongly E_h-b-vex function f. Let $k_1, k_2 \in K$, and $\mu, \alpha \in [0,1]$, then
$\mu(\alpha k_1 + Ek_1) + (1-\mu)(\alpha k_2 + Ek_2) \in K$ and

$f(\mu(\alpha k_1 + Ek_1) + (1-\mu)(\alpha k_2 + Ek_2)) \leq h(\mu b)f(h_1) + h(1-\mu b)f(h_2),$
$\mathrm{G}(f(\mu(\alpha k_1 + Ek_1) + (1-\mu)(\alpha k_2 + Ek_2)) \leq \mathrm{G}\big(h(\mu b)f(h_1) + h(1-\mu b)f(h_2)\big).$

The last inequality holds because G is a non-decreasing function. Using the sublinearity assumption of G, the right-hand side of the last inequality yields,
$(\mathrm{G}of)(\mu(\alpha k_1 + Ek_1) + (1-\mu)(\alpha k_2 + Ek_2)) \leq h(\mu b)(\mathrm{G}of)(h_1) + h(1-\mu b)(\mathrm{G}of)(h_2).$
Thus, Gof is semi strongly E_h-b-vex function on K. Analogue proof proceeds when f is strongly E_h-b-vex function on K. ■

The following definitions are needed in the sequel.

Definition 3.9 Let $M, N \subseteq \mathbb{R}^n$. Then M is said to be strongly slack 2-convex w.r.t. N if for each $x, y \in M \cap N$ and $\mu, \alpha \in [0,1]$ such that
$\mu(\alpha x + Ex) + (1-\mu)(\alpha y + Ey) \in N$ then $\mu(\alpha x + Ex) + (1-\mu)(\alpha y + Ey) \in M$.
An example of strongly slack 2-convex set is given below.

Example3.10 Let $M = M_1 \cup M_2 = \{ (x,y) \in \mathbb{R}^2 : x \leq -1, -1 \leq y \leq 1 \} \cup \{(x,y) \in \mathbb{R}^2 : x \geq 1, -1 \leq y \leq 1 \}$, $N = M_2$ and let $E: \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ is given by $E(x,y) = (x,0)$. Note that M is not strongly E-convex set and N is strongly E-convex. Indeed, if we take $(-1,1), (1,1) \in M$, and $\mu = \frac{1}{2}$. Then $\mu E(-1,1) + (1-\mu)E(1,1) = \frac{1}{2}(-1,0) + \frac{1}{2}(1,0) = (0,0) \notin M$. Thus, M is not E-convex and from Remark 2.8, M is not strongly E-convex. On the other hand, let $(x_1, y_1), (x_2, y_2) \in N$ and $\alpha, \mu \in [0,1]$ then $\mu\big(\alpha(x_1,y_1) + E(x_1,y_1)\big) + (1-\mu)\big(\alpha(x_2,y_2) + E(x_2,y_2)\big) = (\mu(\alpha+1)x_1 + (1-\mu)(\alpha+1)x_2, \mu\alpha y_1 + (1-\mu)\alpha y_2) \in N$.Thus, N is strongly E-convex as required. Since $M \cap N = M_2 = N$, then for each $(x_1,y_1), (x_2,y_2) \in M \cap N$ such that

$\mu\big(\alpha(x_1,y_1) + E(x_1,y_1)\big) + (1-\mu)\big(\alpha(x_2,y_2) + E(x_2,y_2)\big) \in N$ implies

$\mu\big(\alpha(x_1,y_1) + E(x_1,y_1)\big) + (1-\mu)\big(\alpha(x_2,y_2) + E(x_2,y_2)\big) \in M_2 \subseteq M$. Thus, M is strongly slack 2-convex w.r.t. N.

Some properties related to the γ-level sets and $epi\ f$ are given next.

Proposition 3.11 Let K be a strongly E-convex set and f is semi strongly E_h-b-vex. Assume that h is linear and $h(1) = 1$. Then K_γ is strongly E-convex set for each $\gamma \in \mathbb{R}$.

Proof. Let $\gamma \in \mathbb{R}$ and $k_1, k_2 \in K_\gamma$ then $k_1, k_2 \in K$ and $f(k_1) \leq \gamma$, $f(k_2) \leq \gamma$. Since f is semi strongly E_h-b-vex on the strongly E-convex set K then for each $\mu, \alpha \in [0,1]$ we have $\mu(\alpha k_1 + Ek_1) + (1-\mu)(\alpha k_2 + Ek_2) \in K$, (1)
and $f(\mu(\alpha k_1 + Ek_1) + (1-\mu)(\alpha k_2 + Ek_2)) \leq h(\mu b)f(k_1) + h(1-\mu b)f(k_2)$
$\leq h(\mu b)f(k_1) + h(1)f(k_2) - h(\mu b)f(k_2),$
where the previous inequality is obtained from the linearity of h and the assumption $h(1) = 1$.
$\leq h(\mu b)\gamma + \gamma - h(\mu b)\gamma = \gamma$
Hence, $f(\mu(\alpha k_1 + Ek_1) + (1-\mu)(\alpha k_2 + Ek_2)) \leq \gamma$ (2)
From (1) and (2), we get K_γ is strongly E-convex set. ■

Proposition 3.12 Let K is strongly E-convex set and f is semi strongly E_h-b-vex. Assume that E and h are linear and $h(1) = 1$. Then $E - K_\gamma = \{k \in K: f(Ek) \leq \gamma\}$ is strongly E-convex set for each $\gamma \in \mathbb{R}$.

Proof. Let $\gamma \in \mathbb{R}$ and $k_1, k_2 \in E$-K_γ then $k_1, k_2 \in K$ and $f(Ek_1) \leq \gamma$, $f(Ek_2) \leq \gamma$. Since K is strongly E-convex set then for each $\mu, \alpha \in [0,1]$ we have
$$w = \mu(\alpha k_1 + Ek_1) + (1-\mu)(\alpha k_2 + Ek_2) \in K \quad (3)$$
We must show that $f(Ew) \leq \gamma$. From the linearity of E we have
$$f(Ew) = f(\mu(\alpha Ek_1 + E^2k_1) + (1-\mu)(\alpha Ek_2 + E^2k_2)) \quad (4)$$
From Proposition 2.5, $Ek_1, Ek_2 \in E(K) \subseteq K$.Using (4) and the assumptions on f and h, we obtain
$f(Ew) \leq h(\mu b)f(Ek_1) + h(1-\mu b)f(Ek_2) \leq h(\mu b)\gamma + h(1-\mu b)\gamma,$
$= h(\mu b)\gamma + \gamma - h(\mu b)\gamma = \gamma.$
Therefore, $f(Ew) = f(\mu(\alpha Ek_1 + E^2k_1) + (1-\mu)(\alpha Ek_2 + E^2k_2)) \leq \gamma$ (5)

From (3) and (5), the γ-level set $E\text{-}K_\gamma$ is strongly E-convex set. ■

Proposition 3.13 Let K and $E(K)$ are strongly E-convex sets and f is semi strongly E_h-b-vex. Assume that E and h are linear and $h(1) = 1$. Then $K_\gamma^E = \{E(k) \in E(K): f(k) \leq \gamma\}$ is strongly E-convex set for each $\gamma \in \mathbb{R}$.

Proof. Let $\gamma \in \mathbb{R}$ and $E(k_1), E(k_2) \in K_\gamma^E$ then $E(k_1), E(k_2) \in E(K)$ and $f(k_1) \leq \gamma$, $f(k_2) \leq \gamma$. Because $E(K)$ is strongly E-convex set and E is linear then for each $\mu, \alpha \in [0,1]$ we have $\mu(\alpha E k_1 + E^2 k_1) + (1-\mu)(\alpha E k_2 + E^2 k_2)$

$$= E(\mu(\alpha k_1 + E k_1) + (1-\mu)(\alpha k_2 + E k_2)) \in E(K) \quad (6)$$

i.e., $\mu(\alpha k_1 + E k_1) + (1-\mu)(\alpha k_2 + E k_2) \in K$. Using now the assumptions on f and h, we get

$$f(\mu(\alpha k_1 + E k_1) + (1-\mu)(\alpha k_2 + E k_2)) \leq h(\mu b) f(k_1) + h(1-\mu b) f(k_2) \leq \gamma \quad (7)$$

From (6) and (7), we have K_γ^E is strongly E-convex set. ■

Proposition 3.14 Let K is strongly E-convex sets and f is semi strongly E_h-b-vex. Assume that E and h are linear and $h(1) = 1$. Then $K_\gamma^E = \{E(k) \in E(K): f(k) \leq \gamma\}$ is strongly slack 2-convex w.r.t. $E(K)$, for each $\gamma \in \mathbb{R}$.

Proof. Let $\gamma \in \mathbb{R}$ and $E(k_1), E(k_2) \in K_\gamma^E \cap E(K)$ such that for each $\mu, \alpha \in [0,1]$, then as in (6), $\mu(\alpha E k_1 + E^2 k_1) + (1-\mu)(\alpha E k_2 + E^2 k_2) \in E(K)$. Following the same steps of the proof of Proposition 3.13 yields the required result. ■

Proposition 3.15 Let f is semi strongly E_h-b-vex on the strongly E-convex set K. Assume that h is non-negative and $h(\lambda) \leq \lambda \quad \forall \lambda \in [0,1]$. Then $epi\, f$ is strongly $E \times I$-convex set.

Proof. Let $(x, \beta), (y, \omega) \in epi\, f$ and $\mu, \alpha \in [0,1]$, we must show

$(\mu(\alpha x + E(x)) + (1-\mu)(\alpha y + E(y)), \mu(\alpha\beta + \beta) + (1-\mu)(\alpha\omega + \omega) \in epif.$

From the assumptions on K and f, we have

$\mu(\alpha x + E(x)) + (1-\mu)(\alpha y + E(y)) \in K, \quad (8)$

and $f\left(\mu(\alpha x + E(x)) + (1-\mu)(\alpha y + E(y))\right) \leq h(\mu b) f(x) + h(1-\mu b) f(y)$

$$\leq h(\mu b)\beta + h(1-\mu b)\omega$$

Using now the assumptions on h, the last inequality yields

$$\leq \mu b\, \beta + (1-\mu b)\omega$$
$$\leq \mu b\, (\alpha+1)\beta + (1-\mu b)(\alpha+1)\omega \quad (9)$$

From (8) and (9), we obtain $epi\, f$ is strongly $E \times I$-convex set. ■

Proposition 3.16 Let f is semi strongly E_h-b-vex on the strongly E-convex set K. Assume that h is non-negative and $h(\lambda) \leq \lambda \quad \forall \lambda \in [0,1]$. Then $epi\, f$ is strongly slack 2-convex w.r.t $E(K) \times \mathbb{R}$-convex set.

Proof. Let $(x, \beta), (y, \omega) \in epi\, f \cap E(K) \times \mathbb{R}$ such that for $\mu, \alpha \in [0,1]$

$(\mu(\alpha x + E(x)) + (1-\mu)(\alpha y + E(y)), \mu(\alpha\beta + \beta) + (1-\mu)(\alpha\omega + \omega) \in E(K) \times \mathbb{R}$

We aim to show that

$$(\mu(\alpha x + E(x)) + (1-\mu)(\alpha y + E(y)), \mu(\alpha\beta + \beta) + (1-\mu)(\alpha\omega + \omega) \in epi\, f \quad (10)$$

Note that $x, y \in E(K) \subseteq K, f(x) \leq \beta, f(y) \leq \omega$ and

$$\mu(\alpha x + E(x)) + (1-\mu)(\alpha y + E(y)) \in E(K) \subseteq K \quad (11)$$

Since f is semi strongly E_h-b-vex on K and from the assumptions on h, we get

$$f\left(\mu(\alpha x + E(x)) + (1-\mu)(\alpha y + E(y))\right) \leq h(\mu b) f(x) + h(1-\mu b) f(y)$$
$$\leq \mu b\, (\alpha+1)\beta + (1-\mu b)(\alpha+1)\omega \quad (12)$$

From (11) and (12), we obtain (10) as required. ■

The next result follows directly from Propositions 3.5 and 3.16.

Proposition 3.17 Let f strongly E_h-b-vex on the strongly E-convex set K and $f(E(k)) \leq f(k) \;\; \forall k \in K$. Assume that h is non-negative and $h(\lambda) \leq \lambda \quad \forall \lambda \in [0,1]$. Then $epi\, f$ is strongly slack 2-convex w.r.t $E(K) \times \mathbb{R}$-convex set. ■

4. Applications to non-linear optimization problems

In this section, we consider the following non-linear optimization problem which will be denoted as (P)

$$\min f(k)$$
$$s.t.\ k \in K,$$

where K and f are defined as in Assumption (A).

Proposition 4.1 Let , E, and f are defined as in assumption A such that K is strongly E-convex set and f is strongly E_h-b-vex on K and $f(Ek) \leq f(k)\ \ \forall k \in K$. Assume that h is linear and $h(1) = 1$. Assume that $k_0 = E(v)$ is a local minimum of problem (P) then k_0 is a global minimum.

Proof. By contrary, assume that k_0 is not a global minimum of the problem (P), then there exists $w \in K$ such that $f(w) < f(k_0) = f(E(v))$. Since f is strongly E_h-b-vex on K and $f(Ek) \leq f(k)\ \ \forall k \in K$ then

$$f\left(\mu\big(\alpha w + E(w)\big) + (1-\mu)\big(\alpha v + E(v)\big)\right) \leq h(\mu b)f(Ew) + h(1-\mu b)f(Ev)$$
$$\leq h(\mu b)f(w) + h(1-\mu b)f(k_0)$$

Since $f(w) < f(k_0)$, h is linear and $h(1) = 1$, the last inequality yield

$$\leq h(\mu b)f(k_0) + f(k_0) - h(\mu b)f(k_0) = f(k_0)$$

Now, let $\alpha = 0$, then

$$f(\mu E(w) + (1-\mu)k_0) \leq f(k_0) \tag{13}$$

Considering μ small enough such that $\mu E(w) + (1-\mu)k_0 \in B_r(k_0) \cap K$, where $B_r(k_0)$ is an open ball with center k_0 and radius r. Since $k_0$is a local minimum, then $f(k_0) \leq f(\mu E(w) + (1-\mu)k_0)$ which contradicts (13). Thus, k_0 is a global minimum. ∎

Proposition 4.2 Let K, E, and f are defined as in assumption (A) such that K is strongly E-convex set and f is semi strongly E_h-b-vex on K. Assume that h is linear and $h(1) = 1$. Assume that $k_0 = E(k_0)$ is a local minimum of problem (P) then k_0 is a global minimum.

Proof. By contrary, assume that k_0 is not a global minimum of the problem (P), then there exists $w \in K$ such that $f(w) < f(k_0) = f(E(k_0))$. Since f is semi strongly E_h-b-vex on K then

$$f\left(\mu\big(\alpha w + E(w)\big) + (1-\mu)\big(\alpha k_0 + E(k_0)\big)\right) \leq h(\mu b)f(w) + h(1-\mu b)f(k_0).$$

The rest of the proof is similar to that of Proposition 4.1. ∎

Proposition 4.3 Suppose that K is strongly E-convex set and f is strictly semi strongly E_h-b-vex on K. If h is linear and $h(1) = 1$. Then the global optimal solution of problem (P) is unique.

Proof. Let $k_1^*, k_2^* \in K$ be two different global optimal solutions of problem (P), then $f(k_1^*) = f(k_2^*)$. Since f is semi strongly E_h-b-vex on the strongly E-convex set K, we have $w = \mu\big(\alpha k_1^* + E(k_1^*)\big) + (1-\mu)\big(\alpha k_2^* + E(k_2^*)\big) \in K$

$$f\left(\mu\big(\alpha k_1^* + E(k_1^*)\big) + (1-\mu)\big(\alpha k_2^* + E(k_2^*)\big)\right) < h(\mu b)f(k_1^*) + h(1-\mu b)f(k_2^*),$$

$$f(w) = f\left(\mu\big(\alpha k_1^* + E(k_1^*)\big) + (1-\mu)\big(\alpha k_2^* + E(k_2^*)\big)\right) < f(k_1^*)$$

for $\mu \in (0,1)$ and $\alpha \in [0,1]$.

and hence, there exists $w \in K$ where $w \neq k_1^*$, $w \neq k_2^*$ and $f(w) < f(k_1^*)$. Thus, w is a global minimum which is a contradiction. Hence, there is a unique global minimum.

Proposition 4.4 Suppose that K is strongly E-convex set and f is semi strongly E_h-b-vex on K. If h is linear and $h(1) = 1$. Then the set optimal solutions of problem (P) is strongly E-convex.

Proof. The set of optimal solutions of problem (P) is defined as follows.

$argmin_f = \{k^* \in K: f(k^*) \leq f(k) \quad \forall k \in K\}$.

We must show that $argmin_f$ is strongly E-convex set. Let $k_1^*, k_2^* \in K$ and $\mu, \alpha \in [0,1]$ such that f is semi strongly E_h-b-vex on the strongly E-convex set K then

$\mu\big(\alpha k_1^* + E(k_1^*)\big) + (1-\mu)\big(\alpha k_2^* + E(k_2^*)\big) \in K$ and

$$f\left(\mu\big(\alpha k_1^* + E(k_1^*)\big) + (1-\mu)\big(\alpha k_2^* + E(k_2^*)\big)\right) \leq h(\mu b)f(k_1^*) + h(1-\mu b)f(k_2^*)$$
$$\leq f(k) \quad \forall k \in K,$$

where the last inequality follows from the definition of $argmin_f$ and the assumptions on h. Consequently, $\mu\big(\alpha k_1^* + E(k_1^*)\big) + (1-\mu)\big(\alpha k_2^* + E(k_2^*)\big) \in argmin_f$. ∎

Conclusion

In this paper, some types of generalized convex functions, namely strongly E_h-b-vex and semi strongly E_h-b-vex are defined and their basic properties are studied. In addition, optimality properties related to non-linear optimization problem involving these functions are discussed.

Acknowledgments

The author deeply thanks the two anonymous reviewers for their useful and valuable comments which contributed to the improvement of the paper

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