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# Estimation of Stress - Strength Reliability for Parallel Redundant System Via Burr of type X Distribution

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**Abstract:**
Estimation for the reliability of a parallel redundant system with independent stress and strength Burr distribution (type $X$ ) probability density functions is considered. Estimation of the reliability parameters is conducted according to the non-Bayesian estimation methods, namely the maximum likelihood and shrinkage methods. The Bayesian estimation is also considered using the Jeffery and Gamma priors with squared error, quadratic and weighted loss functions. Finally, the reliability estimate is calculated and the best method for estimation for each case is given using the mean squared error criteria.

**Keywords**: Burr type $X$ distribution, Reliability, Stress- Strength, Reliability Estimation.

## تقدير معولية الإجهاد – المتانة لنظام فائض متوازي عبر توزيع Burr من النوع.X

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**الخلاصة**
يتم النظر في تقدير معولية نظام فائض بربط متوازي مع ودالتي كثافة احتمالية للاجهاد والمتانة مستقلتين تتبع Burr من النوع (X). تم تقدير معاملات الموثوقية وفقًا لطرق التقدير غير البايزية، وهي طريقة الامكان الاعظم وطريقة الانكماش. تم النظر أيضًا في تقدير بايزي باستخدام توزيعات سابقة من نوع جيفري وجاما مع دوال الخطأ التربيعي والخسارة الموزونة. أخيرًا ، تم حساب تقدير المعولية وأعطيت أفضل طريقة لتقدير كل حالة باستخدام معيار متوسط الخطأ التربيعي.

## 1. Introduction

The Burr distribution of type $X$ belongs to one of the well-known twelve types of the Burr distribution model. This distribution was first introduced in 1942 by Burr [1]. The cumulative Burr distribution function of type $X$ can be defined by :

$$F_X(x) = \begin{Bmatrix} (1-e^{-(\alpha x)^2})^{\lambda} & x \geq 0 \\ 0 & x < 0 \end{Bmatrix}. \quad (1)$$

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Where $\alpha$ and λ are scale and shape parameters, respectively which are real numbers greater than zero. The probability density function of the Burr distribution of type $X$ is:

$$f_X(x,\alpha,\lambda) = \begin{cases} 2\lambda\alpha^2\, x\, e^{-(\alpha x)^2} \left(1 - e^{-(\alpha x)^2}\right)^{\lambda-1} & x > 0, \alpha > 0, \lambda > 0 \\ 0 & x < 0 \end{cases}. \quad (2)$$

The Burr type $X$ distribution was studied and analysed in many different applications. In (2020), Ibrahim and Abdullah Khaleel [2] proposed a generalization of the Burr type distribution based on the methods of Gamma-G , Beta-G and Weibull-G families of distributions. In (2022), Hassan et al.[3] proposed a model with two parameters based on the Kavya- Manharan –Burr X model. The Burr type $X$ is used in a wide range of survival data and hazard functions [4].

Several studies have been conducted in terms of stress-strength reliability. In (2020), N. S. Karam et.al. [5] studied the reliability of a multicomponent system based on the Lomax stress-strength model. In (2021), F. GülceCüran [6], the reliability of a redundant system with exponentially distributed stress and strength variables.In (2021), N. S. Karam [7] estimated the reliability of a stress-strength model based on the Generalized Inverted Kumaraswamy distribution. In the same year, S. A. Jabr and N. S. Karam [8] discussed the estimation of the reliability for the Gompertz Fréchet stress-strength model. E. Sh. M. Haddad and F. Sh. M. Batah (2021) [9] studied the reliability estimation of the stress-strength Rayleigh Pareto model. A. A. J. Ahmed and F. Sh. M. Batah (2023) [10] estimated the reliability of a stress-strength Power Rayleigh model.

Now for $\alpha = 1$, we suppose that two random variables $X$ and $Y$ take the Burr distribution of type X as strength and stress respectively. For each $X$ and $Y$, the probability density functions from equation (2) are defined as follows:

$$f_X(x,\lambda)= \begin{cases} 2\lambda\, x\, e^{-(x)^2} \left(1 - e^{-(x)^2}\right)^{\lambda-1} & x > 0, \lambda > 0 \\ 0 & o.w. \end{cases}. \quad (3)$$

$$f_Y(y,\mu)= \begin{cases} 2\mu\, y\, e^{-(y)^2} \left(1 - e^{-(y)^2}\right)^{\mu-1} & y > 0, \mu > 0 \\ 0 & o.w. \end{cases}. \quad (4)$$

And from Equation (3) and Equation (4), the cumulative density function (CDF) for the two random variables $X$ and Y , respectively are given as:

$$F_X(x) = \begin{cases} (1 - e^{-(x)^2})^{\lambda} & x \geq 0 \\ 0 & x < 0 \end{cases}. \quad (5)$$

$$F_Y(y) = \begin{cases} (1 - e^{-(y)^2})^{\mu} & y \geq 0 \\ 0 & y < 0 \end{cases}. \quad (6)$$

This work focuses on the system of reliability based on parallel redundant stress-strength modules by using the Burr type X distribution.

## 2. The System of Reliability

A parallel system (redundant system) is composed of $k$ components which is the limit state that does not necessarily indicate a system failure. Reliability and redundancy have been the subject of numerous studies such as [11].

The reliability system $R_c$of the component in the stress and strength system is computed as follows:

$$R_c = P(Y < x) = P(x > Y). \quad (7)$$

Where $P$ is the probability; $X$ and $Y$ are the strength and stress, respectively. Equation (7) can be rewritten as follows:

$$R_c = \int_{-\infty}^{\infty} f(y) \left(\int_{y}^{\infty} f(x)dX\right) dy. \quad (8)$$

Where $f(x)$ is the strength probability density function
Hence, the parallel system reliability $R_P$ of stress-strength model can be obtained if we substitute Equation (9) into Equation (10):

$$R(y) = P(X > y) = \int_y^{\infty} f(X)dX. \quad (9)$$

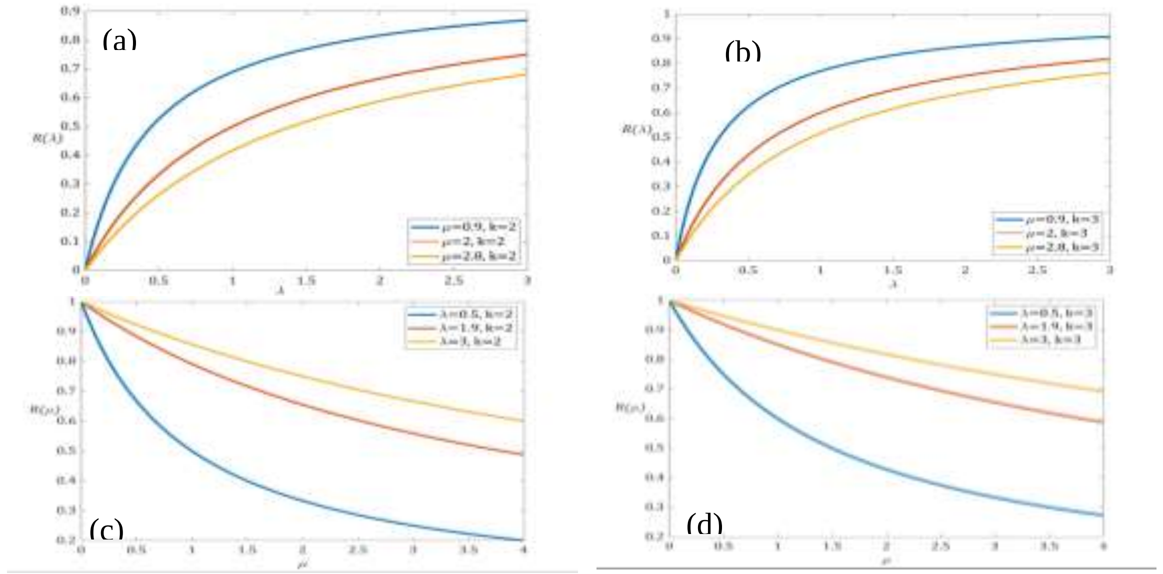


**Figure 1:** The system of reliability based on Burr distribution (type X). In (a) and (b) the reliability is drawn as a function of the parameter $\lambda$, while in (c) and (d) as a function of

$$R_P(y) = 1 - (1 - R(y))^k . \quad (10)$$

Where $k$ is the number of components. The overall reliability of the parallel redundant system under stress [12]:

$$R_0 = \int_0^{\infty} \big(R_P(y). f(y)\big) dy . \quad (11)$$

In our work, the reliability system ($R_0$) for this system is defined by the following formula based on the Burr type of X distribution. By using Equation (3) , Equation (4) and Equation (9), we get

$$R(y) = 1 - F_X(y) = 1 - (1 - e^{-y^2})^{\lambda}, \lambda > 0 . \quad (12)$$

By using the same processing in Equation (9) ,Equation (10) and Equation (12), we have:

$$R_p(y) = 1 - \big(F_X(y)\big)^k = 1 - (1 - e^{-y^2})^{k\lambda}, \lambda > 0, k \in \mathbb{Z}^+. \quad (13)$$

So, the reliability of the parallel redundant system can be defined by using Equation (11) and Equation (12) and Equation (13) :

$$R_0 = \int_0^{\infty} R_p(y). f_Y(y, \mu)dy = \frac{k\lambda}{k\lambda + \mu}, \lambda > 0, k \in \mathbb{Z}^+. \quad 14)$$

Figure (2.1) shows the behavior of the reliability with respect to the parameters.

## 3. Estimation Methods

In this section, we will discuss the non-Bayesian and Bayesian estimators for the reliability model. This section deals with the reliability system model in two considered cases in the stress–strength model. The strength ($X$) and the stress ($Y$) are independent variables having two parameters Burr distribution of type (X). The used estimators include the maximum likelihood method, and Shrinkage estimator method (SEM). Finally, Bayesian estimation methods for the Jeffry and gamma prior with the three difference loss functions such as the squared error loss function, quadratic loss function and weight loss function are used in the reliability model.

### *3.1 Non-Bayesian Estimation methods*

In the following subsections, we will use the non-Bayesian estimators for the parameters λ and $\mu$ with the reliability model $R_0$:

#### *3.1.1 Maximum Likelihood Estimator Method (MLE)*[13]*:*

Suppose that $x_1, x_2, . . ., x_n$ is a random sample from $B(1,\lambda)$ and $y_1, y_2, . . ., y_m$ is a random sample from $(1,\mu)$. The likelihood function is given by:

$L = L(\lambda,\mu; x; y) = \prod_{i=1}^{n} f(x_i) \prod_{j=1}^{m} h(y_j)$.

$L = \prod_{i=1}^{n} 2\lambda x_i e^{-x_i{}^2} (1 - e^{-x_i{}^2})^{\lambda-1} \prod_{j=1}^{m} 2\mu y_j e^{-y_j{}^2} (1 - e^{-y_j{}^2})^{\mu-1}$ .

$$L = \lambda^n \mu^m 2^{n+m} e^{-\sum_{i=1}^{n} x_i{}^2 \sum_{j=1}^{m} y_j{}^2} \prod_{i=1}^{n} x_i \prod_{j=1}^{m} y_j \prod_{i=1}^{n} (1 - e^{-x_i{}^2})^{\lambda-1} \prod_{j=1}^{m} (1 - e^{-y_i{}^2})^{\mu-1} \quad (15\text{ a})$$

Take Ln to both sides of Eq(15) , we have :

$$Ln(L) = n\ ln\lambda + m\ ln\mu + (n+m)\ ln2 + \sum_{i=1}^{n} lnx_i + \sum_{j=1}^{m} lny_j - \sum_{i=1}^{n} x_i{}^2 - \sum_{j=1}^{m} y_j{}^2 + \sum_{i=1}^{n} ln(1 - e^{-x_i{}^2})^{\lambda-1} + \sum_{j=1}^{m} ln(1 - e^{-y_j{}^2})^{\mu-1}. \quad (15\text{ b})$$

Differentiating Equation (15 b) with respect to λ and $\mu$ then put the results in:

$$\frac{\partial Ln(L)}{\partial\lambda} = \frac{n}{\lambda} + \sum_{i=1}^{n} \ln(1 - e^{-x_i{}^2}) . \quad (16)$$

$$\frac{\partial Ln(L)}{\partial\mu} = \frac{m}{\mu} + \sum_{j=1}^{m} \ln(1 - e^{-y_j{}^2}) . \quad (17)$$

Putting $\frac{\partial Ln(L)}{\partial\lambda} = 0$ and $\frac{\partial Ln(L)}{\partial\mu} = 0$, Equation (16) and Equation (17) will be:

$\frac{n}{\lambda} + \sum_{i=1}^{n} \ln(1 - e^{-x_i{}^2}) = 0.$

$\frac{m}{\mu} + \sum_{j=1}^{m} \ln(1 - e^{-y_j{}^2}) = 0$ .

Then, the maximum likelihood estimator for λ and $\mu$ ; $\hat{\lambda}_{MLE}$ and $\hat{\mu}_{MLE}$ are given as follows:

$$\hat{\lambda}_{MLE} = \frac{-n}{\sum_{i=1}^{n} \ln(1-e^{-x_i{}^2})}. \quad (18)$$

$$\hat{\mu}_{MLE} = \frac{-m}{\sum_{j=1}^{m} \ln(1-e^{-y_j{}^2})}. \quad (19)$$

Substituting Equations (18), (19) and (14), we get the estimator $\hat{R}_{0_{MLE}}$ as:

$$\hat{R}_{0_{MLE}} = \frac{k\hat{\lambda}_{MLE}}{k\hat{\lambda}_{MLE} + \hat{\mu}_{MLE}}. \quad (20)$$

#### *3.1.2 The Shrinkage Estimator Method (SEM)*[13]*:*

In this section, we will use the shrinkage estimator of the shape parameters λ and $\mu$ of the Burr distribution:

$$\hat{\lambda}_{sh} = \emptyset(\hat{\lambda})\,\hat{\lambda} + (1 - \emptyset(\hat{\lambda}))\hat{\lambda}_0 . \quad (21)$$

$$\hat{\mu}_{sh} = \emptyset(\hat{\mu})\,\hat{\mu} + (1 - \emptyset(\hat{\mu}))\hat{\mu}_0 . \quad (22)$$

Where $0 \le \emptyset(\hat{\lambda}) \le 1$ and $0 \le \emptyset(\hat{\mu}) \le 1$ with the values of $\hat{\lambda}_0$ and $\hat{\mu}_0$ are close to λ and $\mu$, respectively.

In this work, we consider the following three cases:

**Case(1):** Put $\emptyset(\hat{\lambda}) = \emptyset(\hat{\mu}) = 0.05$ , $\hat{\lambda} = \hat{\lambda}_{MLE}$ , $\hat{\mu} = \hat{\mu}_{MLE}$ , then Equation (21) and Equation (22) can be written as:

$$\hat{\lambda}_{SH1} = 0.05\hat{\lambda}_{MLE} + (0.95)\hat{\lambda}_0 . \quad (23)$$

$$\hat{\mu}_{SH1} = 0.05\hat{\mu}_{MLE} + (0.95)\hat{\mu}_0 . \quad (24)$$

By using Equation (23) and Equation (24) in Equation (14) for funding estimator $SH1$ the reliability system $\hat{R}_{0_{SH_1}}$ will be as the following formula:

$$\hat{R}_{0_{SH_1}} = \frac{k\hat{\lambda}_{SH1}}{k\hat{\lambda}_{SH1}+\hat{\mu}_{SH1}}. \quad (25)$$

**Case (2):** Put $\emptyset(\hat{\lambda}) = \frac{|\sin n|}{n}$ ,$\emptyset(\hat{\mu}) = \frac{|\sin m|}{m}$ ,$\hat{\lambda} = \hat{\lambda}_{MLE}$ ,$\hat{\mu} = \hat{\mu}_{MLE}$ , then Equation (23) and Equation (24) can be written as:

$$\hat{\lambda}_{SH2} = \frac{|\sin n|}{n}\hat{\lambda}_{MLE} + (1 - \frac{|\sin n|}{n})\hat{\lambda}_0. \quad (26)$$

$$\hat{\mu}_{SH2} = \frac{|\sin m|}{m}\hat{\mu}_{MLE} + (1 - \frac{|\sin m|}{m})\hat{\mu}_0. \quad (27)$$

Hence, by using Equation (26) and Equation (27) in Equation (14) for funding estimator $SH2$ the reliability system $\hat{R}_{0_{SH_2}}$ will be as the following formula:

$$\hat{R}_{0_{SH_2}} = \frac{k\hat{\lambda}_{SH2}}{k\hat{\lambda}_{SH2}+\hat{\mu}_{SH2}} . \quad (28)$$

**Case (3):** In this case, we estimation the parameters λ and $\mu$ in the Shrinkage method from $\hat{\lambda}_{MLE}$ and $\hat{\mu}_{MLE}$ in Equation (18-19), respectively:

$$\hat{\lambda}_{SH3} = \begin{Bmatrix} \frac{n}{n+1}\hat{\lambda}_{MLE} + (1 - \frac{n}{n+1})\hat{\lambda}_0 & \hat{\lambda}_0 \in R \\ \hat{\lambda}_{MLE} & \hat{\lambda}_0 \notin R \end{Bmatrix}. \quad (29)$$

$$\hat{\mu}_{SH3} = \begin{Bmatrix} \frac{m}{m+1}\hat{\mu}_{MLE} + (1 - \frac{m}{m+1})\hat{\mu}_0 & \hat{\mu}_0 \in R \\ \hat{\mu}_{MLE} & \hat{\mu}_0 \notin R \end{Bmatrix}. \quad (30)$$

Also, by using Equation (29) and Equation (30) in Equation (14) for funding estimator $SH3$ the reliability system $\hat{R}_{0_{SH_3}}$ will be as the following formula:

$$\hat{R}_{0_{SH_3}} = \frac{k\hat{\lambda}_{SH3}}{k\hat{\lambda}_{SH3}+\hat{\mu}_{SH3}} . \quad (31)$$

### *3.2 Bayesian Estimation Methods (BEM)*

In this section, we will give some important Bayes estimators with types of priors, namely the Jeffery and Gamma for the parameters λ and $\mu$ with their reliability model. They are shown below [14]:

#### *3.2.1 Squared error loss function*

The Bayes estimator for λ, $\mu$ and the risk function based on squared error loss function under Jeffrey's prior:

$$\hat{\lambda}_{JS} = \frac{n}{(\sum_{i=1}^{n} \ln(1-e^{-x_i{}^2}))^{-1}} , \hat{\mu}_{JS} = \frac{m}{(\sum_{j=1}^{m} \ln(1-e^{-y_i{}^2}))^{-1}}, \hat{R}_0 = \frac{k\hat{\lambda}_{JS}}{k\hat{\lambda}_{JS}+\hat{\mu}_{JS}} . \quad (32)$$

While, The Bayes estimator for λ , $\mu$ and the risk function based on the squared error loss function under Gamma's prior:

$$\hat{\lambda}_{GS} = \frac{n+a}{(\sum_{i=1}^{n} \ln(1-e^{-x_i{}^2}))^{-1}+b}, \hat{\mu}_{GS} = \frac{m+a}{(\sum_{j=1}^{m} \ln(1-e^{-y_i{}^2}))^{-1}+b}, \hat{R}_0 = \frac{k\hat{\lambda}_{GS}}{k\hat{\lambda}_{GS}+\hat{\mu}_{GS}} \quad (33)$$

#### *3.2.2 Quadratic loss function*

The Bayes estimator for λ , $\mu$ and the risk function based on quadratic loss function under Jeffrey's prior:

$$\hat{\lambda}_{JQ} = \frac{n-2}{(\sum_{i=1}^{n} \ln(1-e^{-x_i{}^2}))^{-1}}, \hat{\mu}_{JQ} = \frac{m-2}{(\sum_{j=1}^{m} \ln(1-e^{-y_i{}^2}))^{-1}}, \hat{R}_0 = \frac{k\hat{\lambda}_{JQ}}{k\hat{\lambda}_{JQ}+\hat{\mu}_{JQ}} . \quad (34)$$

The Bayes estimator for λ , $\mu$ and the risk function based on quadratic loss function under Gamm's prior:

$$\hat{\lambda}_{GQ} = \frac{n+a-2}{(\sum_{i=1}^{n} \ln(1-e^{-x_i{}^2}))^{-1}+b}, \hat{\mu}_{GQ} = \frac{m+a-2}{(\sum_{j=1}^{m} \ln(1-e^{-y_i{}^2}))^{-1}+b}, \hat{R}_0 = \frac{k\hat{\lambda}_{GQ}}{k\hat{\lambda}_{GQ}+\hat{\mu}_{GQ}}. \quad (35)$$

### 3.2.3 Weighted loss function

The Bayes estimator for $\lambda$ , $\mu$ and the risk function based on the weighted loss function under Jeffrey's prior

$$\hat{\lambda}_{JW} = \frac{n-1}{(\sum_{i=1}^{n} \ln(1-e^{-x_i^2}))^{-1}}, \hat{\mu}_{JW} = \frac{m-1}{(\sum_{j=1}^{m} \ln(1-e^{-y_j^2}))^{-1}}, \hat{R}_0 = \frac{k\hat{\lambda}_{JW}}{k\hat{\lambda}_{JW}+\hat{\mu}_{JW}}. \quad (36)$$

As well as, the Bayes estimator for $\lambda$ , $\mu$ and the risk function based on weighted loss function under Gamm's prior:

$$\hat{\lambda}_{GW} = \frac{n+a-1}{(\sum_{i=1}^{n} \ln(1-e^{-x_i^2}))^{-1}+b}, \hat{\mu}_{GW} = \frac{m+a-1}{(\sum_{j=1}^{m} \ln(1-e^{-y_i^2}))^{-1}+b}, \hat{R}_0 = \frac{k\hat{\lambda}_{GW}}{k\hat{\lambda}_{GW}+\hat{\mu}_{GW}}. \quad (37)$$

## 4. Simulation

A simulation study of size (1000) is used to compare the reliability estimators. The program MATLAB(2018b) is used to generate a complete type of data that is used to acquire the reliability estimates based on methods that are given in section 3. A comparison is then made to test the performance using the mean square error (MSE) criteria. The procedure is done as follows:

- From Equation (5), we let $U = F(x)$, where $U$ is uniformly distributed over $(0,1)$, the random sample generated by:

$$U_x = (1 - e^{-(x)^2})^{\lambda} \rightarrow U_x^{1/\lambda} = (1 - e^{-(x)^2}) \rightarrow x = \left(-\ln\left(1 - U_x^{\frac{1}{\lambda}}\right)\right)^{1/2}.$$

$$U_y = (1 - e^{-(y)^2})^{\mu} \rightarrow U_y^{1/\mu} = (1 - e^{-(y)^2}) \rightarrow y = \left(-\ln\left(1 - U_y^{\frac{1}{\mu}}\right)\right)^{1/2}.$$

- A random sample of size $n, m$ is generated for $x_i$and $y_j$ which are given for small medium and large $(n, m) = (10,30), (30,10), (50,90), (90,50)$ and $k = 2,3$. The real values of the parameters $\lambda, \mu$ are taken to be $(\lambda, \mu) = (0.5, 0.9), (0.5,2), (1.9,0.9), (3,0.9)$. The resulting data sets become 32 data sets for each $x$ and $y$.
- Parametric estimation is then conducted for each data set according to equations (14-37). For each case, the reliability of the system is estimated according to equation (14) resulting in 32 system reliability data sets of size 1000.
- The mean of the data sets for each case is calculated and given for each case in Tables (1-16).
- The MSE is also calculated according to the relation MSE= $\frac{1}{N}\sum_{i=1}^{N}(\hat{R}_i - R)^2$ , where $N = 1000$. The values are also given in Tables (1-16) along with the best method corresponding to the minimum value of the MSE.

**Table 1:** Reliability non-Bayesian estimators values when $\lambda = 0.5, \mu = 0.9, k = 2$ and $R = 0.52631$

| | | MLE | SH1 | SH2 | SH3 | Best |
|---|---|---|---|---|---|---|
| n=10, m=30 | mean | 0.53231 | 0.50265 | 0.50376 | 0.52887 | SH2 |
| | MSE | 8.55E-03 | 6.04E-04 | 5.54E-04 | 7.60E-03 | |
| n=30, m=10 | mean | 0.51382 | 0.50033 | 0.49887 | 0.51505 | SH1 |
| | MSE | 7.99E-03 | 7.13E-04 | 7.91E-04 | 7.05E-03 | |
| n=50, m=90 | mean | 0.52908 | 0.50182 | 0.50004 | 0.52839 | SH1 |
| | MSE | 1.84E-03 | 6.07E-04 | 6.91E-04 | 1.79E-03 | |
| n=90, m=50 | mean | 0.52570 | 0.50148 | 0.50047 | 0.52569 | SH1 |
| | MSE | 1.97E-03 | 6.24E-04 | 6.68E-04 | 1.92E-03 | |

**Table 2:** Reliability Bayesian estimators values when $\lambda = 0.5, \mu = 0.9, k = 2$ and $R = 0.52631$

| | | JS | JQ | JW | GS | GQ | GW | Best |
|---|---|---|---|---|---|---|---|---|
| n=10, m=30 | mean | 0.53231 | 0.49514 | 0.51510 | 0.56796 | 0.54296 | 0.55621 | GQ |
| | MSE | 8.55E-03 | 9.56E-03 | 8.70E-03 | 9.43E-03 | 8.14E-03 | 8.68E-03 | |
| n=30, m=10 | mean | 0.51382 | 0.55098 | 0.53109 | 0.48866 | 0.51406 | 0.50064 | GQ |
| | MSE | 7.99E-03 | 8.35E-03 | 7.84E-03 | 8.34E-03 | 7.09E-03 | 7.60E-03 | |
| n=50, m=90 | mean | 0.52908 | 0.52454 | 0.52684 | 0.53534 | 0.53121 | 0.53331 | GQ |
| | MSE | 1.84E-03 | 1.84E-03 | 1.84E-03 | 1.88E-03 | 1.83E-03 | 1.85E-03 | |
| n=90, m=50 | mean | 0.52570 | 0.53024 | 0.52794 | 0.52189 | 0.52603 | 0.52393 | GQ |
| | MSE | 1.97E-03 | 1.99E-03 | 1.97E-03 | 1.95E-03 | 1.93E-03 | 1.94E-03 | |

**Table 3:** Reliability non-Bayesian estimators values when $\lambda = 0.5, \mu = 2, k = 2$ and $R = 0.33333$

| | | MLE | SH1 | SH2 | SH3 | Best |
|---|---|---|---|---|---|---|
| n=10, m=30 | mean | 0.34374 | 0.29926 | 0.29991 | 0.33957 | SH2 |
| | MSE | 6.79E-03 | 1.19E-03 | 1.15E-03 | 5.94E-03 | |
| n=30, m=10 | mean | 0.33251 | 0.29772 | 0.29652 | 0.33148 | SH1 |
| | MSE | 6.30E-03 | 1.29E-03 | 1.38E-03 | 5.46E-03 | |
| n=50, m=90 | mean | 0.33596 | 0.29851 | 0.29647 | 0.33518 | SH1 |
| | MSE | 1.56E-03 | 1.22E-03 | 1.36E-03 | 1.51E-03 | |
| n=90, m=50 | mean | 0.33345 | 0.29825 | 0.29676 | 0.33315 | SH1 |
| | MSE | 1.59E-03 | 1.24E-03 | 1.34E-03 | 1.54E-03 | |

**Table 4:** Reliability Bayesian estimators values when $\lambda = 0.5, \mu = 2, , k = 2$ and $R = 0.33333$

| | | JS | JQ | JW | GS | GQ | GW | Best |
|---|---|---|---|---|---|---|---|---|
| n=10, m=30 | mean | 0.34374 | 0.31080 | 0.32826 | 0.38333 | 0.35961 | 0.37208 | JW |
| | MSE | 6.79E-03 | 6.61E-03 | 6.45E-03 | 9.10E-03 | 7.00E-03 | 7.97E-03 | |
| n=30, m=10 | mean | 0.33251 | 0.36652 | 0.34809 | 0.32734 | 0.35021 | 0.33803 | GS |
| | MSE | 6.30E-03 | 7.98E-03 | 6.79E-03 | 4.98E-03 | 5.55E-03 | 5.12E-03 | |
| n=50, m=90 | mean | 0.33596 | 0.33191 | 0.33397 | 0.34377 | 0.34003 | 0.34192 | JQ |
| | MSE | 1.56E-03 | 1.53E-03 | 1.54E-03 | 1.66E-03 | 1.58E-03 | 1.61E-03 | |
| n=90, m=50 | mean | 0.33345 | 0.33752 | 0.33545 | 0.33388 | 0.33758 | 0.33570 | GS |
| | MSE | 1.59E-03 | 1.63E-03 | 1.60E-03 | 1.52E-03 | 1.56E-03 | 1.54E-03 | |

**Table 5:** Reliability non-Bayesian estimators values when $\lambda = 1.9, \mu = 0.9, k = 2$ and $R = 0.80851$

| | | MLE | SH1 | SH2 | SH3 | Best |
|---|---|---|---|---|---|---|
| n=10, m=30 | mean | 0.80759 | 0.81819 | 0.81872 | 0.80808 | SH1 |
| | MSE | 3.12E-03 | 1.04E-04 | 1.15E-04 | 2.69E-03 | |
| n=30, m=10 | mean | 0.79526 | 0.81692 | 0.81651 | 0.79758 | SH2 |
| | MSE | 3.92E-03 | 8.39E-05 | 7.80E-05 | 3.37E-03 | |
| n=50, m=90 | mean | 0.80778 | 0.81773 | 0.81804 | 0.80786 | SH1 |
| | MSE | 7.43E-04 | 8.71E-05 | 9.08E-05 | 7.19E-04 | |
| n=90, m=50 | mean | 0.80589 | 0.81756 | 0.81816 | 0.80619 | SH1 |
| | MSE | 7.39E-04 | 8.40E-05 | 9.32E-05 | 7.14E-04 | |

**Table 6:** Reliability Bayesian estimators values when $\lambda = 1.9, \mu = 0.9, k = 2$ and $R = 0.80851$

| | | JS | JQ | JW | GS | GQ | GW | Best |
|---|---|---|---|---|---|---|---|---|
| n=10, m=30 | mean | 0.80759 | 0.78305 | 0.79648 | 0.81436 | 0.79831 | 0.80692 | GS |
| | MSE | 3.12E-03 | 4.37E-03 | 3.54E-03 | 2.41E-03 | 2.79E-03 | 2.52E-03 | |
| n=30, m=10 | mean | 0.79526 | 0.81872 | 0.80640 | 0.77283 | 0.79031 | 0.78119 | JQ |
| | MSE | 3.92E-03 | 3.26E-03 | 3.47E-03 | 4.95E-03 | 3.63E-03 | 4.25E-03 | |
| n=50, m=90 | mean | 0.80778 | 0.80493 | 0.80638 | 0.80831 | 0.80571 | 0.80703 | GS |
| | MSE | 7.43E-04 | 7.72E-04 | 7.55E-04 | 7.07E-04 | 7.29E-04 | 7.16E-04 | |
| n=90, m=50 | mean | 0.80589 | 0.80873 | 0.80730 | 0.80158 | 0.80421 | 0.80288 | JQ |
| | MSE | 7.39E-04 | 7.16E-04 | 7.26E-04 | 7.81E-04 | 7.37E-04 | 7.58E-04 | |

**Table 7:** Reliability non-Bayesian estimators values when $\lambda = 3, \mu = 0.9, k = 2$ and $R = 0.86956$

| | | MLE | SH1 | SH2 | SH3 | Best |
|---|---|---|---|---|---|---|
| n=10, m=30 | mean | 0.86858 | 0.87873 | 0.87913 | 0.86920 | SH1 |
| | MSE | 1.75E-03 | 8.98E-05 | 9.73E-05 | 1.49E-03 | |
| n=30, m=10 | mean | 0.85864 | 0.87775 | 0.87749 | 0.86059 | SH2 |
| | MSE | 2.24E-03 | 7.37E-05 | 7.00E-05 | 1.90E-03 | |
| n=50, m=90 | mean | 0.86807 | 0.87832 | 0.87867 | 0.86819 | SH1 |
| | MSE | 4.35E-04 | 7.77E-05 | 8.29E-05 | 4.21E-04 | |
| n=90, m=50 | mean | 0.86743 | 0.87823 | 0.87875 | 0.86768 | SH1 |
| | MSE | 4.25E-04 | 7.62E-05 | 8.44E-05 | 4.11E-04 | |

**Table 8:** Reliability Bayesian estimators values when $\lambda = 3, \mu = 0.9, k = 2$ and $R = 0.86956$

| | | JS | JQ | JW | GS | GQ | GW | Best |
|---|---|---|---|---|---|---|---|---|
| n=10, m=30 | mean | 0.86858 | 0.85028 | 0.86034 | 0.86497 | 0.85244 | 0.85918 | GS |
| | MSE | 1.75E-03 | 2.53E-03 | 2.02E-03 | 1.33E-03 | 1.81E-03 | 1.51E-03 | |
| n=30, m=10 | mean | 0.85864 | 0.87606 | 0.86695 | 0.83821 | 0.85168 | 0.84467 | JQ |
| | MSE | 2.24E-03 | 1.75E-03 | 1.93E-03 | 3.19E-03 | 2.24E-03 | 2.69E-03 | |
| n=50, m=90 | mean | 0.86807 | 0.86596 | 0.86704 | 0.86654 | 0.86460 | 0.86559 | GS |
| | MSE | 4.35E-04 | 4.57E-04 | 4.45E-04 | 4.23E-04 | 4.49E-04 | 4.35E-04 | |
| n=90, m=50 | mean | 0.86743 | 0.86952 | 0.86846 | 0.86314 | 0.86510 | 0.86410 | JQ |
| | MSE | 4.25E-04 | 4.10E-04 | 4.16E-04 | 4.69E-04 | 4.37E-04 | 4.52E-04 | |

**Table 9:** Reliability non-Bayesian estimators values when $\lambda = 0.5, \mu = 0.9, k = 3$ and $R = 0.625$

| | | MLE | SH1 | SH2 | SH3 | Best |
|---|---|---|---|---|---|---|
| n=10, m=30 | mean | 0.62642 | 0.60241 | 0.60347 | 0.62351 | SH2 |
| | MSE | 6.69E-03 | 5.47E-04 | 5.01E-04 | 5.98E-03 | |
| n=30, m=10 | mean | 0.61589 | 0.60072 | 0.5993 | 0.61709 | SH1 |
| | MSE | 7.67E-03 | 6.26E-04 | 6.96E-04 | 6.76E-03 | |
| n=50, m=90 | mean | 0.62755 | 0.60178 | 0.60004 | 0.6269 | SH1 |
| | MSE | 1.73E-03 | 5.46E-04 | 6.23E-04 | 1.69E-03 | |
| n=90, m=50 | mean | 0.62346 | 0.6014 | 0.60045 | 0.62348 | SH1 |
| | MSE | 1.82E-03 | 5.64E-04 | 6.03E-04 | 1.77E-03 | |

**Table 10:** Reliability Bayesian estimators values when $\lambda = 0.5, \mu = 0.9, k = 3$ and $R = 0.625$

| | | JS | JQ | JW | GS | GQ | GW | Best |
|---|---|---|---|---|---|---|---|---|
| n=10, m=30 | mean | 0.62642 | 0.59079 | 0.61005 | 0.65993 | 0.63672 | 0.64908 | GQ |
| | MSE | 6.69E-03 | 8.33E-03 | 7.14E-03 | 6.93E-03 | 6.21E-03 | 6.46E-03 | |
| n=30, m=10 | mean | 0.61589 | 0.65054 | 0.63212 | 0.59184 | 0.61612 | 0.60335 | GQ |
| | MSE | 7.67E-03 | 7.72E-03 | 7.41E-03 | 8.08E-03 | 6.80E-03 | 7.34E-03 | |
| n=50, m=90 | mean | 0.62755 | 0.62328 | 0.62545 | 0.63342 | 0.62955 | 0.63152 | GQ |
| | MSE | 1.73E-03 | 1.75E-03 | 1.74E-03 | 1.75E-03 | 1.71E-03 | 1.73E-03 | |
| n=90, m=50 | mean | 0.62346 | 0.62773 | 0.62556 | 0.61989 | 0.62379 | 0.62181 | GQ |
| | MSE | 1.82E-03 | 1.81E-03 | 1.81E-03 | 1.82E-03 | 1.78E-03 | 1.80E-03 | |

**Table 11:** Reliability non-Bayesian estimators values when $\lambda = 0.5, \mu = 2, k = 3$ and $R = 0.42857$

| | | MLE | SH1 | SH2 | SH3 | Best |
|---|---|---|---|---|---|---|
| n=10, m=30 | mean | 0.43724 | 0.39048 | 0.39122 | 0.43307 | SH2 |
| | MSE | 8.03E-03 | 1.49E-03 | 1.44E-03 | 7.06E-03 | |
| n=30, m=10 | mean | 0.42279 | 0.38859 | 0.38723 | 0.42217 | SH1 |
| | MSE | 7.48E-03 | 1.63E-03 | 1.74E-03 | 6.52E-03 | |
| n=50, m=90 | mean | 0.42892 | 0.38950 | 0.38728 | 0.42810 | SH1 |
| | MSE | 1.89E-03 | 1.53E-03 | 1.71E-03 | 1.84E-03 | |
| n=90, m=50 | mean | 0.43018 | 0.38946 | 0.38765 | 0.42984 | SH1 |
| | MSE | 1.88E-03 | 1.54E-03 | 1.67E-03 | 1.82E-03 | |

**Table 12:** Reliability Bayesian estimators values when $\lambda = 0.5, \mu = 2,, k = 3$ and $R = 0.42857$

| | | JS | JQ | JW | GS | GQ | GW | Best |
|---|---|---|---|---|---|---|---|---|
| n=10, m=30 | mean | 0.43724 | 0.40093 | 0.42030 | 0.47985 | 0.45461 | 0.46793 | JW |
| | MSE | 8.03E-03 | 8.37E-03 | 7.88E-03 | 1.01E-02 | 8.05E-03 | 8.98E-03 | |
| n=30, m=10 | mean | 0.42279 | 0.45961 | 0.43978 | 0.41798 | 0.44298 | 0.42972 | GW |
| | MSE | 7.48E-03 | 8.67E-03 | 7.71E-03 | 5.96E-03 | 6.21E-03 | 5.93E-03 | |
| n=50, m=90 | mean | 0.42892 | 0.42447 | 0.42673 | 0.43749 | 0.43341 | 0.43548 | GQ |
| | MSE | 1.89E-03 | 1.90E-03 | 1.89E-03 | 1.94E-03 | 1.88E-03 | 1.91E-03 | |
| n=90, m=50 | mean | 0.43018 | 0.43465 | 0.43238 | 0.43066 | 0.43473 | 0.43266 | GS |
| | MSE | 1.88E-03 | 1.92E-03 | 1.90E-03 | 1.80E-03 | 1.84E-03 | 1.82E-03 | |

**Table 13:** Reliability non-Bayesian estimators values when $\lambda = 1.9, \mu = 0.9, k = 3$ and $R = 0.86363$

| | | MLE | SH1 | SH2 | SH3 | Best |
|---|---|---|---|---|---|---|
| n=10, m=30 | mean | 0.86134 | 0.87097 | 0.87137 | 0.86196 | SH1 |
| | MSE | 2.01E-03 | 6.09E-05 | 6.68E-05 | 1.72E-03 | |
| n=30, m=10 | mean | 0.85276 | 0.87001 | 0.86970 | 0.85463 | SH2 |
| | MSE | 2.77E-03 | 4.97E-05 | 4.65E-05 | 2.36E-03 | |
| n=50, m=90 | mean | 0.86313 | 0.87065 | 0.87086 | 0.86319 | SH1 |
| | MSE | 4.68E-04 | 5.04E-05 | 5.23E-05 | 4.53E-04 | |
| n=90, m=50 | mean | 0.86120 | 0.87048 | 0.87095 | 0.86143 | SH1 |
| | MSE | 4.61E-04 | 4.82E-05 | 5.35E-05 | 4.45E-04 | |

**Table 14:** Reliability Bayesian estimators values when $\lambda = 1.9, \mu = 0.9, k = 3$ and $R = 0.86363$

| | | JS | JQ | JW | GS | GQ | GW | Best |
|---|---|---|---|---|---|---|---|---|
| n=10, m=30 | mean | 0.86134 | 0.84225 | 0.85274 | 0.86685 | 0.85449 | 0.86114 | GS |
| | MSE | 2.01E-03 | 2.92E-03 | 2.33E-03 | 1.52E-03 | 1.83E-03 | 1.63E-03 | |
| n=30, m=10 | mean | 0.85276 | 0.87073 | 0.86133 | 0.83563 | 0.84924 | 0.84216 | JQ |
| | MSE | 2.77E-03 | 2.21E-03 | 2.42E-03 | 3.44E-03 | 2.53E-03 | 2.95E-03 | |
| n=50, m=90 | mean | 0.86313 | 0.86095 | 0.86206 | 0.86353 | 0.86156 | 0.86256 | GS |
| | MSE | 4.68E-04 | 4.87E-04 | 4.76E-04 | 4.45E-04 | 4.60E-04 | 4.52E-04 | |
| n=90, m=50 | mean | 0.86120 | 0.86337 | 0.86227 | 0.85790 | 0.85992 | 0.85890 | JQ |
| | MSE | 4.61E-04 | 4.43E-04 | 4.51E-04 | 4.92E-04 | 4.62E-04 | 4.76E-04 | |

**Table 15:** Reliability non-Bayesian estimators values when $\lambda = 3, \mu = 0.9, k = 3$ and $R = 0.90909$

| | | MLE | SH1 | SH2 | SH3 | Best |
|---|---|---|---|---|---|---|
| n=10, m=30 | mean | 0.90735 | 0.91571 | 0.91597 | 0.90800 | SH1 |
| | MSE | 9.31E-04 | 4.68E-05 | 5.03E-05 | 7.89E-04 | |
| n=30, m=10 | mean | 0.90233 | 0.91515 | 0.91497 | 0.90370 | SH2 |
| | MSE | 1.25E-03 | 4.02E-05 | 3.82E-05 | 1.06E-03 | |
| n=50, m=90 | mean | 0.90907 | 0.91550 | 0.91571 | 0.90913 | SH1 |
| | MSE | 2.03E-04 | 4.16E-05 | 4.38E-05 | 1.96E-04 | |
| n=90, m=50 | mean | 0.90653 | 0.91534 | 0.91576 | 0.90673 | SH1 |
| | MSE | 2.39E-04 | 3.97E-05 | 4.44E-05 | 2.31E-04 | |

**Table 16:** Reliability Bayesian estimators values when $\lambda = 3, \mu = 0.9, k = 3$ and $R = 0.90909$

| | | JS | JQ | JW | GS | GQ | GW | Best |
|---|---|---|---|---|---|---|---|---|
| n=10, m=30 | mean | 0.90735 | 0.89373 | 0.90124 | 0.90514 | 0.89586 | 0.90086 | GS |
| | MSE | 9.31E-04 | 1.42E-03 | 1.10E-03 | 7.16E-04 | 1.00E-03 | 8.25E-04 | |
| n=30, m=10 | mean | 0.90233 | 0.91494 | 0.90837 | 0.88725 | 0.89718 | 0.89203 | JQ |
| | MSE | 1.25E-03 | 9.84E-04 | 1.08E-03 | 1.76E-03 | 1.24E-03 | 1.49E-03 | |
| n=50, m=90 | mean | 0.90907 | 0.90754 | 0.90832 | 0.90793 | 0.90653 | 0.90724 | GS |
| | MSE | 2.03E-04 | 2.12E-04 | 2.07E-04 | 1.96E-04 | 2.06E-04 | 2.01E-04 | |
| n=90, m=50 | mean | 0.90653 | 0.90807 | 0.90729 | 0.90338 | 0.90483 | 0.90410 | JQ |
| | MSE | 2.39E-04 | 2.27E-04 | 2.33E-04 | 2.70E-04 | 2.50E-04 | 2.60E-04 | |

## 5. Result Discussion

For the non-Bayesian estimation, the results show that SH1, SH2 perform better in terms of MSE over MLE and SH3 as seen in Tables (1,3,5,7,9,11,13 15). It is interesting to note that even with larger MSE the MLE method had in many cases mean value closer to the real value of the reliability than other methods. This is due to the high variance in the reliability simulation values of the MLE method.

No clear pattern had been seen in terms of Bayesian estimation as to which one performed better except in the case when $\lambda, \mu = 0.5, 0.9$, respectively when the GQ estimator seemed to perform better in each case as seen in Table 2. The data shows various methods performing better in each case as seen in Tables (2,4,6,8,10,12,14,16). It is worth mentioning though that higher MSE JS coincided with the MLE.

The comparison between the Bayesian and non-Bayesian methods is not included since the later considers the existence of some information about the estimated parameter while the first does not.

## 6. Conclusion and Recommendation

Estimation of the reliability parameters had been conducted according to non-Bayesian and Bayesian methods for a parallel redundant system based on Burr of type X distribution. The data was generated using a size 1000 simulation with a different value for the parameters $\lambda$ and $\mu$ and components $k$. The result shows that SH1 and SH2 had better MSE in the non-Bayesian estimation. No clear pattern had been seen in terms of Bayesian estimation as to which one performed better.

Therefore, it is recommended to use the SH1 method in the non-Bayesian estimation as it seems to outperform the other methods in most cases. As to the Bayesian estimation, a slight advantage to Gamma quadratic prior. It is therefore recommended to use GQ method.

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