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# On the Dynamics of Prey-Predator Model Involving Treatment and Infections Disease in Prey Population

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**Abstract**
In this paper, a mathematical model consisting of the prey- predator model with treatment and disease infection in prey population is proposed and analyzed. The existence, uniqueness and boundedness of the solution are discussed. The stability analyses of all possible equilibrium points are studied. Numerical simulation is carried out to investigate the global dynamical behavior of the system.

**Keywords:** eco-epidemiological model, *SIS* epidemics disease, prey-predator model, stability analysis.

# حول ديناميكية نموذج الفريسة – المفترس المتضمن العلاج لمرض معدي في مجتمع الفريسة

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**الخلاصة**
في هذا البحث، تم اقتراح ودراسة نموذج رياضي يتكون من فريسة والمفترس عند وجود مرض معدي ومعالجتة في مجتمع الفريسة. ناقشنا وجود، وحدانية وقيد الحل. قمنا بدراسة وجود و تحليل الاستقرارية لجميع نقاط التوازن الممكنة. كما تم استخـدام المحاكاة العددية لبحث السلوك الديناميكي الشامل للنظام.

## 1. Introduction

Ecological populations suffer from the various infectious diseases and these diseases have a significant role in regulating population size. Thus, it is worthwhile to study the combined effect of epidemiological and demographic features on the real ecological populations. Mathematical study of such eco-epidemiological model has explored various unknown aspects of ecological population [1]. However, in ecosystem, the interaction between the predator and prey is a nonlinear and complex process. This complexity has attracted the attention of both theoretical and mathematical ecologists to have extensive investigation concerning the interaction which calls for development of mathematical models that are essential tools in understanding the interaction mechanisms for persistence or extinction of species in natural systems.

Eco-epidemiology is a new branch in mathematical biology which considers both the ecological and epidemiological issues simultaneously [2]. Since [3] modeled firstly a disease spreading among interacting populations, scientists are paying increasing interests to this new field due to its theoretical and empirical importance [2-4]. As a result, the study of diseases in a prey -predator system has also become a very popular topic in eco-epidemiology and made a significant progress in understanding different scenarios for disease transmission [2-5]. Among these studies, most considered the transmission of disease in prey populations. However, epidemic diseases can attack predators through various means,such as food, mating and parasites, then infectious diseases in prey species has need to

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be explored. Mathematical models had become important tools to analyze the spread and control of disease. These models, which known as epidemiological models, are used to study the spread and control of diseases in human or animal populations. One of the major mathematical model in the field of epidemiology that describe the transition of disease from susceptible to infected and then to removal individuals had been formulated by Kermack and Mckendric in 1927. On the other hand, the Mathematical models which describe the dynamical behavior of an interacting species in ecology are known as ecological model. The first mathematical model in the field of ecology that describes the interactions between biological species was formulated, independently, by Lotka (American physical chemist) in 1925 and Volterra (Italian mathematician) in 1926. The researchers had been studied the dynamics of the mathematical models of these two fields (epidemiology and ecology) independently for long years, see for example [5-14]. However during the last four decades the ideas oriented to study the dynamical behavior of the mathematical models involving both the fields simultaneous, these models are known as an eco-epidemiological models.

On contrast to all of the above studies, in this paper a prey-predator model with treatment and disease (SIS) infection in Prey population is proposed and analyzed. Disease dose not spread outside the specific prey species instead the disease transmitted within the same species by contact, according to ratio-dependent incidence. Instead the disease transmitted within the same species by contact, between susceptible individuals and infected individuals. Further, in this model, non linear type of functional response, represented by Holloing type II is used.

**2. Mathematical Model.**

In this section, an eco-epidemiological model is proposed for study. The model consists of a prey, whose total population density at time $T$ is denoted by $N(T)$, interacting with predator population whose density at time $T$ is denoted by $Z(T)$ and population of infected prey under treatment is denoted by $T_r(T)$. Further, the following assumptions are made in formulating the basic eco-epidemiological model:

1. There is an *SIS* epidemic disease in prey population divides the prey population into two classes namely $X(T)$ that represents the density of susceptible prey species at time $T$ and $Y(T)$, which represents the density of infected prey species at time $T$. Therefore at any $T$ we have
   $N(T) = X(T) + Y(T)$.
2. The susceptible prey is capable of reproducing in logistic fashion with carrying capacity $K > 0$, intrinsic growth rate $r > 0$. In addition the disease has the capability to compete with the susceptible.
3. Disease dose not spread outside the specific species prey instead the disease transmitted within the same species by contact, according to ratio-dependent incidence the susceptible rate with infection rate constant $\lambda > 0$. Further the disease disappears and infected individuals become susceptible again at the recover rate $\gamma > 0$.
4. The disease in prey may causes mortality with a constant mortality rate represented by $d_1 > 0$.
5. The predator consumes the prey according to Holling type-II of functional response with maximum attack rate $\alpha > 0$ and $\beta > 0$ from susceptible prey and infected prey respectively. However the constant $m > 0$ represent the half saturation for the susceptible and infected predator respectively.
6. In the absence of the prey the predator decays exponentially with natural death rate $d_2 > 0$.
7. The infected prey is treated at the rate $a > 0$ and removed without immunity at the rate $\delta > 0$ while $d_3 > 0$ is the death of infected prey under treatment.

Considering the above basic assumptions the prey-predator model can be represented in the following set of differential equations.

$$\begin{aligned}
\frac{dX}{dT} &= rX\left(1-\frac{X+Y}{K}\right)-\frac{\lambda XY}{X+Y}-\frac{\alpha XZ}{m+X}+\delta T_r+\gamma Y \\
\frac{dY}{dT} &= \frac{\lambda XY}{X+Y}-\frac{\beta YZ}{m+Y}-(d_1+a+\gamma)Y \\
\frac{dZ}{dT} &= \frac{e_1\alpha XZ}{m+X}+\frac{e_2\beta YZ}{m+Y}-d_2 Z \\
\frac{dT_r}{dT} &= aY-(\delta+d_3)T_r
\end{aligned} \tag{1}$$

With $X(0)\ge 0; Y(0)\ge 0; Z(0)\ge 0; T_r(0)\ge 0$ and $0<e_i<1$; $i=1,2$ represent the conversion rate constants. Consequently, the flow of the food, disease and treatment in system (1) can be described in the following block diagram.

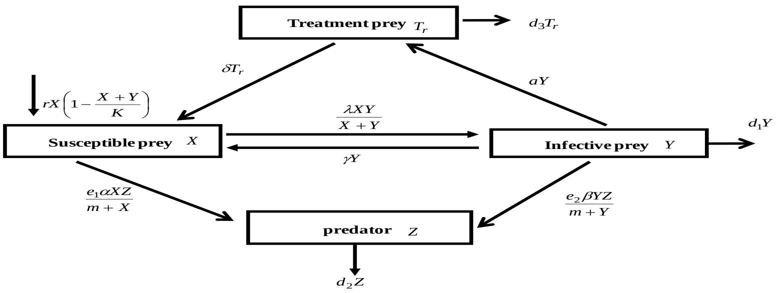


**Figure 1-** Block diagram of the prey –predator model given by system (1).

Cleary, system (1) included (14) parameters which make the analysis difficult. So, in order to simplify the system the number of parameters is reduced by using the following dimensionless variables

$$\mathrm{t}=\mathrm{rT},\ x=\frac{X}{K},\ y=\frac{Y}{K},\ z=\frac{Z}{K},\ w=\frac{T_r}{K}.$$

Thus we obtain the following dimensionless form of the system (1):

$$\begin{aligned}
\frac{dx}{dt} &= x[1-(x+y)]-\tfrac{w_1xy}{x+y}-\tfrac{w_2xz}{w_3+x}+w_4w+w_5y \\
\frac{dy}{dt} &= \tfrac{w_1xy}{x+y}-\tfrac{w_6yz}{w_3+y}-(w_7+w_8+w_5)y \\
\frac{dz}{dt} &= \tfrac{e_1w_2xz}{w_3+x}+\tfrac{e_2w_6yz}{w_3+y}-w_9z \\
\frac{dw}{dt} &= w_8y-(w_4+w_{10})w
\end{aligned} \tag{2}$$

Here:

$$w_1=\frac{\lambda}{r}, w_2=\frac{\alpha}{r}, w_3=\frac{m}{K}, w_4=\frac{\delta}{r}, w_5=\frac{\gamma}{r}, w_6=\frac{\beta}{r},$$
$$w_7=\frac{d_1}{r}, w_8=\frac{a}{r}, w_9=\frac{d_2}{r}, w_{10}=\frac{d_3}{r}$$

Represent the dimensionless parameters of the system (2). Further, the interaction functions $F_i(x,y,z,w), i=1,2,3,4.$ are continuously differentiable on

$Int.R^4_+=\{(x,y,z,w)\in R^4_+, x>0, y>0, z>0, w>0\}.$

In addition to that $\lim\limits_{(x,y,z,w)\to(0,0,0,0)} F_i(x,y,z,w), \forall i=1,2,3,4.$ and $\lim\limits_{(x,y,z,w)\to(x,0,0,0)} F_i(x,y,z,w), \forall i=1,2,3,4, x\in R^4_+$. So, if we define that $F_i(0,0,0,0)=F_i(x,0,0,0)=0, \forall i=1,2,3,4.$ Then with this assumption the interaction functions of system (2), $F_i, i=1,2,3,4.$ are continuously differentiable on the extended domain $R_+^4 = \{(x, y, z, w) \in R_+^4, x \geq 0, y \geq 0, z \geq 0, w \geq 0\}$. In fact, they are Lipschizian on $R_+^4$. Accordingly, the solution of system (2) with non negative initial condition exists and is unique. Therefore $R_+^4$ is invariant for the system (2). Moreover in the following theorem the sufficient condition for uniformly bounded of the solution of system (2) is established.

**Theorem 1.** All solutions of system (2) are uniformly bounded.

**Proof.** Let $(x(t), y(t), z(t), w(t))$ be any solution of the system (2). Define the function $M(t) = x(t) + y(t) + z(t) + w(t)$, then the time derivative of $M(t)$ along the solution of the system (2), gives

$$\frac{dM}{dt} \leq 2 - nM$$

Where $n = \min\{1, w_7, w_9, w_{10}\}$. Now, by using Gronwell lemma, it obtains that:

$$0 < M(t) \leq M(0)e^{-nt} + \frac{2}{n}(1 - e^{-nt})$$

Which yields $\lim_{t\to\infty} M(t) \leq \frac{2}{n}$ that is independent of the initial conditions. Thus the proof is complete.

**3. Existence of equilibrium points.**

It is observed that, system (2) has at most five biologically feasible equilibrium points, namely $E_0$, $E_x, E_{xz}$, $E_{xyw}$ and $E_{xyzw}$ . The existence conditions for each of these equilibrium points are discussed in the following:

1- **The vanishing equilibrium point** $E_0 = (0,0,0,0)$ always exists.

2- **The axial equilibrium point** $E_x = (1,0,0,0)$ always exists.

3- **The disease free equilibrium point** $E_{xz} = (\hat{x}, 0, \hat{z}, 0)$ ,where:

$$\hat{x} = \frac{w_3 w_9}{e_1 w_2 - w_9} \text{ and } \hat{z} = \frac{e_1 w_2 w_3 [e_1 w_2 - (1 + w_3) w_9]}{w_2 (e_1 w_2 - w_9)^2} \tag{3}$$

Exists uniquely in the interior of first quadrant of $xz$ - plane under the following necessary and sufficient condition:

$$e_1 w_2 > (1 + w_3) w_9 \tag{4}$$

4- **The predator free equilibrium point** $E_{xyw} = (\tilde{x}, \tilde{y}, 0, \tilde{w})$ ,where:

$$\tilde{x} = \frac{b\tilde{y}}{w_1 - b} \text{ , } \tilde{w} = \frac{w_8 \tilde{y}}{w_4 + w_{10}} \text{ and } \tilde{y} = \frac{ab(w_1 - b) + (w_1 - b)^2 [w_4 w_8 + a w_5 - ab]}{abw_1} \tag{5}$$

Here $a = (w_4 + w_{10})$ and $b = w_5 + w_7 + w_8$ (5a)

exists uniquely in the interior of first quadrant of $xyw$ - octant under the following necessary and sufficient conditions:

$$w_1 > b \text{ and } w_4 w_8 + a w_5 > ab \tag{6}$$

5- **The positive equilibrium point** $E^* = (x^*, y^*, z^*, w^*)$ Where:

$$w^* = \alpha_1 y^*$$

$$x^* = \frac{w_3 \alpha_3 y^* + \alpha_6}{w_3 \alpha_2 + \alpha_5 y^*} \tag{7}$$

$$z^* = \frac{-b\alpha_5 y^{*3} + \alpha_8 y^{*2} + \alpha_9 y^* + \alpha_{10}}{w_6 [\alpha_5 y^{*2} + \alpha_7 y^* + \alpha_6]}$$

while $y^*$ represents a positive root of the following eighth order polynomial equation

$$I_1 y^8 + I_2 y^7 + I_3 y^6 + I_4 y^5 + I_5 y^4 + I_6 y^3 + I_7 y^2 + I_8 y + I_9 = 0 \tag{8}$$

where:

$I_1 = w_6\alpha_5 a_0 + \alpha_5^2 a_{10} + w_3\alpha_5^3 a_{17};$

$I_2 = w_6\alpha_5 a_1 + w_3\alpha_5^3 a_{23} + a_{17}[2w_3 w_6\alpha_2\alpha_5^2 + \alpha_5^2\alpha_7] + 2w_3\alpha_2\alpha_5 a_{10} + \alpha_5^2 a_{11};$

$$I_3 = \alpha_6 a_0 + w_6\alpha_5 a_2 + \alpha_7 a_1 + w_3\alpha_5^3 a_{25} + a_{23}[2w_3 w_6\alpha_2\alpha_5^2 + \alpha_5^2\alpha_7] + a_{17}a_{21} + w_2^2\alpha_2^2 a_{10} + 2w_3\alpha_2\alpha_5 a_{11} + \alpha_5^2 a_{12};$$

$$I_4 = w_6\alpha_5 a_3 + \alpha_7 a_2 + \alpha_6 a_1 + w_3\alpha_5^3 a_{24} + a_{23}[2w_3 w_6\alpha_2\alpha_5^2 + \alpha_5^2\alpha_7] + a_{22}a_{17} + w_2^2\alpha_2^2 a_{11} + 2w_3\alpha_2\alpha_5 a_{12} + \alpha_5^2 a_{13};$$

$$I_5 = w_6\alpha_5 a_4 + \alpha_7 a_3 + \alpha_6 a_2 + a_{24}[2w_3 w_6\alpha_2\alpha_5^2 + \alpha_5^2\alpha_7] + a_{21}a_{25} + a_{22}\beta_0 + w_3^2\alpha_2^2\alpha_6 a_{17} + w_2^2\alpha_2^2 a_{12} + 2w_3\alpha_2\alpha_5 a_{13} + \alpha_5^2 a_{14};$$

$$I_6 = w_6\alpha_5 a_5 + \alpha_7 a_4 + \alpha_6 a_3 + a_{21}a_{24} + a_{22}a_{25} + w_3^2\alpha_2^2\alpha_6 a_{23} + w_2^2\alpha_2^2 a_{13} + 2w_3\alpha_2\alpha_5 a_{10} + \alpha_5^2 a_{13}$$

$$I_7 = w_6\alpha_5 a_6 + \alpha_7 a_5 + \alpha_6 a_4 + a_{22}a_{24} + w_3^2\alpha_2^2\alpha_6 a_{25} + w_2^2\alpha_2^2 a_{14} + 2w_3\alpha_2\alpha_5 a_{15} + \alpha_5^2 a_{16}$$

$$I_8 = \alpha_7 a_6 + \alpha_6 a_5 + w_3^2\alpha_2^2\alpha_6 a_{24} + w_2^2\alpha_2^2 a_{15} + 2w_3\alpha_2\alpha_5 a_{16}$$

$$I_9 = \alpha_6 a_6 + w_2^2\alpha_2^2 a_{16}$$

And

$$\alpha_1 = \frac{w_8}{w_4 + w_{10}} > 0,\ \alpha_2 = e_1 w_2 - w_9,\ \alpha_3 = w_9 - e_2 w_6,\ \alpha_4 = w_1 - (w_7 + w_8 + w_5)$$

$$\alpha_5 = \alpha_2 + e_2 w_6,$$

$$\alpha_6 = w_3^2 w_9 > 0,\ \alpha_7 = w_3(\alpha_2 + \alpha_3),\ \alpha_8 = w_3\alpha_3\alpha_4 - w_3(w_7 + w_8 + w_5)(\alpha_2 + \alpha_5)$$

$$\alpha_9 = \alpha_4(w_3^2\alpha_3 + \alpha_6) - \alpha_2 w_3^2(w_7 + w_8 + w_5),\ \alpha_{10} = w_3\alpha_4\alpha_6,\ \alpha_{11} = w_3\alpha_5(\alpha_3 + \alpha_5)$$

$$\alpha_{12} = \alpha_5(\alpha_6 + w_3^2\alpha_6) + \alpha_7 w_3(\alpha_3 + \alpha_5),\ \alpha_{13} = \alpha_7(\alpha_6 + w_3^2\alpha_2) + \alpha_6 w_3(\alpha_3 + \alpha_5)$$

$$\alpha_{14} = \alpha_6(\alpha_6 + w_3^2\alpha_2),\ a_0 = -w_3\alpha_3\alpha_5\alpha_{11},$$

$$a_1 = w_3\alpha_3[\alpha_{11}(\alpha_5 - \alpha_7) - \alpha_5\alpha_{12}] - \alpha_5\alpha_6\alpha_{11}$$

$$a_2 = w_3\alpha_3[\alpha_{11}(w_3\alpha_2 - \alpha_6) + \alpha_{12}(\alpha_5 - \alpha_7) - \alpha_5\alpha_{13}] + \alpha_5[\alpha_{11}(\alpha_5 - \alpha_7) - \alpha_5\alpha_{12}]$$

$$a_3 = w_3\alpha_3[\alpha_{12}(w_3\alpha_2 - \alpha_6) + \alpha_{13}(\alpha_5 - \alpha_7) - \alpha_5\alpha_{14}] + \alpha_6[\alpha_{12}(w_3\alpha_2 - \alpha_6) + \alpha_{12}(\alpha_5 - \alpha_7) - \alpha_5\alpha_{13}]$$

$$a_4 = (w_3\alpha_2 - \alpha_6)[w_3\alpha_3\alpha_{12} + \alpha_6\alpha_{12}] + (\alpha_5 - \alpha_7)[w_3\alpha_3\alpha_{14} + \alpha_6\alpha_{13}] - \alpha_5\alpha_6\alpha_{14}$$

$$a_5 = (w_3\alpha_2 - \alpha_6)[w_3\alpha_3\alpha_{14} + \alpha_6\alpha_{12}] + \alpha_6\alpha_{14}(\alpha_5 - \alpha_7),\ a_6 = \alpha_6\alpha_{14}(w_3\alpha_2 - \alpha_6)$$

$$a_7 = -w_1 w_3^2\alpha_3(\alpha_3 + \alpha_5),\ a_8 = -w_1 w_3[\alpha_3(\alpha_6 + w_3^2\alpha_2) + \alpha_6(\alpha_3 + \alpha_5)]$$

$$a_9 = -w_1\alpha_6(\alpha_6 + w_3^2\alpha_2),\ a_{10} = \alpha_5^2 w_2 b w_3\alpha_3,\ a_{11} = w_3\alpha_3[w_2\alpha_5(b\alpha_7 - \alpha_8)] + \alpha_5^2 w_2 b\alpha_6$$

$$a_{12} = w_3\alpha_3\left[w_2[b\alpha_5\alpha_6 - (\alpha_5\alpha_7 + \alpha_7\alpha_8)]\right] + w_2\alpha_6\alpha_5(b\alpha_7 - \alpha_8)$$

$$a_{13} = -w_2 w_3\alpha_3[\alpha_5\alpha_{10} + \alpha_7\alpha_9 + \alpha_6\alpha_8)] + w_2\alpha_6[b\alpha_5\alpha_6 - (\alpha_5\alpha_7 + \alpha_7\alpha_8)]$$

$$a_{14} = -w_2 w_3\alpha_3[\alpha_7\alpha_{10} + \alpha_6\alpha_9)] - w_2\alpha_6[\alpha_5\alpha_{10} + \alpha_7\alpha_9 + \alpha_6\alpha_8)]$$

$$a_{15} = -w_2 w_3\alpha_3\alpha_6\alpha_{10} - w_2\alpha_6[\alpha_7\alpha_{10} + \alpha_6\alpha_9)],\ a_{16} = -w_2\alpha_6^2\alpha_{10}$$

$$a_{17} = \alpha_{11}(w_4\alpha_1 + w_5),\ a_{18} = \alpha_{12}(w_4\alpha_1 + w_5),\ a_{19} = \alpha_{13}(w_4\alpha_1 + w_5)$$

$$a_{20} = \alpha_{14}(w_4\alpha_1 + w_5),\ a_{21} = w_3^2 w_6\alpha_5\alpha_2^2 + 2w_3\alpha_2\alpha_5\alpha_7 + \alpha_5^2\alpha_6$$

$$a_{22} = w_3^2\alpha_2^2\alpha_7 + 2w_3\alpha_2\alpha_5\alpha_6,\ a_{23} = a_7 + a_{18},\ a_{24} = a_9 + a_{20},\ a_{25} = a_8 + a_{19}$$

The positive equilibrium point $E^* = (x^*, y^*, z^*, w^*)$ exists uniquely in $Int.R_+^4$ if and only if the following conditions are hold.

$I_1 > 0, I_2 > 0, I_3 > 0, I_4 > 0, I_5 > 0, I_6 > 0$ and $I_8 < 0$ (9a)

*OR*

$I_1 > 0, I_2 > 0, I_3 > 0, I_4 > 0, I_5 > 0, I_7 < 0$ and $I_8 < 0$ (9b)

*OR*

$I_1 > 0, I_2 > 0, I_3 > 0, I_4 > 0, I_6 < 0, I_7 < 0$ and $I_8 < 0$ (9c)

*OR*

$I_1 > 0, I_2 > 0, I_3 > 0, I_5 < 0, I_6 < 0, I_7 < 0$ and $I_8 < 0$ (9d)

*OR*

$I_1 > 0, I_2 > 0, I_4 < 0, I_5 < 0, I_6 < 0, I_7 < 0$ and $I_8 < 0$ (9e)

*OR*

$I_1 > 0, I_3 < 0, I_4 < 0, I_5 < 0, I_6 < 0, I_7 < 0$ and $I_8 < 0$ (9f)

*OR*

$I_1 < 0, I_2 < 0, I_3 < 0, I_4 < 0, I_5 < 0, I_6 < 0$ and $I_8 > 0$ (9g)

*OR*

$I_1 < 0, I_2 < 0, I_3 < 0, I_4 < 0, I_5 < 0, I_7 > 0$ and $I_8 > 0$ (9h)

*OR*

$I_1 < 0, I_2 < 0, I_3 < 0, I_4 < 0, I_6 > 0, I_7 > 0$ and $I_8 > 0$ (9i)

*OR*

$I_1 < 0, I_2 < 0, I_3 < 0, I_5 > 0, I_6 > 0, I_7 > 0$ and $I_8 > 0$ (9j)

*OR*

$I_1 < 0, I_2 < 0, I_4 > 0, I_5 > 0, I_6 > 0, I_7 > 0$ and $I_8 > 0$ (9k)

*OR*

$I_1 < 0, I_3 > 0, I_4 > 0, I_5 > 0, I_6 > 0, I_7 > 0$ and $I_8 > 0$ (9l)

$e_2 w_6 < w_9 < e_1 w_2$, $w_1 > (w_7 + w_8 + w_5)$ and $b\alpha_5 y^3 < \alpha_8 y^2 + \alpha_9 y + \alpha_{10}$ (9m)

**4. Local Stability Analysis of System (2):**

In this section, the local stability analyses of system (2) around each of the above equilibrium points of system (2) are studied with the help of Linearization method as shown in the following theorems. Note that the symbols $\lambda_{ix}, \lambda_{iy}, \lambda_{iz}$ and $\lambda_{iw}$ denote to the eigenvalues of the Jacobian matrix $J(E_i)$; $i = 0,\ldots,5$ that describe the dynamics in the $x$- direction- $y$ direction, $z$-direction and $w$-direction, respectively.

The Jacobian matrix of system (2) at $E_0$ can be written as:

$$J(E_0) = [\ \psi_{ij}\ ]_{4\times 4} \tag{10}$$

where:

$\psi_{11} = 1 > 0; \psi_{12} = \psi_{13} = \psi_{14} = 0;$

$\psi_{21} = \psi_{23} = \psi_{24} = 0; \psi_{22} = -b;$

$\psi_{31} = \psi_{32} = \psi_{34} = 0; \psi_{33} = -w_9 < 0;$

$\psi_{41} = \psi_{43} = 0, \psi_{42} = w_8; \psi_{44} = -(w_4 + w_{10}) < 0$

Clearly, $J(E_0)$ has the following eigenvalues:

$$\lambda_{0x} = 1 > 0; \lambda_{0y} = -b < 0; \lambda_{0z} = -w_9 < 0; \lambda_{0w} = -(w_4 + w_{10}) < 0$$

Here $b$ define in eq.(5a). Since $J(E_0)$ has one positive eigenvalue in the $x$-direction, then by using the stability theorem, the equilibrium point $E_0$ is unstable saddle point.

The Jacobian matrix of system (2) at $E_x$ can be written as:

$$J(E_x)=[k_{ij}]_{4\times 4} \tag{11}$$

Where:

$$k_{11}=-1<0;\ k_{12}=w_5-(1+w_1);\ k_{13}=\frac{-w_2}{w_3+1}<0;\ k_{14}=w_4;$$

$$k_{21}=k_{23}=k_{24}=0; k_{22}=w_1-b;$$

$$k_{31}=k_{32}=k_{34}=0; k_{33}=\frac{e_1w_2}{w_3+1}-w_9;$$

$$k_{41}=k_{43}=0; k_{42}=w_8; k_{44}=-(w_4+w_{10})$$

Clearly, $J(E_x)$ has the following eigenvalues:

$$\lambda_{1x}=-1<0; \lambda_{1y}=w_1-b;\ \lambda_{1z}=\frac{e_1w_2}{w_3+1}-w_9;\ \lambda_{1w}=-(w_4+w_{10})<0$$

Therefore all the eigenvalues have negative real parts provided that the following conditions are satisfied:

$$w_1<b \tag{12a}$$

$$\frac{e_1w_2}{w_3+1}<w_9 \tag{12b}$$

Hence the axial equilibrium point $E_x=(1,0,0,0)$ of the system (2) is locally asymptotically stable in the $Int.R_+^4$.

The Jacobian matrix of system (2) at the disease free equilibrium point $E_{xz}=(\hat{x},0,\hat{z},0)$ can be written as:

$$J(E_{xz})=\lfloor e_{ij}\rfloor_{4\times 4} \tag{13}$$

Where:

$$e_{11}=\hat{x}\left(\frac{w_2\hat{z}}{(w_3+\hat{x})^2}-1\right); e_{12}=w_5-(\hat{x}+w_1); e_{13}=\frac{-w_2\hat{x}}{w_3+\hat{x}}; e_{14}=w_4;$$

$$e_{21}=e_{23}=e_{24}=0; e_{22}=w_1-\frac{w_6\hat{z}}{w_3}-b$$

$$e_{31}=\frac{e_1w_2w_3\hat{z}}{(w_3+\hat{x})^2}; e_{32}=\frac{e_2w_6\hat{z}}{w_3}; e_{33}=e_{34}=0;$$

$$e_{41}=e_{43}=0;\ e_{42}=w_8; e_{44}=-(w_4+w_{10})$$

Here $b$ define in eq.(5a). Note that the characteristic equation of this Jacobian matrix is given by

$$\left[\lambda_2^2-e_{11}\lambda_2-e_{13}e_{31}\right]\left[(e_{22}-\lambda_{2y})(e_{44}-\lambda_{2w})\right]=0$$

Hence, straightforward computations show that, the eigenvalues of $J(E_{xz})$ satisfy the following relations

$$\lambda_{2x}+\lambda_{2z}=e_{11} \tag{14a}$$

$$\lambda_{2x}.\lambda_{2z}=-e_{13}e_{31} \tag{14b}$$

$$\lambda_{2y}=e_{22} \tag{14c}$$

$$\lambda_{2w}=e_{44} \tag{14d}$$

Clearly according to the following condition all the eigenvalues have negative real parts.

$$\frac{w_3(w_1-b)}{w_6}<\hat{z}<\frac{(w_3+\hat{x})^2}{w_2} \tag{15}$$

Hence the equilibrium point $E_{xz}$ is locally asymptotically stable in $R_+^4$.

**Theorem 2.** The predator free equilibrium point $E_{xyw}=(\tilde{x},\tilde{y},0,\tilde{w})$ of system (2) is locally asymptotically stable in the $R_{+}^{4}$ if and only if the following conditions are satisfied:

$$\frac{e_1 w_2 \tilde{x}}{w_3+\tilde{x}}+\frac{e_2 w_6 \tilde{y}}{w_3+\tilde{y}}<w_9 \tag{16a}$$

$$\frac{w_1\tilde{x}\tilde{y}}{[\tilde{x}^2+w_4\tilde{w}+w_5\tilde{y}]}<(\tilde{x}+\tilde{y})^2<\min\left\{\frac{\tilde{x}\left((\tilde{x}+\tilde{y})^2+w_1\tilde{x}\right)}{w_5},\frac{2w_1\tilde{x}^2 y}{2w_5\tilde{y}+2w_4\tilde{w}+\tilde{x}^2+\tilde{x}\tilde{y}}\right\} \tag{16b}$$

$$Q_1+Q_2>0 \tag{16c}$$

**Proof.** The Jacobian matrix of system (2) at the predator free equilibrium point $E_{xyw}=(\tilde{x},\tilde{y},0,\tilde{w})$ can be written as:

$$J(E_{xyw})=\left[h_{ij}\right]_{4\times 4} \tag{17}$$

Here:

$$h_{11}=-\tilde{x}+\frac{w_1\tilde{x}\tilde{y}}{(\tilde{x}+\tilde{y})^2}-\frac{w_4\tilde{w}}{\tilde{x}}-\frac{w_5\tilde{y}}{\tilde{x}};h_{12}=w_5-\tilde{x}\left(1+\frac{w_1\tilde{x}}{(\tilde{x}+\tilde{y})^2}\right);$$

$$h_{13}=\frac{-w_2\tilde{x}}{w_3+\tilde{x}};h_{14}=w_4;h_{21}=\frac{w_1\tilde{y}^2}{(\tilde{x}+\tilde{y})^2};h_{22}=\frac{-w_1\tilde{x}\tilde{y}}{(\tilde{x}+\tilde{y})^2};h_{23}=\frac{-w_6\tilde{y}}{w_3+\tilde{y}};h_{24}=0;$$

$$h_{31}=h_{32}=h_{34}=0;h_{33}=\frac{e_1w_2\tilde{x}}{w_3+\tilde{x}}+\frac{e_2w_6\tilde{y}}{w_3+\tilde{y}}-w_9;\ h_{41}=h_{43}=0;h_{42}=w_8;h_{44}=\frac{-w_8\tilde{y}}{\tilde{w}}$$

Then the characteristic equation of $J(E_{xyw})$ can be written as:

$$(h_{33}-\lambda_{3z})\left[\lambda_3^{\,3}+B_1\lambda_3^2+B_2\lambda_3+B_3\right]=0$$

Here:

$$B_1=-(R_1+h_{44})$$
$$B_2=R_2+h_{44}R_1$$
$$B_3=-(h_{44}R_2+R_3)$$

With $R_1=h_{11}+h_{22}$, $R_2=h_{11}h_{22}-h_{12}h_{21}$ and $R_3=h_{21}h_{14}h_{42}$

Note that, according to the element of $J(E_{xyw})$, it is easy to verify that:

$$R_1=-\left(\tilde{x}+\frac{w_4\tilde{w}}{\tilde{x}}+\frac{w_5\tilde{y}}{\tilde{x}}\right)$$

$$R_2=\left(-\tilde{x}+\frac{w_1\tilde{x}\tilde{y}}{(\tilde{x}+\tilde{y})^2}-\frac{w_4\tilde{w}}{\tilde{x}}-\frac{w_5\tilde{y}}{\tilde{x}}\right)\left(\frac{-w_1\tilde{x}\tilde{y}}{(\tilde{x}+\tilde{y})^2}\right)-\left(w_5-\tilde{x}\left(1+\frac{w_1\tilde{x}}{(\tilde{x}+\tilde{y})^2}\right)\right)\left(\frac{w_1\tilde{y}^2}{(\tilde{x}+\tilde{y})^2}\right)$$

$$R_3=\left(\frac{w_1w_4w_8\tilde{y}^2}{(\tilde{x}+\tilde{y})^2}\right)$$

Further, it is easy to verify that $\Delta=B_1B_2-B_3=Q_1+Q_2$, where

$Q_1=-(R_1+h_{44})(R_2+h_{44}R_1);\ Q_2=(h_{44}R_2+R_3)$

Clearly, the eigenvalue $\lambda_{3z}$ in $z$- direction has negative real part if and only if condition (16a) holds. However, $B_i>0\quad\forall i=1,3;\ Q_1>0$ provided that conditions 16(b) hold. Finally, condition (16c )

guarantees that $\Delta > 0$. So, according to the (Routh–Hawirtiz) criterion the equilibrium point $L_4$ is locally asymptotically stable and the proof is complete.
Similarly the following theorem for locally stability of $E_{xyzw}$ can be proved easily.

**Theorem 3**. Assume that the positive equilibrium point $E_{xyzw}$ of system (2) exists. Then $E_{xyzw}$ is locally asymptotically stable in the $Int.\, R_+^4$ if the conditions (20a)-(20e) and (20f) are satisfied.

**Proof.** The Jacobian matrix of system (2) at the positive equilibrium point $E^* = (x^*, y^*, z^*, w^*)$ can be written as:

$$J(E_{xyzw}) = \lfloor a_{ij} \rfloor_{4\times 4} \tag{18}$$

where

$$a_{11} = -x^* + \frac{w_1 x^* y^*}{(x^*+y^*)^2} + \frac{w_2 x^* z^*}{(w_3+x^*)^2} - \frac{w_4 w^*}{x^*} - \frac{w_5 y^*}{x^*}; a_{12} = w_5 - x^*\left(1 + \frac{w_1 x^*}{(x^*+y^*)^2}\right);$$

$$a_{13} = \frac{-w_2 x^*}{w_3 + x^*}; a_{14} = w_4; a_{21} = \frac{w_1 y^{*2}}{(x^*+y^*)^2}; a_{22} = y^*\left(\frac{-w_1 x^*}{(x^*+y^*)^2} + \frac{w_6 z^*}{(w_3+y^*)^2}\right);$$

$$a_{23} = \frac{-w_6 y^*}{w_3 + y^*}; a_{24} = 0; a_{31} = \frac{e_1 w_2 w_3 z^*}{(w_3+x^*)^2}; a_{32} = \frac{e_2 w_3 w_6 z^*}{(w_3+y^*)^2}; a_{33} = a_{34} = 0;$$

$$a_{41} = a_{43} = 0; a_{42} = w_8; a_{44} = \frac{-w_8 y^*}{w^*}$$

Accordingly the characteristic equation of $J(E_{xyzw})$ is given by:

$$\lambda_4^4 + A_1 \lambda_4{}^3 + A_2 \lambda_4^2 + A_3 \lambda_4 + A_4 = 0 \tag{19}$$

where

$$A_1 = -(\sigma_1 + a_{44});$$
$$A_2 = \sigma_2 + a_{44}\sigma_1 - \sigma_3;$$
$$A_3 = a_{32}\sigma_5 - a_{31}\sigma_4 + a_{44}(\sigma_3 - \sigma_2) - \sigma_6;$$
$$A_4 = a_{44}(a_{31}\sigma_4 - a_{32}\sigma_5) - \sigma_7;$$

with $\sigma_1 = a_{11} + a_{22}, \sigma_2 = a_{11}a_{22} - a_{12}a_{21}, \sigma_3 = a_{13}a_{31} + a_{23}a_{32},$
$\sigma_4 = a_{12}a_{23} - a_{13}a_{22}, \sigma_5 = a_{11}a_{23} - a_{13}a_{21}, \ \sigma_6 = a_{14}a_{21}a_{42},$
$\sigma_7 = a_{14}a_{23}a_{31}a_{42}.$

Note that, due to Routh-Hurwitz criterion, the necessary and sufficient conditions for $E_{xyzw}$ to be locally asymptotically stable in the $Int.\, R_+^4$, are $A_i > 0$ *for* $i = 1,2,3,4.$ and

$$\Delta = A_1 A_2 A_3 - A_3^2 - A_1^2 A_4 > 0$$

Straightforward computation shows that, if the following condition holds

$a_{11} < 0$ iff

$$x^{*2}\left(w_1 y^*(w_3 + x^*)^2 + w_2 z^*(x^*+y^*)^2\right) < \left(x^{*2} + w_4 w^* + w_5 y^*\right)(w_3 + x^*)^2 (x^*+y^*)^2 \tag{20a}$$

$a_{22} < 0$ iff

$$w_6 z^*(x^*+y^*)^2 < w_1 x^*(w_3 + y^*)^2 \tag{20b}$$

$a_{12} < 0$ iff

$$w_5(x^*+y^*)^2 < x^*\left((x^*+y^*)^2 + w_1x^*\right) \tag{20c}$$

$$w_6(w_3+x^*)(w_3+y^*)\left(w_5(x^*+y^*)^2 - x^*\left((x^*+y^*)^2+w_1x^*\right)\right) < w_2x^*\left(w_6z^*(x^*+y^*)^2 - w_1x^*(w_3+y^*)^2\right) \tag{20d}$$

$$a_{32}\sigma_5 - a_{31}\sigma_4 + a_{44}(\sigma_3-\sigma_2) > \sigma_6 \tag{20e}$$

Conditions (20)-(a-e) guarantees' that $\sigma_i < 0 \; for\, i = 1{,}4$ and $\sigma_i > 0 \; for\, i = 2{,}5$ hence $A_i > 0 \; for\, i = 1,2,3,4.$

Finally, substituting the values of $A_i \; for\, i = 1,2,3,4.$ in $\Delta = A_1A_2A_3 - A_3^2 - A_1^2A_4 > 0$ and then simplifying the resulting term we get that

$$\Delta = [a_{32}\sigma_5 - a_{31}\sigma_4 + a_{44}(\sigma_3-\sigma_2) - \sigma_6][a_{31}\sigma_4 - a_{32}\sigma_5 + \sigma_1(\sigma_3-\sigma_2) + \sigma_6] + (\sigma_1 + a_{44})[a_{44}(a_{44}[a_{32}\sigma_5 - a_{31}\sigma_4 - \sigma_1(\sigma_3-\sigma_2)] + \sigma_1\sigma_6) + (\sigma_1 + a_{44})\sigma_7]$$

Obviously $\Delta > 0$ if and only if in addition to conditions (20)-(a-e) the following condition holds:

$$Max.\left\{\frac{a_{44}(a_{44}[a_{32}\sigma_5 - a_{31}\sigma_4] + \sigma_1\sigma_6) + (\sigma_1 + a_{44})\sigma_7}{a_{44}^2}, a_{32}\sigma_5 - a_{31}\sigma_4 - \sigma_6\right\} < \sigma_1(\sigma_3-\sigma_2) \quad \ldots \tag{20f}$$

**5. Global Stability Analysis of System (2)**

In this section the global stability for the equilibrium points of system (2) is investigated by using the Lyapunov method as shown in the following theorems.

**Theorem 4.** Assume that the axial equilibrium point $E_x$ of system (2) is locally asymptotically stable in the $R_+^4$, and the following conditions are satisfied:

$$\frac{e_2w_1}{x+y+w_1} \le e_1 \le \min.\left\{\frac{e_2bx(x+y)}{(x+y)[x(1+w_5+w_8)-w_1]+w_1x}, \frac{w_3w_9}{w_2}\right\} \tag{21}$$

Then $E_x$ is globally asymptotically stable in the $R_+^4$.

**Proof**. Consider the following function:

$$U_1(x,y,z,w) = c_1(x-1-\ln x) + c_2y + c_3z + c_4w$$

where $c_i;\, i = 1,2,3,4$ are positive constants to be determined. Clearly $U_1 : R_+{}^4 \to R$ is $C^1$ positive definite function. Now since the derivative of $U_1$ along the trajectory of the system (2) can be written as:

$$\frac{dU_1}{dt} = c_1(x-1)\left[1-(x+y) - \frac{w_1y}{x+y} - \frac{w_2z}{w_3+x} + \frac{w_4w}{x} + \frac{w_5y}{x}\right] + c_2y\left[\frac{w_1x}{x+y} - \frac{w_6z}{w_3+y} - (w_7+w_8+w_5)\right] + c_3z\left[\frac{e_1w_2x}{w_3+x} + \frac{e_2w_6y}{w_3+y} - w_9\right] + c_4\,w\left[\frac{w_8y}{w} - (w_4+w_{10})\right]$$

Then straightforward computation gives

$$\frac{dU_1}{dt} = -c_1(x-1)^2 - \left[c_1\left(1+\frac{w_1}{x+y}\right) - \frac{c_2w_1}{x+y}\right]xy - \left[c_1\left(\frac{w_5}{x} - (1+w_5+\frac{w_1}{x+y})\right) + c_2b - c_4w_8\right]y - [c_1 - c_3e_1]\frac{w_2xz}{w_3+x} - \left[c_3w_9 - \frac{c_1w_2}{w_3+x}\right]z - \left[c_4w_{10} + \frac{c_1w_4}{x}\right]w - [c_2 - c_3e_2]\frac{w_6yz}{w_3+y}$$

So by choosing the positive constants as below and using the upper and lower bounds of prey species:

$$c_1 = c_4 = 1,\ c_2 = \frac{e_2}{e_1},\ c_3 = \frac{1}{e_1}$$

It is obtain that:

$$\frac{dU_1}{dt} \le -(x-1)^2 - \left[1 + \frac{w_1}{x+y} - \frac{e_2 w_1}{e_1(x+y)}\right]xy - \left[\frac{w_9}{e_1} - \frac{w_2}{w_3}\right]z \ - [w_{10} + w_4]w$$
$$- \left[\frac{w_5}{x} - (1 + w_5 + \frac{w_1}{x+y}) + \frac{e_2}{e_1}b - w_8\right]y$$

According to condition (21), $\frac{dU_1}{dt} < 0$ then $U_1$ is strictly Lyapunov function. Therefore $E_x$ is globally asymptotically stable in the $R_+^4$.

**Theorem 5.** Assume that the disease free equilibrium point $E_{xz}$ of system (2) is locally asymptotically stable in the $R_+^4$, and the following conditions are satisfied:

$$\hat{z} > \max.\left\{\frac{[1-(x+\hat{x})](w_3+x)(w_3+\hat{x})}{w_2 w_3}, \frac{e_2 z - 1}{e_2}\right\} \quad (22a)$$

$$\left[\hat{x} + (\hat{x}+1)\frac{w_1}{x+y}\right]\frac{x+y}{x+y+w_1} < x < \min.\left\{\frac{w_5(1+\hat{x}) + w_7}{w_5}, \frac{w_4(1+\hat{x}) + w_{10}}{w_4}\right\} \quad (22b)$$

$$\left(\frac{e_1 w_2 w_3 \hat{z}}{(w_3+x)(w_3+\hat{x})} - \frac{w_2 x}{w_3+x}\right)^2 < 4\left(x + \hat{x} + \frac{w_2 w_3 \hat{z}}{(w_3+x)(w_3+\hat{x})} - 1\right)\left(w_9 - \frac{e_1 w_2 x}{w_3 + x}\right) \quad (22c)$$

Then $E_{xz}$ is globally asymptotically stable in the sub region of $R_+^4$ that satisfy the above conditions.

**Proof.** Consider the following function:

$$U_2(x,y,z,w) = \frac{(x-\hat{x})^2}{2} + y + \frac{(z-\hat{z})^2}{2} + w$$

Clearly $U_2 : R_+{}^4 \to R$ is $C^1$ positive definite function. Now since the derivative of $U_2$ along the trajectory of the system (2) can be written as:

$$\frac{dU_2}{dt} = (x-\hat{x})\left[x - x^2 + xy - \frac{w_1 xy}{x+y} - \frac{w_2 xz}{w_3+x} + w_4 w + w_5 y\right]$$
$$+ \left[\frac{w_1 x}{x+y} - \frac{w_6 z}{w_3+y} - (w_7 + w_8 + w_5)\right]y$$
$$+ (z-\hat{z})\left[\frac{e_1 w_2 xz}{w_3+x} + \frac{e_2 w_6 yz}{w_3+y} - w_9 z\right]$$
$$+ w_8 y - (w_4 + w_{10})w$$

Then straightforward computation gives

$$\frac{dU_2}{dt} = -[x + \hat{x} - 1](x-\hat{x})^2 - \left[1 + \frac{w_1}{x+y}\right](x-\hat{x})xy - w_2(x-\hat{x})\left(\frac{xz}{w_3+x} - \frac{\hat{x}\hat{z}}{w_3+\hat{x}}\right)$$
$$+ w_4 w(x-\hat{x}) + w_5 y(x-\hat{x}) + y\left[\frac{w_1 x}{x+y} - \frac{w_6 z}{w_3+y} - b\right]$$
$$+ e_1 w_2(z-\hat{z})\left(\frac{xz}{w_3+x} - \frac{\hat{x}\hat{z}}{w_3+\hat{x}}\right) + \frac{e_2 w_6 yz(z-\hat{z})}{w_3+y}$$
$$- w_9(z-\hat{z})^2 + w_8 y - (w_4 + w_{10})w$$

From which we obtain

$$\frac{dU_2}{dt} = -\varepsilon_{11}(x-\hat{x})^2 + \varepsilon_{13}(x-\hat{x})(z-\hat{z}) \ - \varepsilon_{33}(z-\hat{z})^2$$
$$- \varepsilon_{12} xy - \varepsilon_4 w - \varepsilon_2 y \ - \varepsilon_{23}\frac{yz}{J}$$

here

$$\varepsilon_{11} = x + \hat{x} + \frac{w_2 w_3 \hat{z}}{(w_3+x)(w_3+\hat{x})} - 1; \varepsilon_{13} = \frac{e_1 w_2 w_3 \hat{z}}{(w_3+x)(w_3+\hat{x})} - \frac{w_2 x}{w_3+x}; \varepsilon_{33} = w_9 - \frac{e_1 w_2 x}{w_3 + x};$$

$$\varepsilon_{12} = \left[1 + \frac{w_1}{x+y}\right](x-\hat{x}) - \frac{w_1}{x+y}; \varepsilon_{23} = w_6[1 - e_2(z-\hat{z})]; \varepsilon_2 = w_5[1-(x-\hat{x})] + w_7;$$

$$\varepsilon_4 = w_4[1-(x-\hat{x})] + w_{10};$$

Now according to the conditions (22)-(a-b) then all the values of $\varepsilon_{11}, \varepsilon_{33}, \varepsilon_{12}, \varepsilon_{23}, \varepsilon_2$ and $\varepsilon_4$ are positive values. So by using condition (22c) we obtain

$$\frac{dU_2}{dt} < -\left[\sqrt{\varepsilon_{11}}(x-\hat{x}) - \sqrt{\varepsilon_{33}}(z-\hat{z})\right]^2 - \varepsilon_2 y - \varepsilon_4 w$$

Consequently, $U_2$ is strictly Lyapunov function. Therefore $E_{xz}$ is globally asymptotically stable in the sub region of $R_+^4$ that satisfy the above condition.

**Theorem 6.** Assume that the predator free equilibrium point $E_{xyw}$ of system (2) is locally asymptotically stable in the $R_+^4$, and the following conditions are satisfied

$$\max.\left\{1 - \tilde{x} - \tilde{y} - \frac{w_1 y\tilde{y}}{(x+y)(\tilde{x}+\tilde{y})}, \tilde{x} + e_1\right\} < x < \min.\left\{\frac{b}{w_1\tilde{x}}, (w_5 + w_1 y\tilde{y})\frac{(x+y)(\tilde{x}+\tilde{y})}{(x+y)(\tilde{x}+\tilde{y}) + w_1\tilde{x}}\right\} \quad (23a)$$

$$y > \tilde{y} + e_2 \quad (23b)$$

$$\left(w_5 + w_1 y\tilde{y} - x - \frac{w_1 x\tilde{x}}{(x+y)(\tilde{x}+\tilde{y})}\right)^2 < \left(x + \tilde{x} + \tilde{y} + \frac{w_1 y\tilde{y}}{(x+y)(\tilde{x}+\tilde{y})} - 1\right)(b - w_1 x\tilde{x}) \quad (23c)$$

$$(w_4)^2 < \left(x + \tilde{x} + \tilde{y} + \frac{w_1 y\tilde{y}}{(x+y)(\tilde{x}+\tilde{y})} - 1\right)(w_4 + w_{10}) \quad (23d)$$

$$(w_8)^2 < (b - w_1 x\tilde{x})(w_4 + w_{10}) \; w < \tilde{w} \quad (23e)$$

Then $E_{xyw}$ is globally asymptotically stable in the sub region of $R_+^4$ that satisfy the above conditions.

**Proof.** Consider the following function:

$$U_3(x,y,z,w) = \frac{(x-\tilde{x})^2}{2} + \frac{(y-\tilde{y})^2}{2} + z + \frac{(w-\tilde{w})^2}{2}$$

Clearly $U_3 : R_+{}^4 \to R$ is $C^1$ positive definite function. Now since the derivative of $U_3$ along the trajectory of the system (2) can be written as:

$$\begin{aligned}\frac{dU_3}{dt} &= (x-\tilde{x})\left[x - x^2 + xy - \frac{w_1 xy}{x+y} - \frac{w_2 xz}{w_3+x} + w_4 w + w_5 y\right] \\ &\quad + (y-\tilde{y})\left[\frac{w_1 xy}{x+y} - \frac{w_6 yz}{w_3+y} - by\right] \\ &\quad + z\left[\frac{e_1 w_2 x}{w_3+x} + \frac{e_2 w_6 y}{w_3+y} - w_9\right] \\ &\quad + (w-\tilde{w})[w_8 y - (w_4 + w_{10})w]\end{aligned}$$

Where $b = w_7 + w_8 + w_5$. Then straightforward computation gives

$$\begin{aligned}\frac{dU_3}{dt} &= (x-\tilde{x})^2 - (x+\tilde{x})(x-\tilde{x})^2 - (x-\tilde{x})(xy - \tilde{x}\tilde{y}) - w_1(x-\tilde{x})\left(\frac{xy}{x+y} - \frac{\tilde{x}\tilde{y}}{\tilde{x}+\tilde{y}}\right) \\ &\quad - \frac{w_2 xz(x-\tilde{x})}{w_3 + x} + w_4(x-\tilde{x})(w-\tilde{w}) + w_5(x-\tilde{x})(y-\tilde{y}) + w_1(y-\tilde{y})\left(\frac{xy}{x+y} - \frac{\tilde{x}\tilde{y}}{\tilde{x}+\tilde{y}}\right) \\ &\quad - \frac{w_6 yz}{w_3+y}(y-\tilde{y}) - b(y-\tilde{y})^2 + z\left[\frac{e_1 w_2 x}{w_3+x} + \frac{e_2 w_6 y}{w_3+y} - w_9\right] + w_8(y-\tilde{y})(w-\tilde{w}) \\ &\quad - (w_4 + w_{10})(w-\tilde{w})^2\end{aligned}$$

From which we obtain

$$\frac{dU_3}{dt} < -\lambda_{11}(x-\tilde{x})^2 - \lambda_{22}\ (y-\tilde{y})^2 - \lambda_{44}\ (w-\tilde{w})^2 + \lambda_{12}(x-\tilde{x})(y-\tilde{y})$$
$$+\lambda_{14}(x-\tilde{x})(w-\tilde{w}) + \lambda_{42}(y-\tilde{y})(w-\tilde{w}) - \lambda_{13}\frac{xz}{w_3+x} - \lambda_{23}\frac{yz}{w_3+y}$$

here

$$\lambda_{11} = x+\tilde{x}+\tilde{y}+\frac{w_1 y\tilde{y}}{(x+y)(\tilde{x}+\tilde{y})}-1; \lambda_{22} = b - w_1 x\tilde{x};$$
$$\lambda_{44} = w_4 + w_{10}; \lambda_{12} = w_5 + w_1 y\tilde{y} - x - \frac{w_1 x\tilde{x}}{(x+y)(\tilde{x}+\tilde{y})};$$
$$\lambda_{14} = w_4; \lambda_{42} = w_8;\ \lambda_{13} = w_2[x-\tilde{x}-e_1]; \lambda_{23} = w_6[y-\tilde{y}-e_2]$$

Now according to the conditions (23)-(a-b) then all the values of $\lambda_{11},\lambda_{22},\lambda_{44},\lambda_{12},\lambda_{14},\lambda_{42},\lambda_{13}$ and $\lambda_{23}$ are positive values. So by using condition (23) - (c-e) we obtain

$$\frac{dU_3}{dt} < -\left(\sqrt{\frac{\lambda_{11}}{2}}(x-\tilde{x})-\sqrt{\frac{\lambda_{22}}{2}}(y-\tilde{y})\right)^2 - \left(\sqrt{\frac{\lambda_{11}}{2}}(x-\tilde{x})-\sqrt{\frac{\lambda_{44}}{2}}(w-\tilde{w})\right)^2$$
$$-\left(\sqrt{\frac{\lambda_{22}}{2}}(y-\tilde{y})-\sqrt{\frac{\lambda_{44}}{2}}(w-\tilde{w})\right)^2$$

Consequently, $U_3$ is strictly Lyapunov function. Therefore $E_{xyw}$ is globally asymptotically stable in the sub region of $R_+^4$ that satisfy the above condition.

**Theorem 7.** Assume that the positive equilibrium point $E_{xyzw}$ of system (2) is locally asymptotically stable in the $Int.R_+^4$, and the following conditions are satisfied:

$$\beta_{11} > 0\ ,\beta_{22} > 0, \beta_{33} \text{ and } \beta_{44} > 0 \qquad (24a)$$
$$\beta_{12}^2 < \frac{4}{9}(\beta_{11}\beta_{22}) \qquad (24b)$$
$$\beta_{13}^2 < \frac{4}{6}(\beta_{11}\beta_{33}) \qquad (24c)$$
$$\beta_{14}^2 < \frac{4}{6}(\beta_{11}\beta_{44}) \qquad (24d)$$
$$\beta_{23}^2 < \frac{4}{6}(\beta_{22}\beta_{33}) \qquad (24e)$$
$$\beta_{42}^2 < \frac{4}{6}(\beta_{22}\beta_{44}) \qquad (24f)$$

here $\beta_{ij}(i,j=1,2,3,4)$ are given in proof. Then $E_{xyzw}$ is globally asymptotically stable in the sub region of $Int.R_+^4$ that satisfy the above conditions.

**Proof.** Consider the following function:

$$U_4(x,y,z,w) = \left(x - x^* - x^*\ln\frac{x}{x^*}\right) + \left(y - y^* - y^*\ln\frac{y}{y^*}\right)$$
$$+\left(z - z^* - z^*\ln\frac{z}{z^*}\right) + \left(w - w^* - w^*\ln\frac{w}{w^*}\right)$$

Clearly $U_4 : {R_+}^4 \to R$ is $C^1$ positive definite function. Now since the derivative of $U_4$ along the trajectory of the system (2) can be written as:

$$\frac{dU_4}{dt} = (x - x^*)\left[x - x^2 + xy - \frac{w_1 xy}{x+y} - \frac{w_2 xz}{w_3 + x} + w_4 w + w_5 y\right]$$
$$+ (y - y^*)\left[\frac{w_1 xy}{x+y} - \frac{w_6 yz}{w_3 + y} - by\right]$$
$$+ (z - z^*)\left[\frac{e_1 w_2 xz}{w_3 + x} + \frac{e_2 w_6 yz}{w_3 + y} - w_9 z\right]$$
$$+ (w - w^*)\left[w_8 y - (w_4 + w_{10})w\right]$$

Then after doing some algebraic manipulations we get

$$\frac{dU_4}{dt} = -\beta_{11}(x - x^*)^2 - \beta_{22}(y - y^*)^2 - \beta_{33}(z - z^*)^2 - \beta_{44}(w - w^*)^2$$
$$+ \beta_{12}(x - x^*)(y - y^*) + \beta_{13}(x - x^*)(z - z^*) + \beta_{14}(x - x^*)(w - w^*)$$
$$+ \beta_{23}(y - y^*)(z - z^*) + \beta_{42}(y - y^*)(w - w^*)$$

where

$$\beta_{11} = x + x^* + y^* + \frac{w_1 y y^*}{(x+y)(x^*+y^*)} + \frac{w_2 w_3 z^*}{(w_3+x)(w_3+x^*)} - 1,$$

$$\beta_{22} = \frac{w_3 w_6 z^*}{(w_3+y)(w_3+y^*)} - \frac{w_1 x x^*}{(x+y)(x^*+y^*)} + b,\ \beta_{33} = w_9 - \frac{e_1 w_2 x^*}{w_3 + x^*}\ \ \beta_{44} = w_4 + w_{10},$$

$$\beta_{12} = w_5 + \frac{w_1 y y^*}{(x+y)(x^*+y^*)} - x - \frac{w_1 x x^*}{(x+y)(x^*+y^*)},\ \beta_{14} = w_4$$

$$\beta_{13} = \frac{w_2 w_3 (e_1 z^* - x)}{(w_3+x)(w_3+x^*)},\ \beta_{23} = \frac{w_6[w_3(e_2 z^* - y) - y y^*]}{(w_3+y)(w_3+y^*)},\ \beta_{24} = w_8.$$

Then using the above conditions (24)-(a-f) we obtain that

$$\frac{dU_4}{dt} \le -\left(\sqrt{\frac{\beta_{11}}{3}}(x - x^*) - \sqrt{\frac{\beta_{22}}{3}}(y - y^*)\right)^2 - \left(\sqrt{\frac{\beta_{11}}{3}}(x - x^*) - \sqrt{\frac{\beta_{33}}{2}}(z - z^*)\right)^2$$
$$- \left(\sqrt{\frac{\beta_{11}}{3}}(x - x^*) - \sqrt{\frac{\beta_{44}}{2}}(w - w^*)\right)^2 - \left(\sqrt{\frac{\beta_{22}}{3}}(y - y^*) - \sqrt{\frac{\beta_{33}}{2}}(z - z^*)\right)^2$$
$$- \left(\sqrt{\frac{\beta_{22}}{3}}(y - y^*) - \sqrt{\frac{\beta_{44}}{2}}(w - w^*)\right)^2$$

Consequently, $U_4$ is strictly Lyapunov function.Therefore $E_{xyzw}$ is globally asymptotically stable in the sub region of $Int.R_+^4$ that satisfy the above conditions.

**6. Numerical Analysis of System (2)**.

In this section the dynamical behavior of system (2) is studied numerically for different sets of parameters and different sets of initial points. The objectives of this study are: first investigate the affect of varying the value of each parameter on the dynamical behavior of system (2) and second confirm our obtained analytical results. It is observed that, for the following set of hypothetical parameters that satisfies stability conditions of positive equilibrium point, system (2) has a globally asymptotically stable positive equilibrium point as shown in following figure-2.

$$w_1 = 0.7, w_2 = 0.5, w_3 = 0.5, w_4 = 0.2, w_5 = 0.01, w_6 = 0.3,$$
$$w_7 = 0.05, w_8 = 0.2, w_9 = 0.1, w_{10} = 0.1, \quad e_1 = 0.5, \ e_2 = 0.4 \qquad (25)$$

Note that, in Figure-2, we will use that (—— ) to describe the trajectory starting at (0.8, 0.7, 0.6, 0.5) and (…..) to describe the trajectory starting at (0.5, 0.4, 0.3, 0.2).

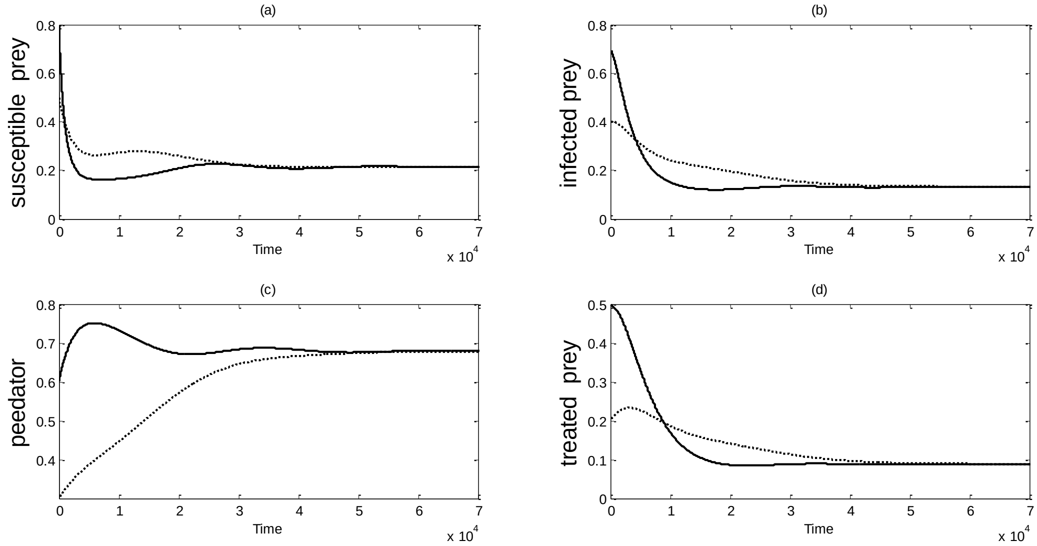


**Figure 2-**Time series of the solution of system (2), (a) trajectories of $x$ as a function of time, (b) trajectories of $y$ as a function of time, (c) trajectories of $z$ as a function of time,(d) trajectories of $w$ as a function of time.

Clearly, Figure-2 shows that the solution of system (2) approaches asymptotically to the positive equilibrium point $L_6 = (0.4, 0.1, 0.2, 0.3)$ starting from two different initial points and this is confirming our obtained analytical results regarding to global stability of the positive equilibrium point.

Now in order to discuss the effect of the parameters values of system (2) on the dynamical behavior of the system, the system is solved numerically for the data given in Eq. (25) with varying one parameter each time. It is observed that for the data as given in Eq. (25) with varying the parameters values $w_i;\, i = 4,10$ do not have any effect on the dynamical behavior of system (2) and the system still approaches to a positive equilibrium point. It is observed that for the data as given in Eq. (25) with $w_1 \le 0.4$, the solution of system (2) approaches asymptotically to $E_{xz} = (\hat{x}, 0, \hat{z}, 0)$ in the interior of positive quadrant of $xz-$plane as shown in Figure-3, however for $0.5 \le w_1$ the system(2) approaches to the positive equilibrium point.

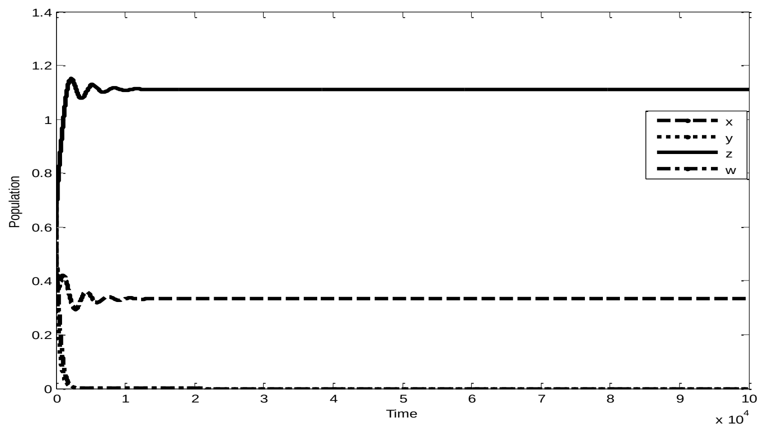


**Figure 3-** Time series of the solution of system (2) for the data given by Eq. (25) with $w_1 = 0.3$, which approaches to (0.33, 0, 1.11, 0) in the interior of positive quadrant of $xz-$plane.

By varying the parameter $w_2$ keeping the rest of parameters values as in Eq. (25), it observed that for $w_2 \le 0.2$ system (2) approaches asymptotically to $E_{xyw} = (\tilde{x}, \tilde{y}, 0, \tilde{w})$ in the interior of positive octant of $xyw-$space as shown in Figure-4 , while for $0.3 \le w_2$ the solution of system (2) approaches to the positive equilibrium point.

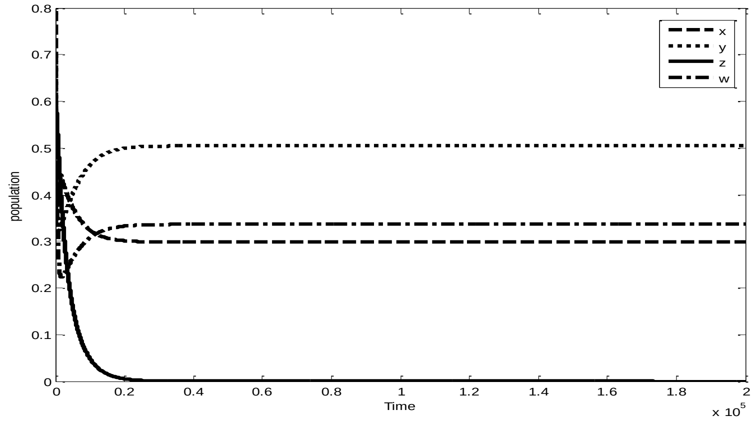


**Figure 4-**Time series of the solution of system (2) for the data given by Eq. (25) with $w_2 = 0.1$, which approaches to (0.29, 0.5, 0, 0.33) in the interior of positive octant of $xyw-$space.

On the other hand varying the parameter $w_3$ keeping the rest of parameters values as in Eq. (25), it observed that for $w_3 \le 0.3$ system (2) has periodic dynamics in $Int.R_+^4$ as shown in Figure-5, while for $w_3 \ge 0.4$ the solution 0f the system transfer to the positive equilibrium point.

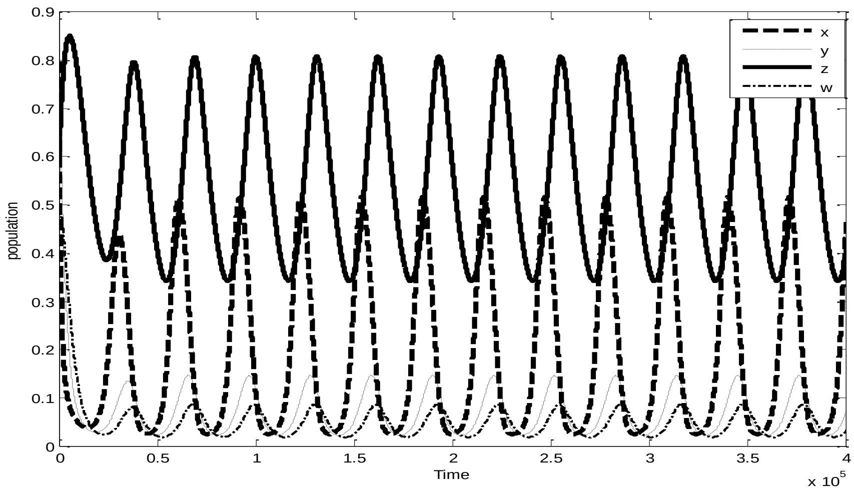


**Figure 5-** Time series of the solution of system (2) for the data given by Eq. (25) with $w_3 = 0.4$, which approaches to periodic dynamics in $Int.R_+^4$.

Varying the parameter $w_5$ keeping the rest of parameters values as in Eq. (25), showed that for $w_5 \le 0.2$ system (2) approaches to a positive equilibrium point , while for $0.3 \le w_5$ the solution of system (2) approaches asymptotically to $E_{xz} = (\hat{x}, 0, \hat{z}, 0)$ in the interior of positive quadrant of $xz-$plane.

For the parameters values given in Eq. (25) with varying $w_6$ in the range $w_6 \le 0.3$ system (2) approaches to a positive equilibrium point , while for $0.4 \le w_6$ the solution of system (2) approaches asymptotically to $E_{xz} = (\hat{x}, 0, \hat{z}, 0)$ in the interior of positive quadrant of $xz-$plane.

Varying the parameter $w_7$ keeping the rest of parameters values as in Eq. (25), showed that for $w_7 \le 0.05$ system (2) approaches to a positive equilibrium point , while for $0.06 \le w_7$ the solution of system (2) approaches asymptotically to $E_{xz} = (\hat{x},0,\hat{z},0)$ in the interior of positive quadrant of $xz$ – plane.

Varying the parameter $w_8$ keeping the rest of parameters values as in Eq. (25), showed that for $w_8 \le 0.2$ system (2) approaches to a positive equilibrium point , while for $0.3 \le w_8$ the solution of system (2) approaches asymptotically to $E_{xz} = (\hat{x},0,\hat{z},0)$ in the interior of positive quadrant of $xz$ – plane.

For the parameters values given in Eq. (25) with varying $w_9$ in the range $w_9 \le 0.09$ system (2) has periodic dynamics in $Int.R_+^4$., however for $w_9 = 0.1$ system(2) approaches to a positive equilibrium point, while for $0.2 \le w_9$ it is observed that the solution of system (2) approaches asymptotically to the equilibrium point $E_{xyw} = (\tilde{x},\tilde{y},0,\tilde{w})$.

For the changing in the value of the parameter $e_1$ keeping the rest of parameters values as in Eq. (25), it observed that for $e_1 \le 0.7$ , system (2) approaches to a positive equilibrium point, however for $0.8 \le e_1$ system (2) has a periodic dynamics in $Int.R_+^4$.

For the parameters values given in Eq. (25) with varying $e_2$ in the range $e_2 \le 0.8$ system (2) approaches to a positive equilibrium point, however for $0.9 \le e_2$ system (2) has a periodic dynamics in $Int.R_+^4$.

Finally for the parameters values given in Eq. (25) with $w_7 = 0.5$ and $w_9 = 0.5$ the solution of system (2) approaches asymptotically to $E_x = (1,0,0,0)$ as shows in Figure-6 .

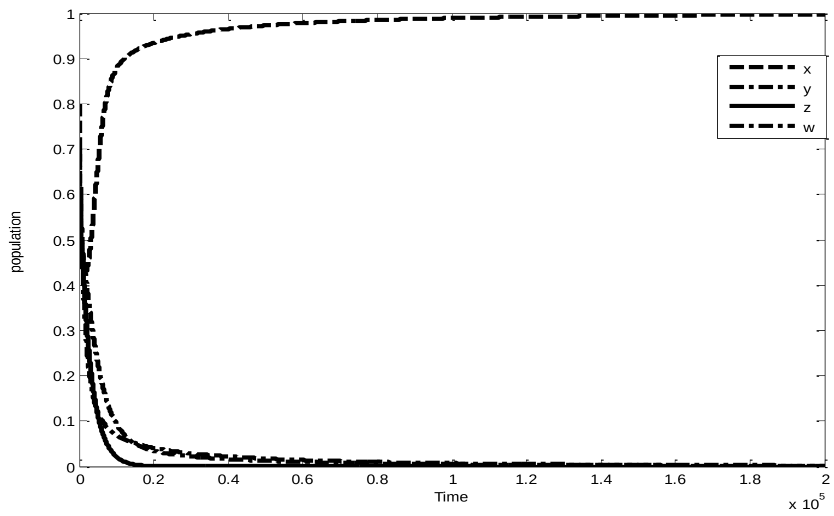


**Figure 6-** Time series of the solution of system (2) for the data given by Eq. (25) with $w_7 = 0.5$ and $w_9 = 0.5$ which approaches asymptotically to $E_x = (1,0,0,0)$

Keeping the above in view, the effect of the other parameters on the dynamics of system (2) is summarized in the following table-1.

**Table 1-** of parameters varied in system (2)

| Parameters varied in system(2) | Numerical behavior of system(2) |
|---|---|
| $w_1 \le 0.4$ | Approaches to stable point in $Int.R^2_{+(xz)}$ |
| $w_1 \ge 0.5$ | Approaches to stable positive point in $Int.R^4_+$ |
| $w_2 \le 0.2$ | Approaches to stable point in $Int.R^3_{+(xyw)}$ |
| $w_2 \ge 0.3$ | Approaches to stable positive point in $Int.R^4_+$ |
| $w_3 \le 0.3$ | Approaches to stable point in $Int.R^3_{+(xyw)}$ |
| $w_3 \ge 0.4$ | Approaches to periodic dynamic in $Int.R^4_+$ |
| for all values of $w_i; i = 4,10$ | Approaches to stable positive point in $\text{Int}.R^4_+$ |
| $w_5 \le 0.2$ | Approaches to stable positive point in $Int.R^4_+$ |
| $w_5 \ge 0.3$ | Approaches to stable point in $Int.R^2_{+(xz)}$ |
| $w_6 \le 0.3$ | Approaches to stable positive point in $Int.R^4_+$ |
| $w_6 \ge 0.4$ | Approaches to stable point in $Int.R^2_{+(xz)}$ |
| $w_7 \le 0.05$ | Approaches to stable positive point in $Int.R^4_+$ |
| $w_7 \ge 0.06$ | Approaches to stable point in $Int.R^2_{+(xz)}$ |
| $w_8 \le 0.2$ | Approaches to stable positive point in $Int.R^4_+$ |
| $w_8 \ge 0.3$ | Approaches to stable point in $Int.R^2_{+(xz)}$ |
| $w_9 \le 0.9$ | Approaches to periodic dynamic in $Int.R^4_+$ |
| $w_9 \ge 0.1$ | Approaches to stable point in $Int.R^3_{+(xyw)}$ |
| $e_1 \le 0.7$ | Approaches to stable positive point in $Int.R^4_+$ |
| $e_1 \ge 0.8$ | Approaches to periodic dynamic in $Int.R^4_+$ |
| $e_2 \le 0.8$ | Approaches to stable positive point in $Int.R^4_+$ |
| $e_2 \ge 0.9$ | Approaches to periodic dynamic in $Int.R^4_+$ |

## 7. Conclusions and Discussion.

In this paper, an eco-epidemiological model of Holloing type II of prey-predator model has proposed and analyzed. The model consists of four non-linear autonomous differential equations that describe the dynamics of four different population namely susceptible prey $x$, infected prey $y$, susceptible predator $z$, infected prey under treatment $w$. The boundedness of the system (2) has been discussed. The dynamical behavior of system (2) has been investigated locally as well as globally.

To understand the effect of varying each parameter on the global dynamics of system (2) and to confirm our obtained analytical results, system (2) has been solved numerically and the following results are obtained:

1. For the set of hypothetical parameters values given Eq. (25), the system (2) approaches asymptotically to globally stable positive equilibrium point $E^* = (x^*, y^*, z^*, w^*)$.
2. It is observed that varying the parameters values $w_i; i = 4,10$ and keeping other parameters as given by Eq. (25) do not have any effect on the dynamical behavior of system (2) and the system still approaches to a positive equilibrium point.

3. As the infection rate of prey $w_1$ decreases keeping other parameters as in Eq. (25) the system (2) approaches asymptotically to the equilibrium point $E_{xz} = (\hat{x},0,\hat{z},0)$ .Otherwise the system still have a globally asymptotically stable positive point in the $Int.R_+^4$.
4. As the susceptible prey's maximum attack rate by predator $w_2$ decreases keeping the rest of parameters as in Eq. (25) the predator will faces extinction and the solution of system (2) approaches asymptotically to the equilibrium point $E_{xyw} = (\tilde{x},\tilde{y},0,\tilde{w})$ . Otherwise the system still have a globally asymptotically stable positive point in the $Int.R_+^4$.
5. As the half saturation constant of the susceptible predator $w_3$ decreases keeping the rest of parameters as in Eq. (25) will causes destabilizing of system (2) and the solution approaches to asymptotically stable positive point in the $Int.R_+^4$. Otherwise the system still have a globally asymptotically stable positive point in the $Int.R_+^4$.
6. As the infected prey's recover rate $w_5$ decreases keeping the rest of parameters as in Eq. (25), system (2) still has a stable positive equilibrium point in the $Int.R_+^4$. However increasing the parameter $w_5$ causes extinction of ( infected and treatment) prey and the solution of system (2) approaches asymptotically to the equilibrium point $E_{xz} = (\hat{x},0,\hat{z},0)$. It is observed that the susceptible prey's maximum attack rate by predator, disease death rate of prey and immunity under treatment rate $w_6, w_7$ and $w_8$ respectively, have the same effect as $w_5$.
7. As natural death rate of predator $w_9$ decreases keeping the rest of parameters as in Eq. (25), the positive equilibrium point will be unstable and the solution of system (2) approaches asymptotically positive point in the $Int.R_+^4$.

   Otherwise the system still have the solution approaches to a stable limit cycle in $\mathrm{Int}R_{+(xyw)}^3$.
8. Finally, the conversion rate from susceptible prey to predator $e_1$ decreases keeping the rest of parameters as in Eq. (25), the system has a globally asymptotically stable positive point in the $Int.R_+^4$. While increasing $e_1$ will causes destabilizing of system (2) and the solution approaches to stable positive equilibrium. It is observed that the conversion rate from infected prey to predator $e_2$ , have the same effect as $e_1$

## References


**1.** Mukhopadhyay B., Bhattacharyya R., **2009**. Role of predator switching in an eco-epidemiological model with disease in the prey. *Elsevier- Ecol. Model.* 220:931-939.

**2.** Bairagi N., Roy P.K., Chattopadhyay J., **2007**. Role of infection on the stability of a predator-prey system with several response functions-Acomparativ study. *J. Theor. Biol.* 248:10-25.

**3.** Hadeler K.P, Freedman H.I, **1989**. Predator-prey populations with parasitic infection. *J. Math. Biol.*27: 609-631.

**4.** Deredec A, Courchamp F, **2006**. *Combined impacts of Allee effects and parasitism. Oikos* 112: 667-679.

**5.** Blanchard P., et. al. **2002**. *Differential equations,* Boston University,

**6.** Han. L., Ma. Z., Hethcote, H. W., **2001**. Four predator prey models with infectious disease. *Math.Comp Model*,34, 849 – 858,.

**7.** Hsich. Y. H., Hsiao,C. K., **2008**. predator – prey with disease infection in both populations. *Math.Med. Biol,* 25(3), 247- 266.

**8.** Haque M., and Venturion E., **2007**.*An ecoepidemiological model with disease in predator: the ratio – dependent case, Mathematical methods in the Applied Sciences*, 30, 1791-1809.

**9.** Ohri A., Trivedi N., Sisodiya A., Sing B. and Jain S., **2010.** Study of a prey – predator model with disease prey, *Int . J. Contemp. Math. Sciences*, 7(10), 489-468.

**10.** Kapur J. N., **1985**. *Mathematical models in biology & medicine*, New Delhi, Affiliated EWP Ltd.

**11.** Lial M. L., et. al., **1985**. *precalculus,* Adison-wesly, New Affiliated EWPLtd.

**12.** May R. M., **1973**. Stability and Complexity in model ecosystem, Princeton, New Jersey: Princeton University press.

**13.** Oksanen. L., Fretwell. S., Arruda. J., Neimela. P., **1980**. Exploitation ecosystems in gradients of primary productivity. *Am. Nat*, 118, 240 – 261.

**14.** Naji. R., Ali.N., **2014**. Modeling and stability of Lotka-Volterra prey-predator system involving infectious disease in each population **,***Iraqi Journal of science*,55 (2A), PP:491-505.