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Fully Polyform Modules and Related Concepts

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Abstract
An R-module M is called a polyform module if every essential submodule of M is rational. The main objective of this paper is to introduce a new concept of modules named fully polyform modules. This kind of module is contained in the class of polyform modules. We study in detail fully polyform modules, so several properties of this concept are investigated. Other characterizations and partial characterisations (i.e., satisfied by certain conditions) of the definition of fully polyform module analogous to those known in the concept of a polyform module are given and discussed. For instance, we proved that a module M is a fully polyform module if and only if $\mathrm{Hom}_R(\frac{V}{N}, M)=0$ for each P-essential submodule N of M and for each V≤M with N⊆V⊆M. Relationships between this class of modules and some other related concepts are discussed such as monoform, QI-monoform, essentially quasi-Dedekind, essentially prime and St-polyform modules. Moreover, useful concepts and their influence or relationships with fully polyform modules such as P-uniform and Pe-prime modules are introduced.

Keywords: Polyform modules, Fully polyform modules, Rational submodules, Essential submodules, P-essential submodules.

المقاسات متعددة الصيغ التامة والمقاسات ذات العلاقة

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الخلاصة:
يُقال للمقاس M بأنه متعدد الصيغ إذا كان كل مقاس جوهري فيه نسبياً. ان الهدف الرئيسي لهذا البحث هو إعطاء نوع جديد من المقاسات تدعى بالمقاسات متعددة الصيغ التامة. ان هذا النوع من المقاسات محتوى في المقاسات متعددة الصيغ. لقد درسنا بشيء من التفصيل هذا الصنف من المقاسات، لذا فأن العديد من الخصائص المهمة قُدمت حول المقاسات متعددة الصيغ التامة. كما تم إعطاء عدد من التشخيصات والتشخيصات الجزئية (اي تتحقق بشروط معينة) للمقاسات متعددة الصيغ التامة مناظرة لتلك التشخيصات المعروفة الخاصة بالمقاسات متعددة الصيغ، على سبيل المثال برهنا ان المقاس M يكون متعدد الصيغ تام اذا وفقط اذا تحقق $\mathrm{Hom}_R\left(\frac{V}{N}, M\right) = 0$ لكل مقاس جزئي جوهري N من النمط P− ولكل V≤M حيث ان N⊆V⊆M.
أيضاً نوقشت علاقة هذا الصنف من المقاسات مع عدد من المقاسات مثل المقاسات احادية الصيغة، مقاسات احادية الصيغة من النمط−QI، المقاسات شبه الديديكاندية الجوهرية، المقاسات الأولية الجوهرية،

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المقاسات شبه الديديكاندية الأولية. فضلا عن ذلك، فقد تم اعطاء مفاهيم مفيدة ودرس تأثيرها او علاقتها مع
المقاسات متعددة الصيغة التامة، مثل المقاسات المنتظمة من النمط –P والمقاسات الاولية من النمط –Pe.

1. Introduction:

Many authors such as J. Zelmanowitiz, H.H. Storrer, and M.A. Ahmed have studied and discussed polyform modules. An R-module M is called injective if for every monomorphism f: M⟶B and every homomorphism g: M⟶C there is a homomorphism h: B⟶C with g=h∘f, [1, P.116]. A non-zero submodule N of M is said to be essential (briefly, $N\leq_e M$) if N∩L≠0 for every non-zero submodule L of M, [2, P.15]. An essential monomorphism is defined as a monomorphism f: S⟶T such that $f(S) \leq_e T$, [1, Definition 5.6.5 (1)]. For any R-module M, an injective hull of M is denoted by E(M), and it is defined as a monomorphism f: M⟶E(M) with E(M) is an injective module and f is an essential monomorphism, [1, P.124]. A submodule N of an R-module M is called rational (simply, $N\leq_r M$) if $\mathrm{Hom}_R(\frac{M}{N}, E(M))=0$, where E(M) is the injective hull of M, [3, P.274]. An R-module M is called a polyform if every essential submodule of M is rational, [4]. A submodule N of M is called pure if N∩IM=IN for every ideal I of R, [5].

This paper consists of three sections. Section two discusses the main properties of fully polyform modules. Among these results are the following:

- Let M be a PIP module. If M is fully polyform then every non-zero pure and P-essential submodule of M is fully polyform, see Proposition 2.7.
- Let M be a multiplication module with a pure annihilator, and N is a pure and P-essential submodule of M. If M is a fully polyform module then N is fully polyform, see Proposition 2.9.

Also, some characterizations of the definition of fully polyform modules are given, for instance:

- Let M be an R-module. The following statements are equivalent:

i. M is a fully polyform module.
ii. $\mathrm{Hom}_R(\frac{V}{N}, M)=0$ for each P-essential submodule N of M, and for each V≤M, with N⊆V⊆M.
See Theorem 2.14.

- The following statements are equivalent:

1. $\mathrm{Hom}_R\left(\frac{M}{N}, E(M)\right)=0$ for each P-essential submodule N of M.
2. For each non-zero homomorphism f:M⟶E(M), the kernel of f is not P-essential submodule of M.
See Theorem 2.15.

- Let M be an R-module satisfying the Condition (⊗). Consider the following:

i. All partial endomorphisms of M have pure closed kernels in their domains.
ii. $\mathrm{Hom}_R\left(\frac{M}{N}, E(M)\right)=0$, for each P-essential submodule N of M.
Then (i) ⇒ (ii).
See Theorem 2.17.

- Let M be an F-regular module. Consider the following:

1. All partial endomorphisms of M have pure closed kernels in their domains.
2. $\mathrm{Hom}_R\left(\frac{M}{N}, E(M)\right)=0$, for each P-essential submodule N of M.
Then (1) ⇒ (2).
See Theorem 2.19.

Section three deals with the relationships between the fully polyform modules and other related concepts such as the following:

- Let M be a multiplication and prime module. Consider the following:
  1. M is a fully polyform module.
  2. M is a polyform module.
  3. M is a quasi-invertibility monoform module.

Then (1) ⇒ (2) ⇔ (3).
See Theorem 3.22.

- Let M be a quasi-injective module with $J(\mathrm{End}_R(M))=(0)$, consider the following:
  1. M is a fully polyform module.
  2. M is a polyform module.
  3. M is a quasi-invertibility monoform module.

Then (1) ⇒ (2) ⇔ (3).
See Theorem 3.23.

- Let R be a quasi-Dedekind ring. Consider the following statements:
  1. R is a fully polyform ring.
  2. R is a polyform ring.
  3. R is a quasi-invertibility monoform ring.
  4. R is a monoform ring.

Then (1) ⇒ (2) ⇒ (3) ⇒ (4).
See Theorem 3.24.

We must keep in mind that all rings R in this work are commutative with identity and all modules are unitary left R-modules.

2. Fully Polyform Modules

In this section, a new class of modules is introduced, and it is named fully polyform modules. The basis of this concept is the P-essential submodules which appeared in [6], and it is mentioned in the following:

Definition 2.1: [6]

A submodule N of M is called P-essential (briefly $N\leq_{pe}M$), if for every pure submodule L of M with N∩L=(0), implying that L=(0).

Remark 2.2: It is clear that every essential submodule is P-essential. The converse is true when M is uniform, where a non-zero module M is called uniform if every non-zero submodule of M is essential, [2].

Definition 2.3: An R-module M is said to be fully polyform if every P-essential submodule of M is rational in M. That is $\mathrm{Hom}_R\left(\frac{M}{N}, E(M)\right)=0$ for every P-essential submodule N of M. A ring R is called fully polyform if R is a fully polyform R-module.

Remark 2.4: It is obvious that every fully polyform module is polyform. We think the converse is not true in general, but we cannot find an example to confirm that.

Remarks and Examples 2.5:

1. The $\mathbb{Z}$-module $\mathbb{Z}$ is fully polyform, since all submodules $n\mathbb{Z}$ of $\mathbb{Z}$ are essential, hence they are P-essential in $\mathbb{Z}$, and $\mathrm{Hom}_R\left(\frac{\mathbb{Z}}{n\mathbb{Z}}, E(\mathbb{Z})\right)=\mathrm{Hom}_R\left(\frac{\mathbb{Z}}{n\mathbb{Z}}, \mathbb{Q}\right)\cong \mathrm{Hom}_R(\mathbb{Z}_n, \mathbb{Q})=0$.

2. $\mathbb{Z}_{P^\infty}$ is not fully polyform $\mathbb{Z}$-module, in fact, in spite of every submodule of $\mathbb{Z}_{P^\infty}$ is P-essential, and each proper submodule A of $\mathbb{Z}_{P^\infty}$ satisfying

$\mathrm{Hom}_{\mathbb{Z}}\left(\frac{\mathbb{Z}_{P^\infty}}{A}, E(\mathbb{Z}_{P^\infty})\right) = \mathrm{Hom}_{\mathbb{Z}}\left(\frac{\mathbb{Z}_{P^\infty}}{A}, \mathbb{Z}_{P^\infty}\right)$. Note that $\frac{\mathbb{Z}_{P^\infty}}{A} \cong \mathbb{Z}_{P^\infty}$. But in contrast, $\mathrm{Hom}_{\mathbb{Z}}\left(\frac{\mathbb{Z}_{P^\infty}}{A}, \mathbb{Z}_{P^\infty}\right) \cong \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}_{P^\infty}, \mathbb{Z}_{P^\infty}) \neq 0$.

3. $\mathbb{Z}_4$ is not fully polyform $\mathbb{Z}$-module. There is a P-essential submodule $<\bar{2}> \leq \mathbb{Z}_4$ and a non-zero homomorphism $f \in \mathrm{Hom}_{\mathbb{Z}}\left(\frac{\mathbb{Z}_4}{<\bar{2}>}, E(\mathbb{Z}_4)\right) = \mathrm{Hom}_{\mathbb{Z}}\left(\frac{\mathbb{Z}_4}{<\bar{2}>}, \mathbb{Z}_{2^\infty}\right)$, [7, P.21].

4. Homomorphic image of fully polyform is not fully polyform, such as the $\mathbb{Z}$-module $\mathbb{Z}$ is fully polyform, but the quotient $\frac{\mathbb{Z}}{4\mathbb{Z}} \cong \mathbb{Z}_4$ is not fully polyform $\mathbb{Z}$-module as verified by (3).

5. Every simple module is fully polyform since the P-essential submodule of a simple module say M is only itself, hence $\mathrm{Hom}_R(\frac{M}{M}, E(M)) \cong \mathrm{Hom}_R(0, E(M)) \cong 0$.

6. For any R-module M with $N \leq M$, if $\frac{M}{N}$ is fully polyform then M may not be fully polyform for example the $\mathbb{Z}$-module $\mathbb{Z}_2$ is simple and by (5), it is fully polyform. On contrast, $\mathbb{Z}_2 \cong \frac{\mathbb{Z}_4}{<\bar{2}>}$, and we verified in (3), that $\mathbb{Z}_4$ is not fully polyform.

An R-module M is called F-regular if every submodule of M is pure, [8].

7. If M is an F-regular module, then the two concepts of polyform and fully polyform coincide.
Proof: If M is F-regular, then it is easy to show that the essential and P-essential concepts are identical. This yields that polyform and fully polyform coincide.

8. For any regular ring R, any R-module M is fully polyform if and only if M is polyform.
Proof: Since every module over a regular ring is regular, [5, P.29]. Then the result is followed by (7).

Remember that a submodule N of an R-module M is called a quasi-invertible submodule of M (we choose the symbols $N \leq_{qu} M$) if $\mathrm{Hom}_R(\frac{M}{N}, M) = 0$, [9].

9. If M is a fully polyform module, then $\mathrm{ann}_R(M) = \mathrm{ann}_R(N)$ for all $N \leq_{pe} M$.

Proof: By assumption, $\mathrm{Hom}_R\left(\frac{M}{N}, E(M)\right) = 0$ for all $N \leq_{pe} M$. This implies that N is a rational submodule of M for all $N \leq_{pe} M$. On the other hand, the rationally of any submodule implies quasi-invertibility, and if $N \leq_{qu} M$ then $\mathrm{ann}_R(M) = \mathrm{ann}_R(N)$, [9, Proposition (1.4), P.7]. Therefore, $\mathrm{ann}_R(M) = \mathrm{ann}_R(N)$ for all $N \leq_{pe} M$.

10. The direct sum of fully polyform is not necessarily fully polyform, for example, both of $\mathbb{Q}$ and $\mathbb{Z}_2$ are fully polyform $\mathbb{Z}$-module, but $\mathbb{Q} \oplus \mathbb{Z}_2$ is not fully polyform, in fact, $(\mathbb{Z} \oplus \mathbb{Z}_2) \leq_e \mathbb{Q} \oplus \mathbb{Z}_2$, [9, Example (3.4), P.15], hence $\mathbb{Z} \oplus \mathbb{Z}_2 \leq_{pe} \mathbb{Q} \oplus \mathbb{Z}_2$, but $\mathbb{Z} \oplus \mathbb{Z}_2 \nleq_r \mathbb{Q} \oplus \mathbb{Z}_2$, [9, Example (3.4), P.15].

11. In the class of uniform modules, obviously, the two concepts fully polyform and polyform modules are identical.

Proposition 2.6: Let M be an R-module, assume that $\bar{R}=\frac{R}{J}$ where J is an ideal of R with $J\subseteq \text{ann}_R(M)$. Then M is a fully polyform R-module if and only if M is a fully polyform $\bar{R}$-module.

Proof: Suppose that M is a fully polyform R-module, so that $\text{Hom}_R(\frac{M}{N}, E(M))=0$ for all non-zero P-essential submodule N of M. Since $\text{Hom}_R(\frac{M}{N}, E(M)) = \text{Hom}_{\bar{R}}(\frac{M}{N}, E(M))$ for all $N\leq M$, [1, P.51], in particular for each P-essential submodule N of M. Then $\text{Hom}_{\bar{R}}(\frac{M}{N}, M)=0$, hence M is a fully polyform $\bar{R}$-module. Similarity for the converse.

Next, we study the hereditary of the fully polyform property, before that, an R-module M has the pure intersection property (simply, PIP) if the intersection of any two pure submodules of M is pure, [10, P.33].

Proposition 2.7: Let M be a PIP module. If M is fully polyform then every non-zero pure and P-essential submodule of M is fully polyform.

Proof: Assume that N is a non-zero pure and P-essential submodule of M. Let $K\leq_{pe}N$, since M has PIP and N is pure in M, then $K\leq_{pe}M$, [6, Theorem 4.4]. But M is fully polyform, therefore $K\leq_{r}M$, hence $K\leq_{r}N$, [2, Proposition 2.25, P.55], that is N is fully polyform.

An R-module M is called multiplication if every submodule of M is of the form IM, for some ideal I of R, [11]. To present another case for the hereditary of a fully polyform property, we need to give the following lemma.

Lemma 2.8: Let M be a multiplication module with a pure annihilator, and let K, N be submodules of M such that $K\leq N\leq M$, where N is a pure submodule of M. If $K\leq_{pe}N$ and $N\leq_{pe}M$ then $K\leq_{pe}M$.

Proof: Let L be a pure submodule of M with $K\cap L=0$, we have to prove L=0. By assumption, L is pure in M. In addition, N is pure in M, and since M is a multiplication module with a pure annihilator, then $L\cap N$ is pure in M, [12, Corollary 1.3], hence $L\cap N$ is pure in N, [5]. Now, $K\cap L=0$ implies to $(L\cap N)\cap K=0$. Since $L\cap N$ pure in N and $K\leq_{pe}N$, so by the definition of P-essential we have $L\cap N=0$. Furthermore, L is pure in M, thus L=0. That is $K\leq_{pe}M$.

Proposition 2.9: Let M be a multiplication module with a pure annihilator, and N is a pure and P-essential submodule of M. If M is a fully polyform module, then N is fully polyform.

Proof: Assume that N is a pure and P-essential submodule of M. Since M is multiplication with pure annihilator then by Lemma 2.8, $L\leq_{pe}M$. But M is fully polyform, then $L\leq_{r}M$, hence $L\leq_{r}N$, [2, Proposition 2.25,P.55]. Thus, N is fully polyform.

Proposition 2.10: If M is a uniform module, then polyform and fully polyform are identical.

Proof: It is followed directly from Remark 2.2.

Recall that an R-module M is called a scalar if for each $f\in \text{End}_R(M)$, there exists $r\in R$ such that f(x)=rx for all $x\in M$, where $\text{End}_R(M)$ is the endomorphism ring of the module M. [13].

Proposition 2.11: Let M be a faithful scalar R-module. Then $End_R(M)$ is a fully polyform ring if and only if R is a fully polyform R-module.

Proof: Since M is a faithful scalar module, then $End_R(M) \cong R$, [14], so if $End_R(M)$ is a fully polyform ring then R is a fully polyform ring and vice versa.

Corollary 2.12: Let M be a finitely generated faithful and multiplication module. Then $End_R(M)$ is a fully polyform ring if and only if R is a fully polyform ring.

Proof: The result is followed by Proposition 2.11 and the fact that every finitely generated multiplication module is a scalar module, [14].

Remember that an R-module M is called quasi-Dedekind if every non-zero submodule of M is quasi-invertible, [9, P.24].

Proposition 2.13: If M be a multiplication and quasi-Dedekind R-module, then $End_R(M)$ is a fully polyform ring.

Proof: Since M is multiplication and quasi-Dedekind R-module, then $End_R(M)$ is an integral domain, [15, Proposition 2.1.27, P.55]. By Remark 2.5 (1), M is fully polyform.

The following theorem gives another characterization of the definition of a fully polyform module.

Theorem 2.14: Let M be an R-module. The following statements are equivalent:
a. M is a fully polyform module.
b. $Hom_R(\frac{V}{N}, M)=0$ for each P-essential submodule N of M, and for each $V \leq M$, with $N \subseteq V \subseteq M$.

Proof:
(a) ⇒ (b): Let $0 \neq f \in Hom_R(\frac{V}{N}, M)$. Consider the following diagram:

$$\begin{tikzcd} 0 \arrow[r] & \frac{V}{N} \arrow[rr, "i_1"] \arrow[dr, "f"'] & & \frac{M}{N} \arrow[dd, "g"] \\ & & M \arrow[dr, "i_2"'] & \\ & & & E(M) \end{tikzcd}$$

where i_1 and i_2 are the inclusion homomorphism. Since E(M) is injective, then $i_2 \circ f = g \circ i_1$, and because of $f \neq 0$, then clearly $g \circ i_1 \neq 0$, hence $g \neq 0$. But M is fully polyform, so we obtain a contradiction. Thus f=0, that is $Hom_R(\frac{V}{N}, M)=0$.
(b) ⇒ (a): Consider a P-essential submodule say N of M, and a non-zero homomorphism $g \in Hom_R(\frac{M}{N}, E(M))$, put $g^{-1}(M) \equiv \frac{V}{N}$ for some submodule V of M with $N \subseteq V \subseteq M$. Restrict g on $\frac{V}{N}$, that is we can define h: $\frac{V}{N} \rightarrow N$ by $h(x+N)=g(x+N)$ for all $x+N \in \frac{V}{N}$. Obviously, h is well-defined and homomorphism. In addition, since $g \neq 0$, then $h \neq 0$ which is a contradiction, therefore M is fully polyform.

As well as the following characterization for the fully polyform module.

Theorem 2.15: The following statements are equivalent:

1. $\mathrm{Hom}_R\left(\frac{M}{N}, E(M)\right)=0$ for each P-essential submodule N of M.

2. For each non-zero homomorphism f: M⟶E(M), the kernel of f is not P-essential submodule of M.

Proof:

(1) ⇒ (2): Assume that $\mathrm{Hom}_R\left(\frac{M}{N}, E(M)\right)=0$ for each P-essential submodule N of M and let f: M⟶E(M) be a homomorphism with kerf is a P-essential submodule of M. We have to show that f=0. Define g: $\frac{M}{\mathrm{kerf}} \longrightarrow$ E(M) by g(m+kerf)=f(m) for all m∈E(M). To show that g is well-defined, assume that m_1+kerf=m_2+kerf, m_1, m_2∈M. This implies that (m_1-m_2)∈kerf that is f(m_1-m_2)=0. Since f is a homomorphism, then f(m_1)-f(m_2)=0, hence f(m_1)=f(m_2). Moreover, since f≠0, then g≠0. That is $\mathrm{Hom}_R(\frac{M}{\mathrm{kerf}}$,E(M))≠0 which is a contradiction, therefore f=0.

(2) ⇒ (1): Let N$\leq_{pe}$M. Suppose there exists a non-zero homomorphism f: $\frac{M}{N}$ ⟶E(M), so we have the following:

$$M \xrightarrow{\pi} \frac{M}{N} \xrightarrow{f} E(M)$$

where π is the natural epimorphism. Consider f∘π: M⟶E(M), put Ψ≡(f∘π), it is clear that Ψ≠0.

Now, N⊆kerΨ, and N$\leq_{pe}$M, so according to, [6, Theorem 4.4 (i)], kerΨ$\leq_{pe}$M, which is a contradiction with (2). Thus f=0, and the proof is complete.

Next, Theorem 2.15 can be applied to prove the following. Before that, an R-module M is called prime if ann_R(M) = ann_R(N) for every non-zero submodule N of M, [16].

Proposition 2.16: Let M be a uniform R-module. If E(M) is a prime R-module, then M is fully polyform where E(M) is the injective hull of M.

Proof: Assume that E(M) is a prime R-module and let f: M →E(M) be a monomorphism. We depend on Theorem 2.15, so to show that kerf$\nleq_{pe}$M. Suppose the contrary, that is kerf$\leq_{pe}$M. Because f is a monomorphism, then f≠0, so there exists 0≠x∈M such that f(x)≠0. Since kerf$\leq_{pe}$M and M is uniform then kerf$\leq_{e}$M, so there exists r∈R with 0≠rx∈kerf, [2, Proposition 2.25, P.55]. Therefore f(rx)=0, this implies that rf(x)=0, hence r∈ann_R(f(x)). Besides that E(M) is prime, then ann_R(f(x))= ann_R(E(M)). So that r∈ann_R(E(M))= ann_R(M), thus rx=0 which is a contradiction, thus kerf$\nleq_{pe}$M. By Theorem 2.15, M is a fully polyform module.

Following [2], a submodule N of an R-module M is called closed if N has no proper essential extension in M, and a submodule N of an R-module M is called pure closed if N has no proper P-essential extension in M, that is if there is a P-essential submodule K of M such that N⊂K, then N=K. briefly, we use the symbol N$\leq_{pc}$M, [6].

We need to consider the following condition.
Condition (⊗): For any submodules A and B of an R-module C with A⊆B⊆C. If B is a P-essential submodule of C, then A is a P-essential submodule of B.

Theorem 2.17: Let M be an R-module satisfying the Condition (⊗). Consider the following:
i. All partial endomorphisms of M have pure closed kernels in their domains.
ii. $\mathrm{Hom}_R\left(\frac{M}{N}, E(M)\right)$=0, for each P-essential submodule N of M.
Then (i) ⇒ (ii).

Proof: Suppose (i) is satisfied and let N be a P-essential submodule of M, f: $\frac{M}{N} \longrightarrow$ E(M) be a homomorphism. If f≠0, then there exists m+N∈$\frac{M}{N}$ such that f(m+N)=ḿ≠0, ḿ ∈E(M). Since M$\leq_e$E(M), so there exists r∈R such that 0≠rḿ∈M. Put rm´=x. Define φ: N+Rm ⟶ Rx by φ(n+rm)=rx ∀n∈N, r∈R. To prove that φ is well-defined, assume that n_1+r_1m=n_2+r_2m where n_1, n_2∈N, r_1,r_2∈R, that is n_1-n_2=(r_1-r_2)m∈N. But:

$$f\,[(r_1\text{-}r_2)(m+N)] = f\,[(r_1\text{-}r_2)m+N] = 0 \;\ldots\ldots\ldots\ldots\; \mathbf{(1)}$$

Also,

$$f(r_1\text{-}r_2)(m+N) = (r_1\text{-}r_2)f(m+N) = (r_1\text{-}r_2)ḿ \;\ldots\ldots\ldots\; \mathbf{(2)}$$

from (1) and (2) we get (r_1-r_2)ḿ=0, that is r_1ḿ=r_2ḿ, then $r_1 r$ḿ=$r_2 r$ḿ, hence r_1x=$r_2 x$. This implies that $\varphi(n_1$+r_1m)=r_1x = $\varphi(n_2$+r_2m)= r_2x, therefore φ is well-defined. Also, φ is a non-zero homomorphism. It remains to prove that N⊆kerφ, let n∈N that is n=n+0m, so that φ(n)=0x=0, that is N⊆kerφ. Now, since N⊆kerφ ⊆M and N is a P-essential submodule of M, then by [6, Theorem 4.4(1)], kerφ is P-essential of M. Now, kerφ ⊆ N+Rm ⊆ M, again by [6, Theorem 4.4 (1)], N+Rm$\leq_{pe}$M. On the other hand, by Condition (⊗), kerφ $\leq_{pe}$N+Rm. From (i), kerφ is pure closed in N+Rm, thus kerφ=N+Rm. This implies that φ=0, which is a contradiction, thus f=0. Hence the proof of (ii) is complete.

We need the following lemma.

Lemma 2.18: If M is an F-regular module then:
1. N$\leq_e$M if and only if N$\leq_{pe}$M.
2. N$\leq_c$M if and only if N$\leq_{pc}$M.

Proof:
1. The necessity is clear. For the converse, assume that N$\leq_{pe}$M. Let L be a submodule of M with N∩L=0. Since M is F-regular, then L is pure, and by assumption L=0, then N$\leq_e$M.
2. Assume that N$\leq_{pc}$M, and let L≤ M with N$\leq_e$L≤ M so that N$\leq_{pe}$L≤ M. By assumption N=L, that is N$\leq_c$M. The sufficiency is clear.

Theorem 2.19: Let M be an F-regular module. Consider the following:
1. All partial endomorphisms of M have pure closed kernels in their domains.
2. $\mathrm{Hom}_R\left(\frac{M}{N}, E(M)\right)$=0, for each P-essential submodule N of M.
Then (1) ⇒ (2).

Proof: Let $N\leq_{pe}M$, by the same argument of Theorem 2.17, we get φ is well defined, homomorphism and $N \subseteq \ker\varphi$. Now, for the chain $N \subseteq \ker\varphi \subseteq M$, since $N\leq_{pe}M$ then by Lemma 2.18(1), $N\leq_{e}M$ this implies that $\ker\varphi\leq_{e}M$, [2, Proposition 1.1, P.16]. Again for the chain $\ker\varphi \leq N+Rm \leq M$, we have $N+Rm\leq_{e}M$, [2, Proposition 1.1, P.16]. This implies that $\ker\varphi\leq_{e}N+Rm$. In contrast, from (1), kerφ is pure closed in N+Rm, hence kerφ is closed in N+Rm, thus kerφ=N+Rm. This implies that φ=0, which is a contradiction, thus f=0.

Proposition 2.20: If M satisfies Condition (⊗), then every submodule of fully polyform module is fully polyform.

Proof: Let N≤M, and let L be a P-essential submodule of N. Assume that f: L→N be a homomorphism. Consider the following sequence of homomorphisms.

$$L \xrightarrow{f} N \xrightarrow{i} M$$

Since M is fully polyform and satisfies Condition (⊗), then by Theorem 2.17, ker(i∘f) is a pure closed submodule of L. But ker(i∘f)=kerf, thus kerf is pure closed in L, hence N is a fully polyform submodule.

Corollary 2.21: For any module M satisfying Condition (⊗), if E(M) is fully polyform then M is fully polyform.

Lemma 2.22: The following statements are equivalent:

1. For every submodule N of M and all homomorphism f: N→M, implies that $\ker f\leq_{pc}N$.

2. For any non-zero submodule N of M and each non-zero homomorphism f: N→ M, implies that kerf is not a P-essential submodule of N.

Proof:

(1) ⇒ (2): Suppose there exists a submodule N of M and a non-zero homomorphism f: N⟶ M such that $\ker f \leq_{pe} N$. By assumption $\ker f \leq_{pc} N$, this implies that kerf=N, hence f=0, which is a contradiction. Thus, kerf is not a P-essential submodule of N.

(2) ⇒ (1): Let 0≠N≤M, and f: N⟶M be a homomorphism, we have to show that $\ker f \leq_{pc}N$. Suppose that $\ker f\nleq_{pc} N$. This implies the existence of a submodule K of N containing kerf such that $\ker f \leq_{pe} K$. Consider the following:

$$K \xrightarrow{i} N \xrightarrow{f} M$$

where i is the inclusion homomorphism. It is clear that f∘i≠0, and since kerf⊂K, then $\ker f=\ker(f\circ i) \leq_{pe}K$. Now, if K=0, then kerf=0, hence $\ker f\leq_{pe}N$ which is a contradiction. Thus K≠0. By (2), we obtain a contradiction, therefore $\ker f\leq_{pc}N$.

From Theorem 2.17 and Lemma 2.22, we have the following:

Theorem 2.23: Let M be an R-module satisfying the Condition (⊗). Consider the following:

i. All non-zero partial endomorphisms f: N→M with N≠0, haven't P-essential kernels in their domains.

ii. All partial endomorphisms of M have pure closed kernels in their domains.

iii. $\mathrm{Hom}_R\left(\frac{M}{N}, E(M)\right)$=0, for each P-essential submodule N of M.

Then (i) ⇒ (ii) ⇒ (iii).

Proof:
(i) ⇒ (ii): It is just Lemma 2.22.
(ii) ⇒ (iii): It is just Theorem 2.19.

Theorem 2.24: Consider the following statements for an R-module M which satisfies Condition (⊗).
1. $\mathrm{Hom}_R\left(\frac{M}{N}, E(M)\right)$=0, for each P-essential submodule N of M.
2. For each non-zero homomorphism f: M⟶E(M), implies kerf is not P-essential submodule of M.
3. All non-zero partial endomorphisms f: N→M with N≠0, haven't P-essential kernels in their domains.
Then (1) ⇒ (2) ⇒ (3).

Proof:
(1) ⇒ (2): It is just Theorem 2.15.
(2) ⇒ (3): Let N be a non-zero P-essential submodule of M, and f: N→M be a non-zero homomorphism. Consider the following sequence of homomorphisms:

$$N \xrightarrow{f} M \xrightarrow{i} E(M)$$

It is clear that i∘f is a non-zero homomorphism. By assumption ker(i∘f) = kerf is not P-essential submodule of N, and we are done.

Theorem 2.25: The following statements are equivalent for any R-module M satisfying the Condition (⊗).
1. $\mathrm{Hom}_R\left(\frac{M}{N}, E(M)\right)$=0 for each P-essential submodule N of M.
2. For each non-zero homomorphism f:M⟶E(M), implies kerf is not P-essential submodule of M.
3. All non-zero partial endomorphisms f: N→M with N≠0, haven't P-essential kernels in their domains.
4. All partial endomorphisms of M have pure closed kernels in their domains.

Proof:
(1) ⇒ (2) ⇒ (3): It is just Theorem 2.24.
(3) ⇒ (4): It is just Lemma 2.22.
(4) ⇒ (1): It is just Theorem 2.17.

Theorem 2.26: For any R-module M satisfies the Condition (⊗), the following are equivalent:
1. All non-zero partial endomorphisms f: N→M with N≠0 haven’t P-essential kernels in their domains.
2. All partial endomorphisms of M have pure closed kernels in their domains.
3. $\mathrm{Hom}_R\left(\frac{M}{N}, E(M)\right)$=0 for each P-essential submodule N of M.
Proof:
(1) ⇒ (2): It is just Lemma 2.22.
(2) ⇒ (3): It is just Theorem 2.17.
(3) ⇒ (1): It follows by Theorem 2.24.

3. Fully Polyform Modules and Related Concepts

This section is dedicated to studying the relationships between fully polyform modules and other modules such as semisimple, monoform, essentially quasi-Dedekind and St-polyform modules.

Recall that an R-module M is called semisimple if every submodule of M is a direct summand of M, [1, P.107].

Proposition 3.1: Every semisimple module is fully polyform.

Proof: Let M be a semisimple module, and N be a P-essential submodule of M. Assume that f: $\frac{V}{N}\longrightarrow$M be a homomorphism, where N$\subseteq$V$\subseteq$M. By assumption, N is a direct summand of M, so there is a submodule K of M such that M=N$\oplus$K. That is N$\cap$K=0. Since K is pure in M and N$\leq_{pe}$M, then K=0. This implies that N=M, therefore N=V, hence f=0.

The converse of Proposition 3.1 is not true in general, for example, the $\mathbb{Z}$-module $\mathbb{Z}$ is a fully polyform module, but not semisimple.

Proposition 3.2: Any monoform module is fully polyform.

Proof: Let M be a monoform module. By assumption, every non-zero submodule of M is rational in M, particularly, every P-essential submodule of M is rational in M. Thus, M is fully polyform.

The converse of Proposition 3.2 is not always true, for example, the $\mathbb{Z}$-module $\mathbb{Z}_6$ is semisimple, hence it is fully polyform by Proposition 3.1, but not monoform since <$\bar{2}$> is not rational in $\mathbb{Z}_6$.

We need to introduce the following definition.

Definition 3.3: A non-zero module M is called P-uniform if every non-zero submodule of M is P-essential.

It is clear that every uniform module is P-uniform. We think that the converse is not true, but unfortunately, we haven't examples to confirm that. In the following proposition, we use a condition under which the converse of Proposition 3.2 will be true.

Proposition 3.4: Let M be a P-uniform module. Then M is fully polyform if and only if M is a monoform module.

Proof: Assume that M is fully polyform, and let N be a non-zero submodule of M. Since M is P-uniform, then N$\leq_{pe}$M. But M is fully polyform, so N$\leq_{r}$M. Thus, M is monoform. The converse is clear.

Corollary 3.5: Let M be a uniform module. Then M is fully polyform if and only if M is a monoform module.

Proof: Since every uniform is P-uniform, then the result follows directly from Proposition 3.4.

Proposition 3.6: Uniform modules cannot be fully polyform.

Proof: Let M be a uniform module. Suppose the converse, that is M is a fully polyform module, so for all $0\neq N\leq M$ and $0\neq f: N\longrightarrow M$, kerf $\nleq_{pe}$N. This implies that kerf$\nleq_{e}$N. But M is uniform, so we have a contradiction. Thus, M is not fully polyform.

Recall that a module M is called an essentially quasi-Dedekind module if $\mathrm{Hom}_R\left(\frac{M}{N}, M\right)=0$ for all $N\leq_e M$, [18].

Proposition 3.7: Every fully polyform is an essentially quasi-Dedekind module**.**

Proof: Let M be a fully polyform module, and N be an essential submodule of M. Suppose that f: $\frac{M}{N}\longrightarrow M$ is a homomorphism. Consider the following sequence of homomorphisms:

$$\frac{M}{N}\xrightarrow{f} M \xrightarrow{i} E(M)$$

where i is the inclusion homomorphism and E(M) is the injective hull of M. Since every essential submodule of M is P-essential, and M is fully polyform, therefore $\mathrm{Hom}_R(\frac{M}{N}$, E(M))=0. So that i∘f=0, hence f=0. Thus $\mathrm{Hom}_R(\frac{M}{N}$, M)=0 for each $N\leq_e M$, that is M essentially quasi-Dedekind.

An R-module M is called K-nonsingular if for each $f\in\mathrm{End}_R(M)$, kerf$\leq_e$M implies that f=0, [19]. Hadi and Ghawi, [18] proved that the two classes essentially quasi-Dedekind and K-nonsingular modules are identical. For that reason, if M is fully polyform then M is a K-nonsingular module.

The converse of Proposition 3.7 is not true in general, for example, the $\mathbb{Z}$-module $\mathbb{Q}\oplus\mathbb{Z}_2$ is essentially quasi-Dedekind, [18, Remark 2.13], but it is not fully polyform. To verify that, if we take the submodule $\mathbb{Z}$ of the $\mathbb{Z}$-module $\mathbb{Q}$, and define η: $\mathbb{Z}\longrightarrow\mathbb{Q}\oplus\mathbb{Z}_2$ by $\eta(x)=(0,\bar{x})$ for each $x\in\mathbb{Z}$. It is clear that η is a non-zero homomorphism. Now, $\ker\eta=\{w\in\mathbb{Z} \mid \eta(w)=(0,\bar{0})\}=\{w\in\mathbb{Z} \mid (0,\bar{w})=(0,\bar{0})\}=2\mathbb{Z}$, therefore $\ker\eta=2\mathbb{Z}$, so that $\ker\eta\leq_e\mathbb{Z}$, hence $\ker\eta\leq_{pe}\mathbb{Z}$. Beside that, $\eta\neq 0$, so by the contrapositive of the part (i)⇒(iii) in Theorem 2.23, M is not fully polyform module.

In Remark 2.5 (9), we proved the following, which can be deduced from Proposition 3.7 as follows.

Corollary 3.8: If M is fully polyform, then $\mathrm{ann}_R M=\mathrm{ann}_R N$, for all $N\leq_{pe}M$.

Proof: Since each fully polyforms module is essentially quasi-Dedekind, hence $\mathrm{ann}_R M=\mathrm{ann}_R N$, [18, Remark 1.2 (4)].

Corollary 3.8 leads us to introduce the following.

Definition 3.9: An R-module M is called a Pe-prime if $\mathrm{ann}_R(M)=\mathrm{ann}_R(N)$ for every P-essential submodule N of M.

Remark 3.10: Every fully polyform module is Pe-prime module.

Following [15, P.47], a module M is called essentially prime if $\mathrm{ann_R}(M)=\mathrm{ann_R}(N)$ for all $N\leq_e M$. Since every essentially quasi-Dedekind module is essentially prime, So we have the following implications:

Fully Polyform Module ⇒ Polyform Modules ⇒ Essentially Quasi-Dedekind Modules ⇒ Essentially Prime Modules

Recall that An R-module M is called fully P-essential if every P-essential submodule of M is essential, [20].

Proposition 3.11: Let M be a fully P-essential module, then M is Pe-prime if and only if M is an essentially prime module.

Remark 3.12: In the class of fully P-essential modules, both of polyform and fully polyform are identical.

A submodule N of an R-module M is SQI if for each $f\in\mathrm{Hom_R}\left(\frac{M}{N},M\right)$, implies that f $\left(\frac{M}{N}\right)$ is small in M, [21]. An R-module M is called an ESQD module if every essential submodule of M is SQI, [22].

Proposition 3.13: Every fully polyform module is ESQD.

Proof: Let M be fully polyform and $N\leq_e M$, so that $N\leq_{pe} M$. Since M is fully polyform, then $N\leq_r M$. Hence N is quasi-invertible, but obviously, every quasi-invertible is SQI. Thus, M is ESQD.

Following [23], a submodule N of an R-module M is St-closed (simply, $N\leq_{stc}M$) if N has no proper semi-essential extensions in M, where a submodule N is said to be semi-essential if N∩P≠0 for every non-zero prime submodule P of M. An R-module M is called St-polyform if for every submodule N of M and all homomorphism f: N→M, the kerf is an St-closed submodule of M. Equivalently, a module M is St-polyform if for each non-zero submodule N of M and each non-zero homomorphism f: N⟶M, the kerf is not semi-essential submodule of N, [24].

Remark 3.14: Since St-polyform and fully polyform modules depend in their construction on prime and pure concepts, and there is no direct implication between prime and pure submodules, then this implies that there is no direct relationship between them. However, under certain conditions, the two classes fully polyform and St-polyform modules coincide as the following proposition shows, before that we need to give the following Lemma.

Lemma 3.15: Let M be an R-module satisfies one of the following:

1. M is a torsion-free module with $(L:_R M)=0$ for all $L\leq M$.
2. M is a prime module with $(L:_R M)=\mathrm{ann_R}(M)$ for all $L\leq M$.

Then any submodule N of M is P-essential if and only if N is semi-essential of M.

Proof:

1. Assume that N is P-essential and let L be a prime submodule of M such that N∩L=0. To prove L=0, since M is torsion-free and $(L:_R M)=0$, [25], then L is pure. But N is P-essential thus L=0. Conversely, suppose that N is a semi-essential submodule of M, and L is a pure submodule of M. By assumption N is prime, [25]. But N is semi-essential, then L=0.

2. It is similar to (1), with used [26] instead of [25].

Proposition 3.16: Let M be a torsion-free module with $(L:_R M) = 0$, for every submodule L of M. Then M is a fully polyform if and only if M is an St-polyform module, provided that M satisfies the Condition (⊗).

Proof: Assume that M is a fully polyform module, and let N be a non-zero submodule of M, $0\neq f: N\to M$ be a homomorphism. Since M is fully polyform, then by Theorem 2.23, kerf is not P-essential in N. But M is torsion-free and $(\mathrm{kerf}:_R M) = 0$, so by the contrapositive of Lemma 3.15(1), the kernel of f is not a semi-essential submodule of N. Thus, M is St-polyform.

Recall that an R-module M is called fully prime if every proper submodule of M is prime, [27].

Proposition 3.17: Let M be a fully prime R-module that satisfies Condition (⊗). If M is fully polyform then M is an St-polyform module. The converse is true if M is a prime module with $(L:_R M) = \mathrm{ann}_R(M)$ for all $L\leq M$.

Proof: Suppose that M is a fully polyform module, and $0\neq N\leq M$, $0\neq f: N\to M$. Since M is fully polyform, then by Theorem 2.23, $\mathrm{kerf}\nleq_{pe}N$, hence $\mathrm{kerf}\nleq_{e}N$. Since M is fully prime, then $\mathrm{kerf}\nleq_{sem}N$, [28]. Thus, M is St-polyform. For the converse, assume that M is St-polyform. Let $0\neq N\leq M$ and $f: N\to M$ be a non-zero homomorphism. So that $\mathrm{kerf}\nleq_{sem}N$. On the other hand, M is a prime module with $(L:_R M) = \mathrm{ann}_R(M)$ for all $L\leq M$, so by Lemma 3.15, $\mathrm{kerf}\nleq_{pe}N$. By Theorem 2.23, M is fully polyform.

Next, the following results deal with the relationship between fully polyform and quasi-invertibility monoform modules. Before that, an R-module M is called a quasi-invertibility monoform (simply, QI-monoform), if every non-zero quasi-invertible submodule of an R-module M is rational in M, [29].

In the category of rings, every quasi-invertible ideal is essential, [9, Corollary 2.3, P,12]. This leads us to give the following.

Proposition 3.18: Every fully polyform ring is a QI-monoform ring.

Proof: Let A be a non-zero quasi-invertible ideal of R. By [9, Corollary 2.3, P.12], $A\leq_{e}R$, this implies that $A\leq_{pe}R$. But R is fully polyform, then A is rational in R. Therefore, R is QI-monoform.

The converse of Proposition 3.18 is not true, for example: $\mathbb{Z}_4$ is a QI-monoform ring. Indeed, there is no non-zero quasi-invertible ideal of the ring $\mathbb{Z}_4$ which is not rational in $\mathbb{Z}_4$. On the other hand, $\mathbb{Z}_4$ is not fully polyform ring see, Example 2.5(3).

Proposition 3.19: Let M be a multiplication module with a prime annihilator. If M is fully polyform then M is QI-monoform.

Proof: Assume that M is a fully polyform module and $N\leq_{qu}M$. Since M is multiplication with prime annihilator, then $N\leq_{e}M$, [9, Theorem 3.11, P.19)]. Hence $N\leq_{pe}M$, and according to assumption, $N\leq_{r}M$, that is M is QI-monoform.

By the same argument of Proposition 3.19, and with replacing [9, Theorem 3.11, P.18], instead of [9, Corollary 3.12, P.19], we can prove the following.

Proposition 3.20: Let M be a multiplication and prime module. If M is a fully polyform module, then M is QI-monoform.

Moreover, we can use [9, Theorem 3.8, P.17] instead of [9, Theorem 3.11, P.18], to prove the following.

Proposition 3.21: Let M be a quasi-injective R-module with $J(End_R(M)) = (0)$, if M is fully polyform, then M is QI-monoform.

Theorem 3.22: Let M be a multiplication and prime module. Consider the following statements:

1. M is a fully polyform module.
2. M is a polyform module.
3. M is a QI-monoform module.

Then (1) ⇒ (2) ⇔ (3).

Proof:
(1) ⇒ (2): It is obvious.
(2) ⇔ (3): Since M is a multiplication and prime module, then the result followed by [29, Proposition 4.4]

Theorem 3.23: Let M be a quasi-injective module with $J(End_R(M))=(0)$. Consider the following statements:

1. M is a fully polyform module.
2. M is a polyform module.
3. M is a QI-monoform module.

Then (1) ⇒ (2) ⇔ (3).

Proof:
(1) ⇒ (2): It is clear.
(2) ⇔ (3): It is just [29, Proposition 4.5].

Theorem 3.24: Let R be a quasi-Dedekind ring. Consider the following statements:

1. R is a fully polyform ring.
2. R is a polyform ring.
3. R is a QI-monoform ring.
4. R is a monoform ring.

Then (1) ⇒ (2) ⇒ (3) ⇒ (4).

Proof:
(1) ⇒ (2): It is straightforward.
(2) ⇒ (3) ⇒ (4): Since R is a quasi-Dedekind ring, then the result is obtained by [29, Theorem 4.13].

Theorem 3.25: Let R be an essentially quasi-Dedekind ring. Consider the following statements:

1. M is a fully polyform ring.

2. M is a QI-monoform ring.
3. M is a polyform ring.

Then (1) ⇒ (2) ⇒ (3).

Proof:
(1) ⇒ (2): It is just Proposition 3.18.
(2) ⇒ (3): Since R is an essentially quasi-Dedekind, then from [29, Theorem 4.9] the result is obtained.

As a consequence of Theorem 3.25, we have the following result.

Corollary 3.26: Let R be an integral domain. Consider the following statements:

1. R is a fully polyform ring.
2. R is a polyform ring.
3. R is a QI-monoform ring.
4. R is a monoform ring.

Then (1) ⇒ (2) ⇒ (3) ⇒ (4).

Proof: Since every integral domain is quasi-Dedekind, [9, Example 1.4, P.24], then the result follows directly from Theorem 3.24.

Theorem 3.27: Let M be a uniform and essentially quasi-Dedekind module. The following statements are equivalent:

1. M is a fully polyform module.
2. M is monoform module.
3. M is a QI-monoform module.
4. M is a polyform module.

Proof:
(1) ⇔ (2): Suppose that M is fully polyform, since M is a uniform module, then by Corollary 3.5, M is monoform. The converse is clear.
(2) ⇒ (3) ⇒ (4): Since R is a uniform and essentially quasi-Dedekind module, then the result follows by [29, Theorem 4.11].
(4) ⇒ (1): Assume that M is a polyform module. Since M is uniform, so by Proposition 2.10, M is a fully polyform module.

Since every nonsingular module is essentially quasi-Dedekind, [29, Remark 4.8(3)], then from Theorem 3.27, we deduce the following.

Corollary 3.28: Let M be a uniform and nonsingular module. The following statements are equivalent:

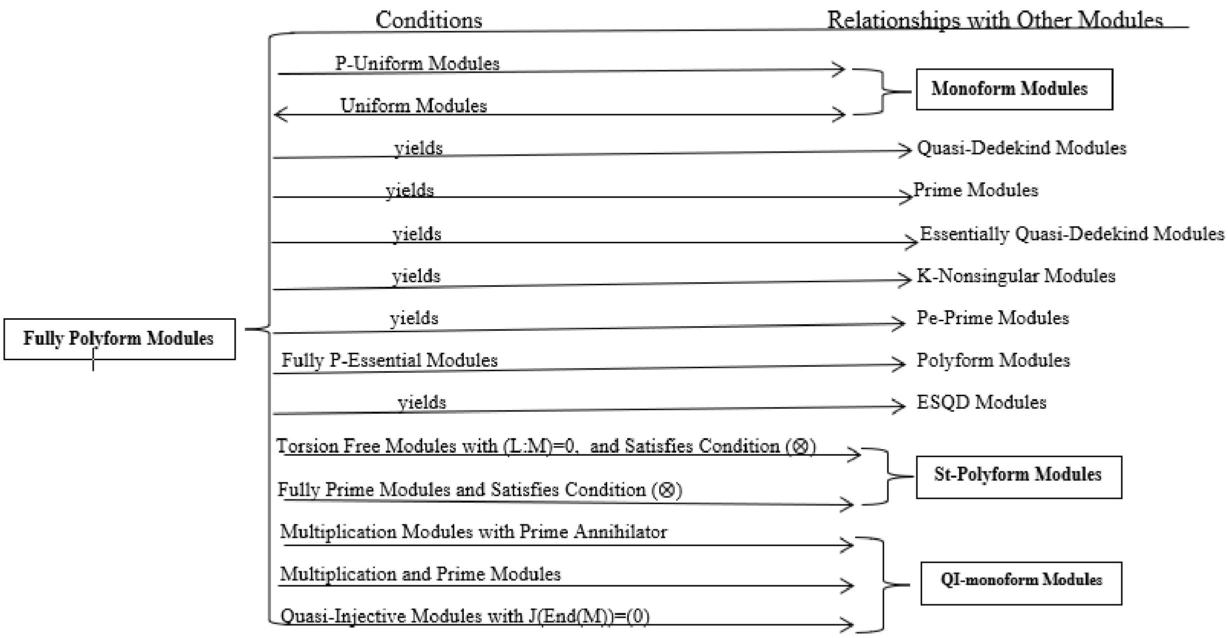
1. M is a fully polyform module.
2. M is monoform module.
3. M is a QI-monoform module.
4. M is a polyform module.

4. Conclusions:

In this work, the class of polyform modules has been restricted to a new class. It is called fully polyform modules. The main results of this paper can be summarized as follows:

1. Many results are given, that describe the main properties of fully polyform modules.
2. Other characterizations and partial characterizations of fully polyform modules are considered.
3. Sufficient conditions are given under which fully polyform and polyform modules are identical.
4. Some modules containing fully polyform modules are examined, such as quasi-Dedekind, Pe-prime and ESQD modules.
5. The connections between fully polyform and related concepts are studied such as monoform and St-polyform modules.

However, all these relationships can be represented in the following diagram:



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