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Generalizations of Supplemented and Lifting Semimodules

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Abstract
In this paper, principally supplemented (δ-supplemented), and principally lifting (δ-lifting) semimodules are defined as generalizations of principally supplemented (δ-supplemented), and principally lifting (δ-lifting) modules. Let R be a semiring. An R-semimodule A is called a principally supplemented (δ-supplemented) semimodule, if for all $a \in A$ there exists a subsemimodule N of A with $A = Ra + N$ and $(Ra) \cap N$ small (δ-small) in N. In this paper, we examine properties of principally δ-supplemented semimodules and generalize results on principally δ-supplemented modules to semimodules. Besides, we characterize δ-semiperfect semimodules as a generalization of δ-semiperfect modules.

Keywords: Supplemented (δ-supplemented) semimodules, Principally supplemented (δ-supplemented) semimodules, Principally lifting (δ-lifting) semimodules, Semiperfect semimodules.

تعميمات شبه المقاسات التكميلية وشبه مقاسات الرفع

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الخلاصة
في هذا البحث، تم تعريف شبه المقاسات التكميلية (التكميلية من النمط $-\delta$) الرئيسية، وشبه مقاسات الرفع (الرفع من النمط$-\delta$) الرئيسية كتعميم للمقاسات التكميلية (التكميلية من النمط$-\delta$) الرئيسية، ومقاسات الرفع (الرفع من النمط-دلتا) الرئيسية. لتكن R شبه حلقة، شبه المقاس A على شبه الحلقة R يدعى تكميلي (تكميلي من نمط$-\delta$) رئيسي اذا كان لكل عنصر في شبه المقاس يوجد شبه مقاس جزئي N من A بحيث $A = Ra +$ N و $(Ra)\cap N$ صغيرة في N. في هذا البحث، نمتحن خصائص شبة المقاسات التكميلية من النمط$-\delta$ و نعمم نتائج من المقاسات التكميلية الرئيسية من النمط$-\delta$ الى شبه المقاسات. أيضا، نميز شبه المقاسات المثالية من النمط$-\delta$ كتعميم للمقاسات المثالية من النمط$-\delta$.

1. Introduction

Firstly, let us point that, R will indicate a commutative semiring with identity and A with indicate an unitary left R-semimodule throughout this article. A (left) R-semimodule A (denoted by ${}_RA$) is a commutative additive semigroup which has a zero element 0_A, together with a mapping from $R \times A$ into A (sending (r,a) to ra) such that $(r+s)a = ra + sa$, $r(a+b) =$

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$ra + rb$, $r(sa) = (rs)a$ and $0a = r0_A = 0$ for all $a, b \in A$ and $r, s \in R$. Let N be a subset of A. We say that N is an R-subsemimodule of A denoted by $N \leq A$, precisely when N is itself an R-semimodule with respect to the operations for A [1-3]. $L \leq A$ is said to be essential in A, denoted by $L \leq_e A$, if $L \cap N \neq 0$ for each nonzero subsemimodule $N \leq A$ [4]. A semimodule A is said to be singular if $A \cong \frac{N}{L}$ for some semimodule N and an essential subsemimodule $L \leq_e N$. Also, we call A singular if $A = Z(A)$, where $Z(A) = \{x \in A : l_R(x)$ is essential in ${}_RR\}$, and $l_R(x) = \{a \in R \mid ax = 0\}$. For a semimodule A, $Z(A)$, and $Z_2(A)$ are the singular subsemimodule and the Goldie torsion subsemimodule of A, respectively. $Z_2(A)$ is defined by $Z(A/Z(A)) = Z_2(A)/Z(A)$. If $A = Z_2(A)$, we say that A is Goldie torsion. If $Z(A) = 0$, A is called non-singular [5].

The subsemimodule N of A is called small in A (we write $N \ll A$), if for every subsemimodule $X \leq A$, with $N + X = A$ involves that $X = A$ [6]. The radical of an R-semimodule A, symbolized by $Rad(A)$, is the sum of all small subsemimodules of A [6]. A is called hollow, if every proper subsemimodule of A is small in A. And, A is called local, if it has a unique maximal subsemimodule, i.e., a proper subsemimodule which contains all other subsemimodules. If A has no proper subsemimodule then A is named simple, and if A is a direct sum of its simple subsemimodules then A is semisimple[4]. The socle of A, symbolized by $Soc(A)$, is the sum of all simple subsemimodules of A [4]. Let L, $K \leq A$. K is called a supplement of L in A if it is minimal with respect to $A = L + K$. A subsemimodule K of A is a supplement (weak supplement) of L in A if and only if $A = L + K$ and $L \cap K \ll K$ ($L \cap K \ll A$) [7]. A is supplemented (weakly supplemented) if each subsemimodule L of A has a supplement in A. Openly, supplemented semimodules are weakly supplemented. $L \leq A$ has ample supplements in A if each subsemimodule K of A such that $A = L + K$ contains a supplement of L in A. A semimodule A is named amply supplemented if every subsemimodule of A has ample supplements in A. Hollow semimodules are ample supplemented. $L \leq A$ is named a δ-supplement of N in A if $A = N + L$ and $N \cap L$ is δ-small in L, and A is named δ-supplemented in case every subsemimodule of A has a δ-supplement in A [5]. A is named lifting (δ-lifting) if, for all $N \leq A$, there exists a decomposition $A = X \oplus Y$ such that $X \leq N$ besides $N \cap Y$ is small (δ-small) in A [5]. $N \leq A$ is a subtractive subsemimodule of A if $a, a + b \in N$ then $b \in N$ [3]. If every $N \leq A$ is subtractive, then A is named subtractive. If C is a subtractive subsemimodule, then $\frac{A}{C}$ is an R-semimodule [3, p.165].

In this work, principally supplemented and lifting semimodules are introduced. In addition, we explore their properties. Besides, we define principally semiperfect (δ-semiperfect) semimodules. A is called principally semiperfect (δ-semiperfect) if, for each $a \in A$, A/Ra has a projective cover (δ-cover). Original descriptions of principally δ-semiperfect semimodules are obtained via principally δ-supplemented semimodules. In addition, we introduce the notion of $\oplus$-supplemented semimodules.

In whatever follows, by $\mathbb{N}$, $\mathbb{N}_0$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{Z}_n$ besides $\mathbb{Z}/n\mathbb{Z}$ we indicate, respectively, natural numbers, non-negative integers, integers, rational numbers, the semiring of integers modulo n besides the $\mathbb{Z}$-semimodule of integers modulo n.

2. δ-Small and δ-supplement subsemimodules

In this section, we revision and state some points of δ-supplement subsemimodules which are vital later. In [8], δ-small submodules are introduced. Small (resp., δ-small) subsemimodules are considered in [6 and 5].

Definition 2.1: Let $N \leq A$. N is said to be δ-small in A if $N + K \neq A$ for any proper subsemimodule K of A with A/K singular. We use $N \ll_{\delta} A$ to indicate that N is a δ-small subsemimodule of A.

Let $f: A \longrightarrow B$ be an epimorphism of left semimodules; f is called δ-small if $Ker(f) \ll_{\delta} A$.

All small subsemimodule or non-singular semisimple subsemimodule of A is δ-small in A. The δ-small subsemimodules of a singular semimodule are small subsemimodules.

Lemma 2.2 [5]: Let A be a subtractive R-semimodule and $N \leq A$. The next are equivalent:
(1) $N \ll_{\delta} A$;
(2) If $A = X + N$, then $A = X \oplus Y$ for a projective semisimple subsemimodule Y with $Y \leq N$;
(3) If $X + N = A$ with A/X Goldie torsion, then $X = A$.

Lemma 2.3 [5]: Let A be an R-semimodule.
(1) For subsemimodules N, K, L of A with $K \leq N$, we have
i. $N \ll_{\delta} A$ if and only if $K \ll_{\delta} A$ and $N/K \ll_{\delta} A/K$.
ii. $N + L \ll_{\delta} A$ if and only if $N \ll_{\delta} A$ and $L \ll_{\delta} A$.
(2) $K \ll_{\delta} A$ and $f: A \longrightarrow N$ is a homomorphism, then $f(K) \ll_{\delta} N$. In particular, if $K \ll_{\delta} A \leq N$, then $K \ll_{\delta} N$.
(3) Let $L_1 \leq A_1 \leq A$, $L_2 \leq A_2 \leq A$ and $A = A_1 \oplus A_2$. Then $L_1 \oplus L_2 \ll_{\delta} A_1 \oplus A_2$ if and only if $L_1 \ll_{\delta} A_1$ and $L_2 \ll_{\delta} A_2$.

Definition 2.4 [5]: Let $\wp$ be the class of all singular simple semimodules. For a semimodule A, let $\delta(A) = Rej_T(\wp) = \cap \{N \leq A | \, A/N \in \wp\}$ be the reject in A of $\wp$.

Lemma 2.5: Let A and B be semimodules.
(1) $\delta(A) = \sum\{L \leq A | L$ is a δ-small subsemimodule of $A\}$.
(2) If $f: A \longrightarrow N$ is an R-homomorphism, then $f(\delta(A)) \leq \delta(B)$. Therefore, $\delta(A)$ is a fully invariant subsemimodule of A and $\delta({}_RR)A \leq \delta(A)$.
(3) If $A = \oplus_{i \in I} A_i$, then $\delta(A) = \oplus_{i \in I} \delta(A_i)$.
(4) If every proper subsemimodule of A is contained in a maximal subsemimodule of A, at that time $\delta(A)$ is the unique largest δ-small subsemimodule of A.

Proof: See [5]. □

Next, we give some descriptions of $\delta({}_RR)$, and certain properties of R related to $\delta({}_RR)$. From now on, let $\delta(R) = \delta({}_RR)$ and $\text{Soc}(R) = \text{Soc}({}_RR)$.

Theorem 2.6 [5]: Given a semiring R, both of the next sets are equal to $\delta(R)$:
1. $R_1 =$ the intersection of all essential maximal left ideals of R.
2. $R_2 =$ the unique largest δ-small left ideal of R.

We mean by $J(R)$ and $J(R/\text{Soc}(R))$ to be the Jacobson radical of R and $R/\text{Soc}(R)$), respectively.

Proposition 2.7 [5]: For a subtractive semiring R, $\delta(R)/\text{Soc}(R) = J(R/\text{Soc}(R))$. In particular, $R = \delta(R)$ if and only if R is a semisimple semiring.

Similar to [9, Lemma 2.2], we provide the next lemma.

Lemma 2.8: The next are equivalent for a subtractive semimodule A and $m \in A$.
(1) Rm is not δ-small in A;
(2) There is a maximal subsemimodule N of A such that $m \notin N$ and A/N singular.

Proof: (1) ⇒ (2) Assume $\Gamma = \{B \leq A \mid B \neq A, Rm + B = A,\ A/B \text{ singular}\}$. Because Rm is not δ-small in A, there exists a proper subsemimodule $B \lneqq A$ such that $Rm + B = A$ and A/B singular. Thus Γ is non-empty. Assume Ω be a nonempty totally ordered subset of Γ and $B_0 = \cup_{B \in \Omega} B$. If m is in B_0 then there is a $B \in \Omega$ with $m \in B$. At that moment $B = Rm + B = A$ which is a contradiction. So, we have $m \notin B_0$ and $B_0 \neq A$. Since $Rm + B_0 = A$ and A/B_0 singular, B_0 is upper bound in Γ. By Zorn's Lemma, Γ has a maximal element, say N. If N is a maximal subsemimodule of A there is not anything to do. Suppose that there exists a subsemimodule K containing N properly. Since N is maximal in Γ, K is not in Γ. Since $A = Rm + N$ and $N \leq K$, so $A = Rm + K$. A/K as a homomorphic image of singular semimodule A/N is singular. From now K must belong to Γ. This is the vital contradiction.

(2) ⇒ (1) Let N be a maximal submodule with $m \in A \backslash N$ and A/N singular. We assume $A = Rm + N$. Then $N \neq A$, thus Rm is not δ-small in A. □

Lemma 2.9: Let A be a semimodule and $K, L, H \leq A$. If $L \ll_\delta K$, then $L \ll_\delta K + H$.

Proof: Assume that $L \ll_\delta K$. Let $U \leq A$ with $K + H = L + U$ and $(K + H)/U$ singular. Then $K/(U \cap K) \cong (K + U)/U = (K + H)/U$ is singular. On the other hand, we get $K = L + (K \cap U)$. Since L is δ-small in K, $K = K \cap U \leq U$. So $K + H = U$. □

Lemma 2.10: Let $L \leq A$. If L is δ-supplement and $U \ll_\delta A$ with $U \leq L$, then $U \ll_\delta L$.

Proof: Let $A = K + L$ with $K \cap L \ll_\delta L$ besides $L = U + V$ with L/V singular. We prove that $L = V$. Then $A = K + U + V$ and $A/(K + V) = (K + L)/(K + V) = ((K + V) + L)/(K + V) \cong L/(L \cap (K + V))$ which is a homomorphic image of singular semimodule L/V. By suggestion $A = K + V$. Then $L = (L \cap K) + V$ and thus $L = V$. □

Lemma 2.11: Let $C \leq B$ and K be subsemimodules of A and $A = C + K$. If $B \cap K \ll_\delta A$, then $B/C \ll_\delta A/C$.

Proof: Let $A/C = B/C + L/C$ with A/L singular. We have $A = B + L$ and $B = C + B \cap K$. Then $A = C + B \cap K + L = B \cap K + L$. Hence $A = L$ since $B \cap K \ll_\delta A$ besides A/L is singular. □

Lemma 2.12: Assume A is an R-semimodule besides K, L, $F \leq A$. At that time, we get the next.
a) If K is a δ-supplement of F in A besides $T \ll_\delta A$, then K is a δ-supplement of $F + T$ in A.
b) Let $f: A \to F$ be an epimorphism such that $Kerf \ll_\delta A$. If $L \leq A$ is a δ-supplement in A, then $f(L)$ is a δ-supplement in F. The reverse holds if $Ker(f) \ll_\delta L$.

Proof: (a) If K is a δ-supplement of F in A. At that time $A = F + K$ and $F \cap K \ll_\delta K$. We verify $(F + T) \cap K \ll_\delta K$. For if, let $L \leq K$ with $K = L + (F + T) \cap K$ and K/L singular, then $A = L + F + T$ and $A/(L + F) = (K + F)/(L + F) \cong K/(K + (L \cap F))$ is singular as an homomorphic image of the singular semimodule K/L. As $T \ll_\delta A$, $A = L + F$. Hence $K = L + K \cap F$. Since $K \cap F \ll_\delta K$ and K/L is singular we get $K = L$.

(b) If L is a δ-supplement of K in A. At that time L is a δ-supplement of $K + Kerf$ by (1). By Lemma 2.10, $f(L) = f(L + Kerf)$ is also a δ-supplement of $f(K) = f(K + Kerf)$ in F. Conversely, let $F = f(L) + U$ with $f(L) \cap U$ is δ-small in $f(L)$ and $K = f^{-1}(U)$. Then $A = L + K$. To end the proof, we show that $L \cap K \ll_\delta L$. For if $L = V + L \cap K$ with L/V singular, then $f(L) = f(V) + f(L) \cap f(K) = f(V) + f(L) \cap U$ since $Kerf \leq K$, $f(L \cap K) = f(L) \cap f(K)$. $f(L)/f(V)$ is singular as a homomorphic image of singular semimodule L/V. Thus, $f(L) = f(V)$. So $L = V + Kerf$. So $L = V$. □

3. Principally supplemented and principally lifting semimodules

Now we introduce two definitions, principally supplemented and principally lifting semimodules as generalization of principally supplemented and principally lifting modules.

Similar to [10], we introduce the following definition in semimodules.

Definition 3.1: A semimodule A is called principally lifting (or has (PD_1) for short) if for all $a \in A$, A has a decomposition $A = N \oplus S$ with $N \leq Ra$ and $Ra \cap S \ll A$.

Definition 3.2: A non-zero semimodule A is called a principally hollow (briefly, P-hollow) if every proper cyclic subsemimodule is small in A. Observe that every P-hollow semimodule satisfies the condition (PD_1).

Proposition 3.3: The condition (PD_1) is inherited by summands.

Proof: Let A have the condition (PD_1) and K a direct summand of A, if $k \in K$, then A has a decomposition $A = N \oplus S$ with $N \leq Rk$ and $Rk \cap S \ll A$. It follows that $K = N \oplus (K \cap S)$, and $Rk \cap (K \cap S) \leq Rk \cap S \ll A$, so $Rk \cap (K \cap S) \ll K$ (due to K a direct summand of A). Thus, K has (PD_1). □

It is known that an indecomposable semimodule is lifting if and only if it is a hollow semimodule [14], the next Lemma gives a similarity to this point.

Lemma 3.4: The following are equivalent for an indecomposable semimodule A:
(1)A has (PD_1).
(2)A is a P-hollow semimodule.

Proof: Follows in a straight line from the defining condition of (PD_1). □

Lemma 3.5: The next are equivalent for a semimodule A.
(1)A has (PD_1);
(2)Every cyclic subsemimodule C of A can be written as $C = N \oplus S$ with N is a direct summand in A and $S \ll A$;
(3)For each $a \in A$, there exist principal ideals I and J of R such that $Ra = Ia \oplus Ja$, where Ia is a direct summand in A and $Ja \ll A$.

Proof: (1)⇒ (2) It is clear.
(2)⇒ (1) Let C be a cyclic subsemimodule of A, then by (2) $C = N \oplus S$ with N is a direct summand in A and $S \ll A$. Write $A = N \oplus N'$, it follows that $C = N \oplus C \cap N'$. Now let $\pi: N \oplus N' \to N'$ be the natural projection, we get $C \cap N' = \pi(C) = \pi(N \oplus S) = \pi(S) \ll A$, [7, Lemma 2.4]. Thus A has (PD_1).
(2) ⇔ (3) Clear. □

Similar to [11, Lemma 2], we give the following lemma.

Lemma 3.6: Let N and L be subsemimodules of A. Then the next are equivalent:
(1)$A = N + L$ and $N \cap L$ is small in L;
(2)$A = N + L$ and for any proper subsemimodule K of L, $A \neq N + K$.

Proof: Clear. □
Similar to [11], we give the following definition in semimodule theory.

Definition 3.7: Let N be a cyclic subsemimodule of A. A subsemimodule L is called a principally supplement of N in A if N and L satisfy the conditions in Lemma 3.6 and the semimodule A is called principally supplemented if every cyclic subsemimodule of A has a principally supplement in A.

Clearly, every supplemented semimodule and every lifting semimodule, and so every principally lifting semimodule is principally supplemented. Also, there is principally supplemented semimodules but neither supplemented nor principally lifting.

Examples 3.8: The $\mathbb{Z}$-semimodule $\mathbb{Q}$ of rational numbers has no maximal subsemimodules. At that point $\mathbb{Q}$ is not supplemented. Every cyclic subsemimodule of $\mathbb{Q}$ is small. However, $\mathbb{Q}$ is principally supplemented $\mathbb{Z}$-semimodule.

Lemma 3.9: Consider the next conditions for an indecomposable semimodule A:
(1)A is a principally lifting semimodule.
(2)A is a principally supplemented semimodule.
(3)A is a principally hollow semimodule.
Then (1) ⇔ (3) and (3) ⇒ (2).

Proof: (1)⇔ (3) By Lemma 3.4.
(3)⇒ (2) Let $a \in A$. By (2) all cyclic subsemimodule is hollow. Then $A = Ra + A$ and $(Ra) \cap A \ll A$. □
Reminder that (3) ⇒ (2) in Lemma 3.9 does not hold in general as in modules see [10].

4. Principally δ-supplemented and principally δ-lifting semimodules

Here, we present the notion of principally δ-supplemented semimodules. We verify that certain marks of supplemented besides δ-supplemented semimodules can be lengthy toward principally δ-supplemented semimodules.
Similar to [12, Lemma 3.1], we give the next lemma.

Lemma 4.1: Let $a \in A$ and L a subsemimodule of A. Then the following are equivalent.
(1)$A = Ra + L$ and $Ra \cap L \ll_\delta L$;
(2)$A = Ra + L$ and for any proper subsemimodule K of L with L/K singular, $A \neq Ra + K$.

Proof: (1) ⇒ (2) Let $K \leq L$ and $A = Ra + K$ where L/K singular. Then $L = (L \cap Ra) + K$. Since $L \cap Ra$ is δ-small in L, $L = K$.
(2) ⇒ (1) If $L = (Ra \cap L) + K$ where $K \leq L$ and L/K singular, then $A = Ra + L = Ra + K$. By (2), $K = L$. So $Ra \cap L$ is δ-small in L. □

Lemma 4.2: Suppose L is a δ-supplement of K in A and K is a δ-supplement of H in A, then K is a δ-supplement of L in A.

Proof: Let $A = K + L = K + H$, $K \cap L \ll_\delta L$ and $K \cap H \ll_\delta K$. To show $K \cap L \ll_\delta K$. Let $X \leq A$ such that $K \cap L + X = K$ besides K/X is singular. Now $A = (K \cap L) + X + H$. Since

$K \cap L \ll_\delta A$, using Lemma 2.2, there exists a projective semisimple subsemimodule Y in $K \cap L$ with $A = Y \oplus (X + H)$. Henceforth $K = (Y \oplus X) + (K \cap H)$. Since $K/(X + Y)$ is singular and $K \cap H \ll_\delta K$, again by Lemma 2.2, $K = X \oplus Y$. Hence $Y = 0$ as K/X is singular besides Y is semisimple projective. □

Definition 4.3: Let A be a semimodule and $a \in A$. A subsemimodule L is named a principally δ-supplement of Ra in A, if Ra and L satisfy Lemma 4.1 besides the semimodule A is named principally δ-supplemented if every cyclic subsemimodule of A has a principally δ-supplement in A, equivalently, for all $a \in A$ there exists a subsemimodule C of A with $A = Ra + C$ and $Ra \cap C \ll_\delta C$.

Similar to [12], we give the following definition.

Definition 4.4: A semimodule A is defined to be principally δ-lifting if, for all $a \in A$, there exists a decomposition $A = M \oplus N$ such that $M \leq Ra$ and $Ra \cap N$ is δ-small in N (equivalently, in A).

Obviously, supplemented semimodules besides principally δ-lifting semimodule is principally δ-supplemented. All singular δ-supplemented semimodule is supplemented, since every factor semimodule of a singular semimodule is singular. There are semimodules which are not supplemented besides not δ-supplemented but principally δ-supplemented.

Example 4.5: Let $\mathbb{N}_0$ and $\mathbb{Q}$ symbolize the semiring of non-negative integers and rational numbers respectively. $\mathbb{Q}$ is not supplemented, besides $\mathbb{Q}$ is not δ-supplemented as it is singular $\mathbb{N}_0$-semimodule. But the $\mathbb{N}_0$-semimodule $\mathbb{Q}$ has no maximal subsemimodules. Any cyclic subsemimodule of $\mathbb{Q}$ is small, so $\mathbb{Q}$ is δ-supplemented as form principally.

Lemma 4.6: If $f: A \to A'$ is a homomorphism besides N is a δ-supplement in A with $Kerf \leq N$, at that point $f(N)$ is a δ-supplement in $f(A)$.

Proof: Let $A = N + K$ with $N \cap K$ δ-small in N. Then $f(M) = f(N + K) = f(N) + f(K)$. Since $Kerf \leq N$, we have $f(N) \cap f(K) = f(N \cap K)$. By Lemma 2.3 (2), $f(N \cap K) = f(N) \cap f(K)$ is δ-small in $f(N)$. Hence $f(N)$ is a δ-supplement of $f(K)$ in $f(M)$. □

Lemma 4.7: Let A be a subtractive principally δ-supplemented semimodule and $N \leq A$. If every cyclic subsemimodule Rx has a δ-supplement B with $N \leq B$, then A/N is principally δ-supplemented.

Proof: Since A is a subtractive semimodule, so we have A/N is an R-semimodule. Let K/N be a cyclic subsemimodule of A/N. Then $K = Ra + N$ for some $x \in A$. There exists $L \leq A$ such that $N \leq L$, $A = Rx + L$ with $Rx \cap L$ δ-small in L. Let $\pi: A \to A/N$ natural epimorphism. Using Lemma 4.6, $\pi(L)$ is δ-supplement of $\pi(Rx) = K/N$, indeed $A/N = L/N + (Rx + N)/N = L/N + K/N$ besides $(N + (L \cap Rx))/N \ll_\delta L/N$ as it is a homomorphic image of $L \cap Rx$ where $L \cap Rx \ll_\delta L$. □

Lemma 4.8: Assume A is a semimodule, N a δ-supplemented subsemimodule of A and F a cyclic subsemimodule of A. If $N + F$ has a δ-supplement T in A, then $N \cap (T + F)$ has a δ-supplement U in N. Specific, $T + U$ is a δ-supplement of F in A.

Proof: Clearly $A = (N + F) + T$ and $(N + F) \cap T$ is δ-small in T, $N \cap (F + T) + U = N$ and $(F + T) \cap U$ is δ-small in U. Then $A = N + F + T = F + N \cap (F + T) + U = F + T + U$. As finite sum of δ-small subsemimodules is δ-small using part (3) of Lemma 2.3, $F \cap (T + U) \leq T \cap (F + U) + U \cap (F + T) \leq T \cap (F + N) + U \cap (F + T)$, and so $F \cap (T + U) \ll_\delta T + U$. □

Recall that [5] a semimodule A is named distributive, if for $K, L, N \leq A$, we have $N \cap (K + L) = N \cap K + N \cap L$ or $N + (K \cap L) = (N + K) \cap (N + L)$.

Lemma 4.9: Let $A = A_1 \oplus A_2 = K + N$ and $K \leq A_1$. If A is distributive and $K \cap N \ll_\delta N$, then $K \cap N \ll_\delta A_1 \cap N$.

Proof: Let $A_1 \cap N = (K \cap N) + L$ with $(A_1 \cap N)/L$ singular. Since A is distributive, $N = A_1 \cap N \oplus A_2 \cap N$. We get $A = K + N = K + A_1 \cap N + A_2 \cap N = K + L + (A_2 \cap N)$ and $N = K \cap N + L + (A_2 \cap N)$. Now
$N/(L \oplus (A_2 \cap N)) = ((N \cap A_1) \oplus (N \cap A_2))/(L \oplus (A_2 \cap N)) \cong (N \cap A_1)/L$ is singular. Hence $N = L \oplus (A_2 \cap N)$. Thus $N = (N \cap A_1) \oplus (N \cap A_2)$ and $L \leq A_1 \cap N$ imply $L = A_1 \cap N$. So $K \cap N \ll_\delta A_1 \cap N$. □

Theorem 4.10: In principally δ-supplemented distributive semimodule each direct summand is principally δ-supplemented.

Proof: Assume $A = A_1 \oplus A_2$, $x \in A_1$. There exists $N \leq A$ with $A = Rx + N$ besides $Rx \cap N \ll_\delta N$. Then $A_1 = Rx + (A_1 \cap N)$ and by Lemma 4.9, $Rx \cap (A_1 \cap N)$ is δ-small in $A_1 \cap N$. □

Proposition 4.11: Let A_1 and A_2 be principally δ-supplemented semimodules and $A = A_1 \oplus A_2$. If A is a distributive semimodule, then A is principally δ-supplemented.

Proof: Let $A = A_1 \oplus A_2$ be a distributive semimodule besides $Rx \leq A$. Then $Rx = (Rx \cap A_1) \oplus (Rx \cap A_2)$. Since $Rx \cap A_1$ and $Rx \cap A_2$ are cyclic subsemimodules of A_1 and A_2 respectively, there exists $M \leq A_1$ such that $A_1 = (Rx \cap A_1) + M$ and $M \cap (Rx \cap A_1) = M \cap Rx$ is δ-small in M, and $N \leq A_2$ such that $A_2 = (Rx \cap A_2) + N$, $N \cap (Rx \cap A_2) = N \cap Rx$ is δ-small in N. Then $A = Rx + M + N$.

We now claim that $Rx \cap (M + N) = (Rx \cap M) + (Rx \cap N)$. The inclusion $(Rx \cap M) + (Rx \cap N) \leq Rx \cap (M + N)$ always holds. For the inverse inclusion, $Rx \cap (M + N) \leq M \cap (Rx + N) + N \cap (Rx + M) = M \cap ((Rx \cap A_1) + A_2) + N \cap (A_1 + (Rx \cap A_2))$. On the other hand $M \cap ((Rx \cap A_1) + A_2) \leq (Rx \cap A_1) \cap (M + A_2) + A_2 \cap ((Rx \cap A_1) + M) = Rx \cap M$. Similarly $N \cap (A_1 + (Rx \cap A_2)) \leq Rx \cap N$. Hence $(Rx \cap (M + N) \leq Rx \cap M + Rx \cap N$. Therefore, the claim $(Rx \cap (M + N) = Rx \cap M + Rx \cap N$ is defensible. Since $Rx \cap M \ll_\delta M$ and $Rx \cap N \ll_\delta N$, by Lemma 2.3(3), we have $Rx \cap (M + N) \ll_\delta M + N$. Hence, A is principally δ-supplemented. □

Similar to that of module theory in [13], if every cyclic subsemimodule is a direct summand of A, we say that a semimodule A is principally semisimple. However, in semimodules, one can say that (semisimple semimodule → principally semisimple). Any principally semisimple semimodule is principally δ-lifting, besides as a result principally δ-supplemented.

Lemma 4.12: Assume a subtractive semimodule A is principally δ-supplemented besides distributive. At that time $A/\delta(A)$ is a principally semisimple semimodule.

Proof: Let $\bar{a} \in A/\delta(A)$. There exists a $N \leq A$ with $A = Ra + N$ and $Ra \cap N \ll_\delta N$, so $Ra \cap N \ll_\delta A$. Using the distributivity of A we get $Ra \cap (N + \delta(A)) = (Ra \cap A) + Ra \cap \delta(A) = \delta(A)$. Now
$A/\delta(A) = ((Ra + \delta(A))/\delta(A) + ((N + \delta(A))/\delta(A) = (R\bar{a}/\delta(A)) \oplus ((N + \delta(A))/\delta(A)$.
□

Theorem 4.13: Assume a subtractive semimodule A is principally δ-supplemented. Then A has a subsemimodule A_1 wherever A_1 has an essential socle as well as $\delta(A) \oplus A_1$ is an essential in A.

Proof: We may find a subsemimodule A_1 of A such that $\delta(A) \oplus A_1$ is essential in A by Zorn's Lemma. Toward prove $Soc(A_1) \leq_e A_1$, we prove that any cyclic subsemimodule of A_1 has a simple subsemimodule. Let $a \in A_1$. There exists a subsemimodule N of A such that $A = Ra + N$ besides $Ra \cap N \ll_\delta N$ since A is principally δ-supplemented. Then $Ra \cap N = 0$. Suppose K be a maximal subsemimodule of Ra. If K is unique maximal subsemimodule in Ra, then $K \ll Ra$, thus $K \ll_\delta Ra$ and so $K \ll_\delta A$. This is not likely since $Ra \cap \delta(A) = 0$. So, there exists $x \in Ra$ with $Ra = K + Rx$. We claim that $K \cap Rx = 0$.

Otherwise, let $0 \neq x_1 \in K \cap Rx$. By hypothesis there exists B_1 such that $Rx_1 \cap B_1 \leq K \cap \delta(A) = 0$. Hence $Ra = Rx_1 \oplus (Ra \cap B_1)$ and $K = Rx_1 \oplus (K \cap B_1)$. If $K \cap B_1$ is nonzero, let $0 \neq x_2 \in K \cap B_1$. By hypothesis there exists B_2 such that $A = Rx_2 + B_2$ with $Rx_2 \cap B_2$ is δ-small in A. So $= Rx_2 \oplus B_2$, since $Rx_2 \cap B_2 \leq K \cap \delta(A) = 0$ and A is subtractive semimodule. Then $K \cap B_1 = Rx_2 \oplus (K \cap B_1 \cap B_2)$. Hence $Ra = Rx_1 \oplus Rx_2 \oplus (Ra \cap B_1 \cap B_2)$ and $K = Rx_1 \oplus Rx_2 \oplus (K \cap B_1 \cap B_2)$, by using subtractive condition of A [4]. If $K \cap B_1 \cap B_2$ is nonzero, similarly there exists $0 \neq x_3 \in K \cap B_1 \cap B_2$ and $B_3 \leq A$ such that $A = Rx_3 \oplus B_3$. Then $Ra = Rx_1 \oplus Rx_2 \oplus Rx_3 \oplus (Ra \cap B_1 \cap B_2 \cap B_3)$ and $K = Rx_1 \oplus Rx_2 \oplus Rx_3 \oplus (K \cap B_1 \cap B_2 \cap B_3)$. This process must terminate at a finite step, give or take t. At this step $Ra = Rx_1 \oplus Rx_2 \oplus Rx_3 \oplus \cdots \oplus Rx_t$ and so $Ra = K$ since at t^{th} step we must have $K \cap B_1 \cap B_2 \cap \cdots \cap B_t \leq Ra \cap B_1 \cap B_2 \cap \cdots \cap B_t = 0$. This is a illogicality. There exists $x \in Ra$ such that $Ra = K \oplus Rx$. At that point Rx is a simple semimodule. □

Now, under some conditions direct summands are principally δ-supplemented.
Lemma 4.14: Assume $A = A_1 \oplus A_2$ be a decomposition of a subtractive semimodule A. Then A_2 is principally δ-supplemented iff for every cyclic subsemimodule N/A_1 of A/A_1, there exists a subsemimodule K of A_2 such that $A = K + N$ and $N \cap K \ll_\delta K$.

Proof: Assume A_2 is principally-supplemented. Lease N/A_1 be a cyclic subsemimodule of A/A_1. Let $N/A_1 = (Rx + A_1)/A_1$ and $x = m_1 + m_2$ where $m_1 \in A_1$, $m_2 \in A_2$. Then $N/A_1 = (Rm_2 + A_1)/A_1$. By supposition there exists a $K \leq A_2$ such that $A_2 = (Rm_2) + K$ with $(Rm_2) \cap K$ is δ-small in K. Then $N = Rm_2 + A_1$ and $A = N + K$. Now, $N \cap K = ((Rm_2) + A_1) \cap K \leq (Rm_2) \cap (A_1 + K) + A_1 \cap (K + (Rm_2)) \leq K \cap (A_1 + (Rm_2)) + A_1 \cap (Rm_2 + K)$. $A_1 \cap (Rm_2 + K) = 0$ implies $(A_1 + Rm_2) \cap K = (Rm_2) \cap ((Rm_1) + K)$. As a result $N \cap K \leq Rm_2$. Since $(Rm_2) \cap K \ll_\delta K$, $N \cap K \ll_\delta K$.

In opposition, let $N \leq A_2$ be a cyclic subsemimodule. Assume the cyclic subsemimodule $(N + A_1)/A_1$ of A/A_1. By hypothesis, there exists $K \leq A_2$ such that $A = (N + A_1) + K$ and $K \cap (N + A_1) \ll_\delta K$. Then $A_2 = N + K$. We need to whole the proof to show that $K \cap (A_1 + N) = N \cap (A_1 + K) = N \cap K$. Now $N \cap (A_1 + K) \leq A_1 \cap (K + N) + K \cap (N + A_1) = K \cap (N + A_1) \leq N \cap (A_1 + K) + A_1 \cap (K + N) = N \cap (A_1 + K)$ since $A_1 \cap (K + N) = 0$.

Then $N \cap (A_1 + K) = K \cap (N + A_1)$. But $(A_1 + K) \cap N = K \cap (N + A_1) = N \cap K$ is clear now. So $N \cap K \ll_\delta K$. □

Proposition 4.15: Let A_1 and A_2 be principally δ-supplemented semimodules with $A = A_1 \oplus A_2$. Then A is principally δ-supplemented if and only if any cyclic subsemimodule N of A such that $A = N + K$ for any proper subsemimodule K of A has a supplement in A.

Proof: One side is evident. Conversely, assume that for each cyclic subsemimodule N of A with $A = N + K$ for any proper direct summand K of A has a supplement in A. Let $N = Rn$ be a cyclic subsemimodule. If $A = N + A_i$ or $N \leq A_i$ we have done. Otherwise, we may take up $n = n_1 + n_2$ and n_1 and n_2 are nonzero. By supposition there are $K_1 \leq A_1$ and $K_2 \leq A_2$ such that $A_1 = (Rn_1) + K_1$, $A_2 = (Rn_2) + K_2$ and $(Rn_1) \cap K_1 \ll_\delta K_1$ and $(Rn_2) \cap K_2 \ll_\delta K_2$. $Rn_1 + Rn_2 = N + Rn_2 = N + Rn_1$ and $= N + Rn_1 + K_1 + K_2 = N + A_1 + K_2$. Similarly $A = N + A_2 + K_1$. Assume $A = A_1 + K_2$. Then $A_2 = K_2$ and so $n_2 = 0$ and $N \leq A_1$. It leads us to a contradiction. Hence $A_1 + K_2$ is a proper subsemimodule of A. Similarly, $A_2 + K_1$ is proper. Hence N has a supplement in A. □

Definition 4.16: Recall [14] A non-zero semimodule A is named δ-hollow if any proper subsemimodule is δ-small in A.

In [9] principally δ-lifting (and principally δ-hollow) modules are defined we now give the following definition similar to [9].

Definition 4.17: A non-zero semimodule A is named principally δ-hollow if every proper cyclic subsemimodule is δ-small in A.

Remark 4.18: A finite direct sum of δ-small subsemimodules is δ-small [5], A is finitely δ-hollow if and only if A is principally δ-hollow. There are principally δ-hollow semimodules nonetheless not δ-hollow. Consider $\mathbb{N}_0$ and $\mathbb{Q}$ symbolize the semiring of non-negative integers and rational numbers, respectively. At that time the $\mathbb{N}_0$-semimodule $\mathbb{Q}$ is principally δ-hollow because any finitely generated $\mathbb{N}_0$-subsemimodule of $\mathbb{Q}$ is small, so δ-small in $\mathbb{Q}$. Assume $\mathbb{Q}_1 = \{\frac{a}{b} \in \mathbb{Q} |$ 2 does not divide $b\}$ and $\mathbb{Q}_2 = \{\frac{a}{b} \in \mathbb{Q} |$ 2 divides $b\}$. Thus $\mathbb{Q} = \mathbb{Q}_1 + \mathbb{Q}_2$. Since $\mathbb{Q}/\mathbb{Q}_1$ and $\mathbb{Q}/\mathbb{Q}_2$ are singular $\mathbb{N}_0$-semimodules, $\mathbb{Q}_1$ and $\mathbb{Q}_2$ are not δ-small subsemimodules in $\mathbb{Q}$.

Definition 4.19: A non-zero semimodule A is named principally δ-lifting if for each one cyclic subsemimodule has the δ-lifting property, i.e., for each $a \in A$, A has a decomposition $A = M \oplus N$ with $M \leq Ra$ besides $Ra \cap N \ll_\delta N$.

Remark 4.20: If A is a principally δ-lifting semimodule then A is principally δ-supplemented. Note there are semimodules not principally δ-lifting but principally δ-supplemented. By way of a design, we record here Example 4.21.

Example 4.21: Consider $A_1 = \mathbb{Z}/2\mathbb{Z}$ and $A_2 = \mathbb{Z}/8\mathbb{Z}$ as a $\mathbb{Z}$-semimodules. As A_1, A_2 are principally δ-hollow, so principally δ-supplemented semimodules. Let $A = A_1 \oplus A_2$. It is stated in [9] that A is not a principally δ-lifting $\mathbb{Z}$-module and so is not principally δ-lifting $\mathbb{Z}$-semimodule. $M_1 = (\bar{1}, \bar{2})\mathbb{Z}$, $M_2 = (\bar{1}, \bar{1})\mathbb{Z}$, $M_3 = (\bar{0}, \bar{4})\mathbb{Z}$ and $M_4 = (\bar{0}, \bar{2})\mathbb{Z}$ are the alone proper subsemimodules of A and all of them are cyclic. $M_3 \ll_\delta A$ and $M_4 \ll_\delta A$ besides $A = M_1 + M_2$. Now $M_1 \cap M_2 = M_3$ is δ-small in both M_1 as well as M_2. Henceforth, A is principally

δ-supplemented. For any prime integer p, by the same reasoning, the $\mathbb{Z}$-semimodule $A = (\mathbb{Z}/p\mathbb{Z}) \oplus (\mathbb{Z}/p^3\mathbb{Z})$ is not principally δ-lifting but it is principally δ-supplemented.

Example 4.22: Assume $\mathbb{N}_0$ is the semiring of non-negative integer numbers and assume the $\mathbb{N}_0$-semimodules $A_1 = \mathbb{N}_0/p\mathbb{N}_0$ and $A_2 = \mathbb{N}_0/p^3\mathbb{N}_0$, for any prime integer p, by the same reasoning in Example 4.21, the $\mathbb{Z}$-semimodule $A = A_1 \oplus A_2$ is not principally δ-lifting but is principally δ-supplemented.

Lemma 4.23: Consider the following conditions for an indecomposable semimodule A.
(1)A is a principally δ-lifting semimodule.
(2)A is a principally δ-supplemented semimodule.
(3)A is a principally δ-hollow semimodule.
Then (1) ⇔ (3) and (3) ⇒ (2).

Proof: (3) ⇔ (1) The proof similar to those for modules in [9]. (3) ⇒ (2) Let $x \in A$. Any cyclic subsemimodule is δ-hollow by (3). Then $A = Rx + A$ and $Rx \cap A \ll_\delta A$. Thus A is principally δ-supplemented. □

Reminder that (3) ⇒ (2) in Lemma 4.23 does not hold in general.

We now give the following definition similar to [14, p. 95].

Definition 4.24: Let R be a semiring. An R-semimodule A is called $\oplus$-supplemented if for all subsemimodule N of A there is a direct summand K of A with $A = N + K$ and $N \cap K \ll K$. Clearly $\oplus$-supplemented semimodules are supplemented.

Definition 4.25: An R-semimodule A is called $\oplus$-δ-supplemented semimodule if for all subsemimodule N of A there exists a direct summand K with $A = N + K$ and $N \cap K \ll_\delta K$.

Remark 4.26: In the similar method δ-$\oplus$-supplemented semimodule means for each subsemimodule N of A there is a direct summand K with $A = N + A$ and $N \cap A \ll_\delta K$. It is the same as $\oplus$-δ-supplemented semimodule.

Now we give the following definitions similar to [12].

Definition 4.27: A semimodule A is called principally $\oplus$-supplemented if for all $a \in A$ there exists a direct summand B of A such that $A = Ra + B$ and $Ra \cap B \ll_\delta B$.

Definition 4.28: A semimodule A is called principally $\oplus$-δ-supplemented semimodule if for all $a \in A$ there exists a direct summand B of A such that $A = Ra + B$ and $Ra \cap B \ll_\delta B$.

Definition 4.29: A semimodule A is called a weak principally $\oplus$-δ-supplemented if for all $a \in A$ there exists a direct summand B such that $A = Ra + B$ and $Ra \cap B \ll_\delta A$.
Weakly supplemented semimodule ⟹ weak principally δ-supplemented. $\oplus$-supplemented semimodule ⟹ principally $\oplus$-δ-supplemented. As well as it is obvious that principally $\oplus$-supplemented ⟹ weak principally δ-supplemented. In a succeeding article, the author examines the interconnections among principally δ-supplemented, weakly principally δ-supplemented besides principally $\oplus$-δ-supplemented semimodules in feature.
Similar to modules in [15], we say a semimodule A is said to have the *summand intersection property* if the intersection of any two direct summands of A is again a direct summand of A. Similar to [16], a semimodule A is named *refinable* if for any subsemimodule U, V of A with $A = U + V$ there is a direct summand U' of A such that $U' \leq U$ and $A = U' + V$.

Theorem 4.30: Consider the following conditions for a refinable semimodule A.
(1)A is principally δ-lifting.
(2)A is principally $\oplus$-δ-supplemented.
(3)A is principally δ-supplemented.
(4)A is weak principally δ-supplemented.
Then (1) ⇒ (2) and (2) ⇔ (3) ⇔ (4). If A has the summand intersection property then (4) ⇒ (1).

Proof: (1) ⇒ (2) ⇒ (3) ⇒ (4) By definitions continuously hold.
(4) ⇒ (2) Assume A is weakly principally δ-supplemented besides $a \in A$. There exists a $B \leq A$ such that $A = Ra + B$ besides $Ra \cap B \ll_{\delta} A$ by (4). By assumption, there exists a direct summand U of A with $U \leq B$ and $A = Ra + U = U' \oplus U$ for some $U' \leq A$. We claim that $Ra \cap U \ll_{\delta} U$. Assume that $Ra \cap U + L = U$ for some $L \leq U$ with U/L singular. Since $A/(U' + L)$ is singular as it is isomorphic to the singular U/L. Then $A = U' + (Ra \cap U) + L$ implies $A = U' \oplus L$ as $Ra \cap U \ll_{\delta} A$. Thus $L = U$. Hence A is principally $\oplus$-δ-supplemented.
(4) ⇒ (1) Let $a \in A$ besides A has the summand intersection property. Using (4) there exists a subsemimodule B with $A = Ra + B$ besides $Ra \cap B \ll_{\delta} A$. Using assumption, there exists a direct summand U' of A with U_1 is contained in A besides $A = Ra + U_1 = U_1{}' \oplus U_1$.
Since U_1 is direct summand besides $Ra \cap B \ll_{\delta} A$, $Rm \cap U_1 \ll_{\delta} U_1$ by Lemma 2.3 (3). Yet again via assumption, there is a direct summand U_2 of A such that U_2 is contained in Ra and $A = U_2 + U_1 = U_2 \oplus U_2{}'$. By the summand intersection property $U_2 \cap U_1$ is a direct summand of A, $A = (U_2 \cap U_1) \oplus K$ for some subsemimodule K of A. Then $U_1 = (U_2 \cap U_1) \oplus (K \cap U_1)$ and $A = U_2 \oplus (K \cap U_1)$. By Lemma 2.3 (1), $Ra \cap (K \cap U_1) \ll_{\delta} U_1$ since $Ra \cap (K \cap U_1) \leq Ra \cap U_1 \leq U_1$ and $Ra \cap U_1 \ll_{\delta} U_1$. By Lemma 2.3 (3), $Ra \cap (K \cap U_1)$ is δ-small in $K \cap U_1$ as $K \cap U_1$ is direct summand of U_1. □

Definition 4.31 [6]: A homomorphism $f: A \longrightarrow B$ of left R-semimodules is called k-quasiregular if whenever $K \leq A$, $a \in A \backslash K$, $a' \in K$, and $f(a) = f(a')$ there exists $s \in \mathrm{Ker}(f)$ such that $a = a' + s$.

Definition 4.32 [6]: Let A be a semimodule. A semimodule P together with an R-homomorphism $f: P \longrightarrow A$ is named a projective cover of A if:
(1) P is projective,
(2) f is small, epimorphism and k-quasiregular.

Definition 4.33 [5]: Let A be a left R-semimodule. A left R-semimodule P together with an R-homomorphism $f: P \longrightarrow A$ (A pair (P, p)) is named a projective δ-cover of A if:
(1) P is projective,
(2) f is δ-small, epimorphism and k-quasiregular.

Definition 4.34: A semimodule A is called semiperfect if every factor semimodule of A has a projective cover. Also, A is called δ-semiperfect if every factor semimodule of A has a projective δ-cover.
Definition 4.35: A semimodule A is called principally semiperfect if every factor semimodule of A by a cyclic subsemimodule has a projective cover. Also, A is named principally δ-semiperfect if every factor semimodule of A by a cyclic subsemimodule has a projective δ-cover.

Now, similar to [9, Theorem 4.3], we give the following theorem.

Theorem 4.36: Let A be a principally δ-semiperfect semimodule. Then
(1)A is principally δ-supplemented.
(2)All factor semimodule of A is principally δ-semiperfect, henceforth any homomorphic image besides any direct summand of A is principally δ-semiperfect.

Proof: Similar to the proof in the case of modules in [9, Theorem 4.3]. □

Similar to [12, Theorem 3.20], we have the following theorem.

Theorem 4.37: The next conditions are equivalent for a subtractive projective semimodule A.
(1)A is principally δ-supplemented.
(2)A is principally δ-lifting.
(3)A is principally δ-semiperfect.

Proof: (3) ⇒ (1) By Theorem 4.36.

(1) ⇒ (3) Let $a \in A$. Using (1) there exists a subsemimodule B with $A = Ra + B$ besides $Ra \cap B \ll_{\delta} B$. Let $f: A \to A/Ra$ defined by $f(y) = b + Ra$, where $y = ra + b \in A$ with $ra \in Ra$, $b \in B$, and $\pi: A \to A/Ra$ the natural epimorphism, (since A is a subtractive semimodule we can say that A/Ra is an R-semimodule [3, p. 165]). There exists $g: A \to A$ such that $fg = \pi$. Then $A = g(A) + Ra \cap B$. Since $Ra \cap B \ll_{\delta} B$, $Ra \cap B \ll_{\delta} A$. By Lemma 2.2, there exists a semisimple projective subsemimodule Y of $Ra \cap B$ such that $A = g(A) \oplus Y$ and so that $g(A)$ is projective. Hence $g(A) \cong A/Ker(g)$ implies $A = Ker(g) \oplus C$ for some subsemimodule C of A and C is projective. Let $(fg)_{|C}$ indicate the restriction of fg on C. Then $Ker(fg)_{|C} \leq Ra \cap B$. So, $Ker(fg)_{|C} \ll_{\delta} C$ and hence $(fg)_{|C} : C \to A/Ra$ is a projective δ-cover of A.

(2) ⇔ (3) Similar to [9, Theorem 4.1]. □

4. Conclusions

In this paper, we have defined and studied principally supplemented (δ-supplemented), and principally lifting (δ-lifting) semimodules as generalizations of principally supplemented (δ-supplemented), and principally lifting (δ-lifting) modules. We studied principally supplemented and principally lifting semimodules. We proved that if A is an indecomposable semimodule, then A is principally lifting if and only if A is principally supplemented if and only if A is a principally hollow semimodule. Also, we proved that if A is a subtractive projective semimodule, then A is principally δ-supplemented if and only if A is principally δ-lifting if and only if A is principally δ-semiperfect.

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