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Weakly 2-Prime Sub-Modules

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Abstract

Let R be a commutative ring containing a unit, and let M be a left R-module. We define a proper sub-module N of an R-module M to be a weakly 2-prime sub-module if whenever $0 \neq rm \in N, r \in R, m \in M$, then either $m \in N$ or $r^2 \in [N:M]$. This concept is an expansion of the idea of a weakly 2-prime ideal, where an ideal P of R is said to be a weakly 2-prime ideal if for all $a, b \in R, 0 \neq ab \in P$ implies $a^2 \in P$ or $b^2 \in P$. Several characteristics of sub-modules that are weakly 2-prime are taken into account.

Keywords: prime sub-module, weakly prime sub-module, 2-prime sub-module, weakly 2-prime sub-module, proper sub-module

المقاسات الجزئية الاولية الضعيفة من النمط −2

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الخلاصة

لتكن R حلقه ابدالية ذا محايد وليكن M مقاسا ايسر على R . تعرف ان مقاسا جزئيا فعليا N في M يكون اوليا ضعيفا من النمط 2 اذا كان لكل $0 \neq rm \in N, r \in R, m \in M$ يؤدي الى $m \in N$ or $r^2 \in [N:M]$. في الحقيقة ان هذا المفهوم هو تعميم لمفهوم مثالي اولي ضعيف من النمط −2 , اذ ان مثاليا فعليا P في R يسمى اوليا ضعيفا من النمط −2 اذا كان لكل $a, b \in R, 0 \neq ab \in P$ يؤدي الى $a^2 \in P$ او $b^2 \in P$. خواص مختلفة عن المقاسات الجزئية الاولية الضعيفة من النمط −2 قد اعطيت.

1. Introduction:

Throughout this paper, R be a commutative ring with identity and M be a unity R-module. A sub-module N of M is called a prime sub-module if every time $r \in R, m \in M, rm \in N$, implies $m \in N$ or $r \in [N:M]$, where $[N:M] = \{r \in R, \; rM \subseteq N\}$, see [1] and [2]. The authors in [3] introduced 2-prime sub-module when $rm \in N, r \in R, m \in M$, either $m \in N$ or $r^2 \in [N:M]$, then N is a 2-pirme sub-module, where N is a generalization of prime. The term "weakly prime sub-module" was coined in 2007 by S.E. Atani and F. Farzalipour [4] and in 2009 by I. M. A. Hadi [5]. Keep in mind that if whenever $0 \neq rm \in N, r \in R, m \in M$, then

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either $m \in N$ or $r \in [N: M]$, implies N is a weakly prime sub-module of M. Every prime sub-module is also a weak prime sub-module, as should be evident.

In this paper, we introduced the idea of a weakly 2-prime sub-module. A suitable sub-module N of an R-module M is weakly 2-prime if and only if for every $0 \neq rm \in N, r \in R, m \in M$, then either $m \in N$ or $r^2 \in [N: M]$. As a service to the academic community, we provide an R-sub-module M of type R that is weakly 2-prime. Therefore, $[N_{\bar{R}}: M]$ is a weakly 2-pirme ideal of $\overline{R}$, where $\overline{R} = R|annM$. In 1999, the quasi-prime sub-module was introduced and studied in [6] by Muntaha, anywhere a suitable sub-module if $r_1 r_2\, m \in N$, for $r_1, r_2 \in R$, $m \in M$ implies $r_1\, m \in N$ or $r_2\, m \in N$, then N of M is a quasi-prime sub-module of M. In addition, the idea of a weakly primary sub-module was developed by S.E. Atani and F. Farzalipour in [4]: a proper sub-module N of M is a primary sub-module if whenever $rm \in N, r \in R, m \in M$, then either $m \in N$ or $r^n \in [N: M]$. A valid sub-module N of M is a weakly primary sub-module.

2. Weakly 2-prime sub-modules

Here we present the idea of a weakly 2-prime sub-module as an extension of a 2-prime sub-module, where a valid sub-module N of M is a 2-prime sub-module if whenever $rm \in N, r \in R, m \in M$, then either $m \in N$ or $r^2 \in [N: M]$, and vice versa (see [3]).

Definition 2.1:

A proper sub-module N of an R-module M is a weakly 2-prime if, whenever, $0 \neq rm \in N, r \in R, m \in M$, then either $m \in N$ or $r^2 \in [N: M]$ holds.

Remarks and Examples 2.2:

1. Every 2-prime sub-module is weakly 2-prime sub-module.

Proof: It is clear.

2. The converse of (1) is not always true for example: The zero sub-module of the Z-module Z_4 is weakly 2-prime sub-module (since it is weakly prime sub-module [2]). However, it is a 2-prime sub-module, because $2.\bar{2} \in (\bar{0}), \bar{2} \notin (0)$ and $2^2 = 4 \notin [(0): Z_4]$.
3. Every weakly 2-prime ideal of R is a sub-module that is weakly 2-prime.
4. All weakly prime sub-modules are weakly 2-prime.

Proof:

Let there N be a weakly prime submodule of an R-module M, and let $0 \neq rm \in N$, where $r \in R, m \in M$. So, either $m \in N$ or $r \in [N: M]$. Thus either $m \in N$ or $r^2 \in [N: M]$. Therefore, N weakly 2-prime sub-module.

5. The convers of (4) is not always true for example: The sub-module $N = (\bar{4})$ of the Z-module Z_8 is weakly 2-prime sub-module (since it is 2- prime sub-module). But it is a weakly prime sub-module, since $2.\bar{2} \in (N)$, but $\bar{2} \notin (N)$ and $2 \notin [N: Z_8]$.
6. It's not necessary for a weakly 2-prime sub-module to be a quasi-prime, for example: a weakly 2-prime sub-module of the Z-module Z_{12} it is zero sub-modules. However, is not quasi-prime, because $(\bar{0}:_Z \bar{3}) = 4Z$ is not the prime ideal of Z, (using [3]). Also, quasi-prime need not be a weakly 2-prime sub-module, for example, the Z-module $Z \oplus \bar{Z}$, $N = 2Z \oplus (0)$, N is a quasi-prime sub-module. But N is not a weakly 2-prime sub-module. Since $(0,0) \neq 2(3,0) \in N$ and $(3,0) \notin N$, $2^2 \notin [2Z \oplus (0):_Z Z \oplus Z]$.
7. Every weakly 2-prime sub-module is weakly primary sub-module.

Proof:

If N is a sub-module that is weakly 2-prime, and $0 \neq rm \in N,$ where $r \in R, m \in M$ are real numbers. As a result, either $rm \in N$ or $r^2 \in [N: M]$.
As a result, N is a weakly primary sub-module.

In general, the opposite of (7) can not be true, as the following demonstrates: let M be the Z-module Z_6 , $N = (\overline{0})$. Clearly, N is a weakly primary sub-module, but it is not a weakly 2-prime sub-module due to the fact that $2.\overline{3} \in (\overline{0})$, and $\overline{3} \notin (\overline{0}), 2^2 \notin [N:_z Z_6]$.

Theorem 2.3:

Let N be a proper sub-module of an R −module M. Then the following statements are equivalent.
1. N is a sub-module that is a weakly 2-prime;
2. $r^2 \in [N_R: M], r \in R$ if and only if, for each $c \in M, c \notin N$, $r^2 \in [N_R: (c)]$;
3. $r^2 \in [N_R: M], r \in R$ if and only if, $r^2 \in [N_R: K]$, for any sub-module K of M such that, $N \subseteq K$.

Proof:

1 ⟹)2 Let $c \in M/N$, if $r^2 \in [N_R: M]$, then $r^2 \in [N_R: (c)]$, therefore, $0 \neq r(rc) \in N$. It follows that either $rc \in N$ or $r^2 \in [N_R: M]$ holds if N is a sub-module that is a weakly 2-prime. Nothing can be done if $r^2 \in [N_R: M]$. If $0 \neq rc \in N$, where N is a weakly prime sub-module, and $c \in N$, then $r^2 \in [N_R: M]$, so the result is $r^2 \in [N_R: M]$.
2⟹)3 Clear.
3⟹)1 Let $0 \neq rm \in N$ and suppose $m \notin N$, where $r \in R, m \in M$. But $K = N + (m)$, so $N \subseteq K$, then $r^2K = r^2(N + \langle m\rangle) = r^2N + r^2\langle m\rangle \subseteq N$. Which means, $r^2 \in [N_R: K]$. Therefore, N is a weakly 2-prime sub-module of M if and only if (by condition 3) $r^2 \in [N_R: M]$.

Remark 2.4:

It is generally known that $[N_R: M]$, is the prime ideal of an R-module if and only if N is a prime sub-module of M. However, the (weak) a logos of this statement is not always holding true for example: The zero sub-module of the Z-module Z_6 is weakly 2-prime sub-module, but $(\overline{0}_Z: Z_6) = 6Z$ not weakly 2-prime ideal, since$2.\overline{3} \in 6Z$, but 2^2 and $3^2 \notin 6Z$.

Recall that an R-module M is called a faithful module if$ann(M) = 0$, where $ann(M) = \{r \in R \mid rx = 0, \quad \forall x \in M\}$, see [7].

The last remark satisfy under certain condition as the following proposition shows:

Proposition 2.5:

If N is a weakly 2-prime sub-module of a faithful R-module M, then $[N_R: M]$ is a weakly 2-prime ideal of R.

Proof:

Let $a, b \in R$, if $0 \neq ab \in [N_R: M]$, then $abM \subseteq N$. Sinec, M is faithful, $abM \neq 0$ hence $0 \neq abM \subseteq N$, so by Theorem (2.3), either $bM \subseteq N$ or $a^2 \in [N_R: M]$, that is either $a^2 \in [N_R: M]$ or $b^2 \in [N_R: M]$. Thus $[N_R: M]$ is a weakly 2-pirme ideal of R.

Remark 2.6:

As the following example shows, the opposite of the statement (2.4) is not always true: The Z-module $Z \oplus \bar{Z}$, $N = (0) \oplus 2Z$, then $[N_R: M] = (0)$, which is a weakly 2-pirme ideal of R. Since $(0,0) \neq 2(0,3) \in N$ and$(0,3) \notin N$, $2^2 \notin [(0) \oplus 2Z:_Z Z \oplus Z]$.

Proposition 2.7:

Let N be a sub-module of M over a ring R that is a proper sub-module. If for each $r \in R$, $[N_R: (r)]$ is a sub-module of M that is weakly 2-prime, then N is a sub-module of M that is weakly 2-prime

Proof:

$\Longrightarrow$) If $0 \neq am \in [N_M: (r)]$, where $a \in R, m \in R$. Then $0 \neq arm \in N$. Since N is a sub-module of M that is weakly 2-prime, we get either $mr \in N$ or $a^2 \in [N_R: M]$. If $mr \in N$, then $m \in [N_M: (r)]$ and if $a^2 \in [N_R: M]$, hence $a^2Mr \subseteq N$. So $a^2Mr \subseteq Nr \subseteq N$. This implies that $a^2Mr \subseteq N$. So $a^2M \subseteq [[N_R: (r)]\ _R: M]$. Thus $[N_R: (r)]$ is a sub-module of M that is weakly 2-prime, for every $r \in R$.
$\Longleftarrow$) Let $0 \neq am \in N$, where $a \in R, m \in R$, so $0 \neq amr \in Nr \subseteq N$ and thus $0 \neq amr \in N$. Therefore $0 \neq am \in [N_M: (r)]$. But $[N_M: (r)]$ is a weakly 2-prime sub-module, we get either $m \in [N_M: (r)]$ or $a^2 \in [[N_R: (r)]\ _R: M]$. If $m \in [N_M: (r)]$, take $r = 1$. Then
$m \in N$. And if $a^2 \in [[N_R: (r)]\ _R: M] = [a^2: M]$. Therefore N is a sub-module of M that is weakly 2-prime.

Using Theorem 2.3, we get the following conclusion:

Proposition 2.8:

N is a weakly 2-prime R-sub-module of M only if and only if N is a weakly 2-prime R/I-sub-module of M, where $I \subseteq annN$.

Proof:

$\Rightarrow$) If $(r + I) \in R/I$ and $m \in M$ and let $I \neq (r + I)x \in N$, so $I \neq rm + I \in N$, i.e. $rm \notin I$. Thus $0 \neq rm \in N$. However, N is a R-sub-module that is weakly 2-prime, therefore, $m \in N$ or $r^2 \in [N_R: M]$, must hold $r^2M \subseteq N$ and hence $(r^2 + I)M \subseteq N$ (given that $I \subseteq annN$). Therefore, $(r^2 + I) \in \left[N_{R/I}: M\right]$, i.e. N is a weakly 2-prime R/I-submodule.
$\Longleftarrow$) Clear

Correct standard Weakly 2-prime ideal P of R is an ideal such that for any a,b in R, $0 \neq ab \in P$ implies $a^2 \in P$ or $b^2 \in P$, [8].

Proposition 2.9:

For every R-module M, if N is a weakly 2-prime R-sub-module of M, then $[N_{\bar{R}}: M]$ is a weakly 2-prime ideal of $\overline{R}$, where $\overline{R} = R/annM$.

Proof:

Where N is a weakly 2-prime R-sub-module, this implies that N is a weakly 2-prime $\bar{R}$-submodule, by Proposition 2.8. However, because R is a faithful, we can prove that $[N_{\bar{R}}: M]$ is a weakly 2-pirme ideal of $\overline{R}$ by Proposition 2.5.
Recall that R-module M is a multiplication module if N=IM for any ideal I of R, see [9].

In the class of finitely generated of faithful and multiplication modules, we have the following:

Theorem 2.10:

Let M be a faithful finitely generated multiplication R-module, and let N be a proper sub-module of M. Thus, the following statements are equivalent.

1. N is a sub-module of M that is a weakly 2-prime;
2. The ideal $[N_R: M]$ of R is a weakly 2-prime ideal;
3. For a weakly 2-prime ideal I of R, N=IM.

Proof:

1 $\Longrightarrow$)2 By Proposition 2.5.
2$\Longrightarrow$)3 As $N = [N_R: M]M$ where $[N_R: M]$ is a weakly 2-prime ideal of R, this is self-evident.

3⟹)1 By (3) $N = IM$ and I is a weakly 2-prime ideal of R. Since $N = [N_R: M]M$, andd $I = [N_R: M]$, follows from [9, Theorem 3.1], M is a finitely produced faithful multiplication R-module. Now, set $r \in R$ amd $m \in M$ such that $0 \neq rm \in N$. But $(m) \leq M$, so that $(0) \neq rKM \subseteq N = [N_R: M]M$ and by [9, Theorem 3.1] $rK \subseteq [N_R: M]$. Moreover, $rK \neq (0)$. But $[N_R: M] = I$ which is a weakly 2-prime ideal, so either $r^2 \in [N_R: M]$ or $K \subseteq [N_R: M]$, that is either $r^2 \in [N_R: M]$ or $(m) = KM \subseteq N$. This means either $r^2 \in [N_R: M]$ or $m \in N$. As a result, N is a sub-module of M that is a weakly 2-prime.

Proposition 2.11:

Consider the sub-module N of an R-module M that is a weakly 2-prime then $[N_R: M]N = 0$ if and only if N is not a 2-prime.

Proof:

Assuming that$[N_R: M]N \neq 0$, we shall demonstrate that N is a weakly 2-prime sub-module. Let $rm \in N$. Suppose $rm \neq 0$, since N is a weakly 2-prime sub-module, so either $m \in N$ or $r^2 \in [N_R: M]$. Now, suppose $rm = 0$, first suppose $rN \neq 0$, so there exists $t \in N$, $0 \neq rt \in N$. Hence $0 \neq rt = r(m + t) \in N$. In other words, either $m + t \in N$ or $r^2 \in [N_R: M]$. Hence either $m \in N$ or $r^2 \in [N_R: M]$. Now, we can assume that $rN = 0$ and $[N_R: M]m = 0$. Since $[N_R: M]N \neq 0$, there exists $s \in [N_R: M]$ and $t \in N$ such that $0 \neq st \in N$. Then $(r + s)(m + t) = rm + sm + rt + st = 0 + 0 + 0 + st$. That is
$0 \neq (r + s)(m + t) = st \in N$. But N is a weakly 2-prime sub-module, so either $m + t \in N$ or $(r + s)^2 \in [N_R: M]$. Since $m \in N$ or $r^2 \in [N_R: M]$ then N is a 2-prime sub-module.

3. Some properties of weakly 2-prime sub-modules

In this section, we will give some basic results and properties for weakly 2-prime sub-modules.

Proposition 3.1:

Let $f: M \to M'$ be an R-epimorphism, and N is a weakly 2-prime sub-module of M containing $kerf$. Then $f(N)$ is a weakly 2-prime sub-module of M'.

Proof:

Let $0 \neq rm \in f(N)$, for same $r \in R, m \in M'$. Therefore, there is$x \in N$, the likes of which $0 \neq rm = f(x)$, since f is an R-epimorphism, we can write $m = f(x_1)$, for some $x_1 \in M$. Thus $f(rx_1 - x) = 0$ and so $rx_1 - x \in kerf \subseteq N$, we have $0 \neq rx_1 \in N$. If $x_1 \in N$ or $r^2 \in [N_R: M]$, then $m = f(x_1) \in f(N)$ or $r^2 \in [f(N)_R: M']$, because N is a sub-module of M that is weakly 2-prime. Therefore, f(N) is a sub-module of M' that is weakly 2-prime.

Proposition 3.2:

Let $f: M \to M'$ be an R-monomorphism, and let N' be a submodule of M' that is weakly 2-prime. Then $f^{-1}(N')$ is a sub-module of M that is weakly 2-prime.

Proof:

Let $0 \neq rm \in f^{-1}(N')$, for some $r \in R, m \in M$. Therefore, there is$x \in N$, such that $0 \neq rm = f(x)$, since f is an R-monomorphism, then $0 \neq f(rx_1 - x) \subseteq N'$ for some $x_1 \in M$. Thus $0 \neq f(r)f(x_1 - x) \in N'$. Since N' is a submodule of M' that is weakly 2-prime, we get either $f(x_1 - x) \in N'$ or $f(r)^2 \in [N'_R: M']$ and thus yields $m = x_1 - x \in f^{-1}(N')$ or $r^2 \in [f^{-1}(N')_R: M]$. Accordingly, $f^{-1}(N')$ is a sub-module of M that is weakly 2-prime.

Proposition 3.3:

If N is a submodule of M that is weakly 2-prime and contains another submodule of M, K, then $N/K$is a sub-module of M/K that is weakly 2-prime.

Proof:

Consider the epimorphism $\pi: M \to M/K$, which is define as $\pi(m) = m + K$, for every $m \in M$. Also, keep in mind that $Ker\pi = K \subseteq N$. By Proposition 3.1 N/K is a sub-module of M/K that is a weakly 2-prime.

Proposition 3.4:
Let $K \subseteq N$ be two are sub-modules of M. If N/K is a sub-module of M/K that is weakly 2-prime and K is a 2-prime sub-module of M. Then N is a 2-prime sub-module of M.
Proof:
Let $rm \in N$ for some $r \in R, m \in M$. If $rm \in K$, it follows that $m \in K$ or $r^2 \in [K_R: M]$. Since K is a sub-module of M with 2-prime. Now let's say that $rm \notin K$, this implies that $0_{M/K} \neq (r + K)(m + K) \in N/K$. As N/K is a sub-module of M/K that is a weakly 2-prime, we get either $(m + K) \in N/K$ or $(r + K)^2 = r^2 + K \in [N/K\ _R: M/K]$, which implies that $m \in N$ or $r^2 \in [N_R: M]$. As a result, N is a sub-module of M which is a 2-prime.

Proposition 3.5:
Le t$K \subseteq N$ be two are sub-modules of M. If N/K is a sub-module of M/K that is a weakly 2-prime, and K is a sub-module of M that is a weakly 2-prime. Then N is a sub-module of M which is a weakly 2-prime.
Proof:
It is similar to Proposition 3.4.

Proposition 3.6:
Let N be a sub-module of M that is a weakly 2-prime, and let M'be a subring of M with $M' \subseteq N$. Then $N \cap M'$ is a sub-module of M' that is a weakly 2-prime.
Proof:
Consider the monomorphism $i: M' \to M$, defined as $i(m) = m$, for any $m \in M'$. Since N is a submodule of M that is a weakly 2-prime, by Proposition 3.2 $i(N) = N \cap M'$is a sub-module of M' that is weakly 2-prime .

Proposition 3.7:
Let M be an R-module. A sub-module N of M is a weakly 2-prime if and only if $[N_R: I]$ is a weakly 2-prime, for every ideal I of R.

Proof:
Let $0 \neq rm \in [N_M: I]$, where $r \in R, m \in M$ and I be any ideal of R. Then $0 \neq arm \in N$, for all $a \in I$. However, as N is a weakly 2-prime sub-module of$M$, we get $am \in N$ or $r^2 \in [N_R: M]$. Therefore, either $m \in [N_R: I]$ or $r^2M \subseteq N$. But $N \subseteq [N_R: I]$ and hence $r^2M \subseteq [N_R: I]$. It is follows that $r^2 \in [[N_R: I]\ _R: M]$. And hence for any ideal I of R, $[N_R: I]$ is a weakly 2-prime sub-module.

Proposition 3.8:
Let N be a weakly 2-prime R-sub-module of M, and S be a multiplicative subset of R with $[N_R: M] \cap S = \emptyset$. Then N_s is a weakly 2-prime R_s-sub-module of M_s .

Proof:
Let$0 \neq \frac{a}{b}\frac{m}{c} \in N_S$, where $\frac{a}{b} \in R_s, \frac{m}{c} \in M_s$. Hence $0 \neq \frac{am}{bc} \in N_S$ and so there exists $y \in N \in$ and $d \in S$ such that $\frac{am}{bc} = \frac{y}{d}$ and suggests the presence of $t \in S$ so $tadm = tbcy$. On the other hand, $\frac{am}{bc} \neq \frac{0}{1} = 0_s$, which implies that $fam \neq 0$ for all $f \in S$. Hence $0 \neq tadm \in N$. Nonetheless, N is an R-sub-module of M that is a weakly 2-prime, we get either $tdm \in N$ or

$a^2 \in [N_R: M]$ and hence either $\frac{tdm}{tdc} \in N_S$ or $\frac{a^2}{b^2} \in [N_R: M]_S$. Because $[N_R: M]_S \subseteq \left[N_{S_{R_S}}: M_S\right]$, we have either $\frac{m}{c} \in N_S$ or $\frac{a^2}{b^2} \in \left[N_{S_{R_S}}: M_S\right]$. As a result,
N_s is a R_s-submodule of M_s that is a weakly 2-prime.

Theorem 3.9:
Let us assume that A and B are two different modules, and that N is a valid sub-module of M. Then, $W = N \oplus B$ is a sub-module of $M = A \oplus B$ that is weakly 2-prime if and only if N is a sub-module of A that is weakly 2-prime, and for $r \in R, m \in A$ with $rm = 0, m \notin N$, $r^2 \notin [N_R: A]$.

Proof:
⟹) Let $m \in A, r \in R$, such that $0 \neq rm \in N$. Then, $(0,0) \neq r(m, 0) \in W$. However, W is a sub-module that is weakly 2-prime. We get either $(m, 0) \in W$ or $r^2 \in [W_R: M]$. Thus either $m \in N$ or $r^2 \in [N_R: M]$, so that N is a sub-module that is weakly 2-prime. Now if $r \in R, m \in A$ such that $rm = 0, m \notin N$, $r^2 \notin [N_R: A]$. Assume that $r \notin annB$, so there exists $a \in B$ such that $ra \neq 0$. Thus $r(m, a) = (rm, ra) = (0, ra) \neq (0,0)$. Hence $(0,0) \neq r(m, a) \in N \oplus B$ =W. Since W is a sub-module weakly 2-prime, we get either $(m, a) \in N \oplus B$ or $r^2 \in [N \oplus B_R: A \oplus B\ ]$. Thus either $m \in N$ or $r^2 \in [N_R: A]$. Which is a contradiction with hypothesis.
⟹) Let $r \in R, (m, a) \in M$. Assume $(0,0) \neq r(m, a) \in N \oplus B$, so if $rm \neq 0$. Thus either $m \in N$ or $r^2 \in [N_R: A]$, since N is a 2-prime sub-module, it is weakly, we obtain either $(m, a) \in N \oplus B$ or $r^2 \in [N \oplus B_R: A \oplus B\ ]$. If $rm = 0$. Suppose that $m \notin N$, $r^2 \notin [N_R: A]$, then by hypothesis $r \in annB$ and so $r(m, a) = (0,0)$. That is an apparent contradiction. Thus either $m \in N, r^2 \in [N_R: A]$ and hence either $(m, a) \in N \oplus B$ or $r^2 \in [N \oplus B_R: A \oplus B\ ]$. Therefore, $W = N \oplus B$ is a sub-module of $M = A \oplus B$ that is weakly 2-prime

Theorem 3.10:
Let us assume that A and B are two different modules, and that N is a valid sub-module of$M$. Then, $W = N \oplus B$ is a sub-module of $M = A \oplus B$ that is weakly 2-prime if and only if N is a sub-module of A that is weakly 2-prime.

Corollary 3.11:
Let A, B be are two modules. If (0) is a sub-module of A with 2-prime, then $(0) \oplus B$ is a sub-module of $M = A \oplus B$ that is weakly 2-prime.

Proof:
Let $r \in R$, and $(a, b) \in A \oplus B$, such that If $(0,0) \neq (a, b) \in (0) \oplus B$, then $ra = 0$ and $rb \in B$. Since (0) is a 2-prime sub-module of A, then either $a = 0$ or $r^2 \in annA$.
Thus either $(a, b) = (0, b) \in (0) \oplus B$ or $r^2 \in [(0) \oplus B_R: A \oplus B\ ]$. Therefore $(0) \oplus B$ is a sub-module of $M = A \oplus B$ that is weakly 2-prime.

Proposition 3.12:
Let A and B be two different modules and let $N = U \oplus W$ be a weakly 2-prime sub-module in $M = A \oplus B$, then U, W are sub-modules of A and B that are weakly 2-prime.

Proof:
The proof is a straight forward, so it is omitted.

In general, the opposite of claim (3.12) is not true, as the following example shows: In the Z-modale $0, 2Z$ (0) are weakly 2-prime in Z-module Z (since there are weakly 2-prime in Z-

module Z and by commenting and illustrating (2.2.1), but $(0) \oplus 2Z$ is not weakly 2-pirme sub-module in the Z-module $Z \oplus Z$, since $(0,0) \neq 2(1,0) \in (0) \oplus 2Z$, but $(1,0) \notin (0) \oplus 2Z$ and$2^2 \in [(0) \oplus 2Z_R : Z \oplus Z\,] = (0)$.

As a generalization of Cohen theorem, the following was given in [10].
Let M be a finitely generated R-module, then M is noetherian if every prime sub-module is finitely generated.
The following holds because every weakly prime is a sub-module of a weakly 2-prime.

Proposition 3.13:
Let M be a finitely generated, then M is a noetherian if every weakly 2-prime is finitely generated.

Remark 3.14:
The requirement that M is a finitely generated, cannot be omitted from the previous Proposition 3.13, as the following example shows: In the Z-module $Z_{P\infty}$ is not finitely generated, also it is not noetherian. The zero sub-module which is clearly finitely generated is the only weakly 2-prime sub-module of $Z_{P\infty}$, $G = \langle \frac{1}{pi} + Z \rangle$ for some $i \in Z_+$ and $0 \neq (\frac{1}{pi+1} + Z) \in G$, but $p \in [G : Z_{P\infty}] = 0$, so $p^2 \notin [G : Z_{P\infty}]$, $\frac{1}{pi+1} + Z \notin G$, that is G is not weakly 2-prime sub-module.

In the following three results, we will assume that $R = R_1 \times R_2$ and $M = M_1 \times M_2$ be the R-module

Proposition 3.15:
If N is a proper R_1-sub-module of M_1 and M_2 is an R_2-module, the following statements are equivalent:
1. N is a 2-pirme R_1-sub-module of M_1;
2. $N \times M_2$ is a 2-prime R-sub-module of$M = M_1 \times M_2$.
3. $N \times M_2$ is a weakly 2-prime R-sub-module of $M = M_1 \times M_2$

Proof:
$1 \Longrightarrow)2$ Let $(r_1, r_2) \in R, (m_1, m_2) \in M_1 \times M_2$ such that $(r_1, r_2)(m_1, m_2) \in N \times M_2$, then $r_1 m_1 \in N$ and $r_2 m_2 \in M_2$. But $r_1 m_1 \in N$ and N is what's known as a 2-prime R-sub-module, so either $m_1 \in N$ or $r_1{}^2 \in [N_R : M_1]$. Hence either $(m_1, m_2) \in N \times M_2$ or $(r_1{}^2, r_2)(1, r_2) \in [N \times M_2 : M_1 \times M_2], (r_1{}^2, r_2{}^2) \in [N \times M_2 : M_1 \times M_2]$. Thus $N \times M_2$ constitutes a 2-prime R-sub-module of the module M.
$2 \Longrightarrow)3$ It is clear.
$3 \Longrightarrow)1$ To show that N is a 2-pirme R_1-sub-module of M_1. Let $r \in R_1$, $m \in M_1$ such that $rm \in N$. Thus for each $w \in M_2, a \neq 0$, $(0,0) \neq (r,1)(m,w) \in N \times M_2$. However, $N \times M_2$ is a R-sub-module of M that is a weakly 2-prime, so either $(r^2, 1) \in \left[N \times M_{2_R} : M_1 \times M_2\right]$ or $(m, w) \in N \times M_2$ and therefore either $r^2 \in [N_R : M_1]$ or $m \in N$, that is N is a 2-prime R_1-sub-module of M_1.
To a similar extent, we have

Proposition 3.16:
If N is a proper R_2-sub-module o f M_2, the following statements are similar.
1. N is a 2-pirme R_2-sub-module of M_2.
2. $M_1 \times N$ is a 2-prime R-sub-module of $M = M_1 \times M_2$.
3. $M_1 \times N$ is a weakly 2-prime R-sub-module of $M = M_1 \times M_2$.

Proposition 3.17:

Let M_1, M_2 represent the $R_1$and R_2-modules respectively. If $N = N_1 \times N_2$ is a weakly 2-prime R-sub-module of $M = M_1 \times M_2$, then either $N = 0$ or N is a 2-prime R-sub-module

Proof:

Assume $N \neq 0$, so either $N_1 \neq 0$ or $N_2 \neq 0$. Suppose that $N_2 \neq 0$, hence there exists $a \in N_2, a \neq 0$. Let $r \in \left[{N_1}_{R_1} : M_1\right]$ and let $m \in M_1$, then $(0,0) \neq (r,1)(m,a) = (rm,a) \in N_1 \times N_2 = N$. Since N is a weakly 2-prime R-sub-module of M, we get either $(m,a) \in N$ or $(r^2,1) \in [N_1 \times N_2 : M_1 \times M_2]$. Hence if $(m,a) \in N$, then $m \in N_1$ and so $M_1 = N_1$. Which implies that $N = M_1 \times N_2$. If $(r^2,1) \in [N_1 \times N_2 : M_1 \times M_2]$, then $M_2 = N_2$. Which implies that $N = N_1 \times M_2$. Hence by propodition (3.15), (3.16), N is a 2-prime R-sub-module of M.

Conclusions:

In this work, a generalization of a 2-prime sub-module has been introduced which is called a weakly 2-prime sub-module. We also show that if every sub-module of an R-module M is 2-prime sub-module, then M is called a weakly 2-prime sub-module. Moreover, many results and properties of this concept are given and discussed.

References

[1] S. A. Saymach, "On prime submodules", *University Noc. Tucumare Ser. A,* vol. 29**,** pp.121-136, 1979.

[2] C. P. Lu, "Prime submodule of modules", *Comment. Math. Univ. St, Paul*, vol.33, pp. 61-69, 1984.

[3] F. D. Jasem and A. A. Elewi, "2-prime submodule of modules," *Iraqi* journal of science, vol. 36, no. 8, pp. 3605-3611, 2022.

[4] S. E. Atani and F. Farzalipour, *Georgien Mathematical Journal*, vol. 12, pp.1-7, 2005.

[5] I. M. A. Hadi, "On weakly prime submodules" *Ibn AL-Haitham J. For Pure&Appl. Sci*, vol. 22 , no. 3, 2009.

[6] M. A. Hassin, "Quasi-prime modules and Quasi-prime submodules" M.Sc Thesis, Univ. of Baghdad, 1999.

[7] F. Kasch, "*Modules and Rings" Acad. Press, London*, 1982.

[8] S. Koc, "On weakly 2-prime ideals in commutative rings" *Communications in Algebra*, Dol: 10.1080|00927872.1897133, 2021.

[9] Z. A. El-Bast and P. F. Smith, "Multipliction modules" Comm. In algebra, 16:755-779, 1988.

[10] E. A. Athab, "Prime and semiprime submodules" MS.c Thesis, Univ. of Baghdad 1996.