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Small Pointwise M-Projective Modules

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Abstract

Let R be a ring and let M be a left R-module. In this paper introduce a small pointwise M-projective module as generalization of small M- projective module, also introduce the notation of small pointwise projective cover and study their basic properties.

Keywords: projective, M- projective, small M- projective.

المقاسات الاسقاطية الصغيرة نقطيا نسبة لمقاس M

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الخلاصة

لتكن R حلقة ولتكن M مقاس ايسر معرف على R .قدمت في هذا البحث مفهوم المقاسات الاسقاطية الصغيرة نقطيا نسبة لمقاس M بصفته تعميما لمفهوم المقاسات الاسقاطية الصغيرة نسبة لمقاس M . كذلك قدمت مفهوم غطاء المقاسات الاسقاطية الصغيرة نقطيا. ودرست بعض الخواص الاساسية.

الكلمات المفتاحية: المقاسات الاسقاطية – المقاسات الاسقاطية نسبة لمقاس M – المقاسات الاسقاطية الصغيرة نسبة لمقاس M

Introduction

Let R be a ring and M be a left R-module. A submodule N of an R-module M is called small submodule of M if N+L = M for any submodule L of M implies L = M [1]. An epimorphism $g:A\rightarrow B$ is called small provided ker g is small submodule in B [2].An R-module M is called small projective module if for each small epimorphism $g:A\rightarrow B$ where A and B are any R-modules and for each homomorphism $f:M\rightarrow B$ there exists a homomorphism $h:M\rightarrow A$ such that $g \circ h = f$ [2].Let U and M be modules, then U is M-projective if for each epimorphism $g:M\rightarrow N$ and each homomorphism $f:U\rightarrow N$,there exists a homomorphism $h:U\rightarrow M$ such that $g\circ h = f$ [3].Let M and N be modules. N is called pointwise M-projective, if for every epimorphism $g:M\rightarrow B$ and any homomorphism $f:N\rightarrow B$,then for every $m\in M$ there exists a homomorphism $h:N\rightarrow M$, such that $g\circ h(m) = f(m)$.Let M and N be modules. N is called small M-projective, if for every small epimorphism $g:M\rightarrow B$ and any homomorphism $f:N\rightarrow B$,there exists a homomorphism $h:N\rightarrow M$, such that $g\circ h = f$. In this paper introduce the concept of small pointwise M-projective module as follows: Let M and N be modules. N is called small pointwise M-projective, if for every small epimorphism $g:M\rightarrow B$ and any homomorphism $f:N\rightarrow B$, then for every $m\in M$ there exists a homomorphism $h:N\rightarrow M$,such that $g\circ h(m) = f(m)$.A submodule V of M is called supplemented of a submodule U of M if V is a minimal element in the set of submodules L of M with U+L = M [1].An R-module M is called supplemented if for every submodule of M has supplemented in M [1]. An R-module M is called small pointwise projective module if for each small epimorphism $g:A\rightarrow B$ where A and B are any R-modules and for each

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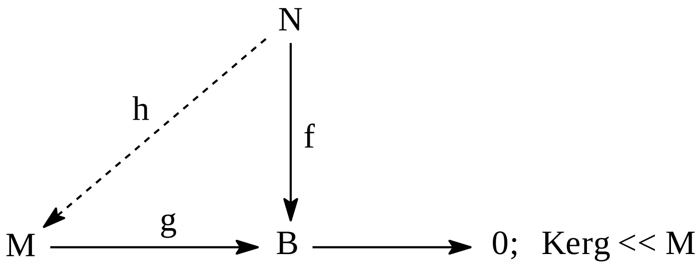
homomorphism f:M→B then for every m∈M there exists a homomorphism h:M→A such that g ∘ h(m) = f(m). A pair (P,f) is a small pointwise projective cover for a module M, if there exists a small epimorphism from P onto M, where P is a small pointwise projective module. Finally, introduce the concept of a small pointwise projective cover as generalization of small projective cover and give some properties of this notion.

1. Some Characterization of Small pointwise M-Projective Modules

In this section, introduce the concept of small pointwise M-projective modules and give some characterizations of small pointwise M-projective modules. Let start with the following:

Let M and N be modules. N is called small pointwise M-projective, if for every small epimorphism g:M→B and any homomorphism f:N→B, then for every m∈M there exists a homomorphism h:N→M, such that g∘h(m) = f(m) (i.e.) the following diagram is commutative:



Every small M-projective module is small pointwise M-projective module since every M-projective module is a pointwise M-projective module.

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Every pointwise M-projective module is small pointwise M-projective module since every M-projective module is small M-projective module.

A module M is self-projective if M is M-projective [3].

A module M is called self-small pointwise projective, if M is small pointwise M-projective.

A module M is called self-pointwise projective, if M is pointwise M-projective.

Every self-small projective module is self-small pointwise projective module.

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Every self-pointwise projective module is self-small pointwise projective module.

The following proposition gives a characterization for small pointwise M-projective module.

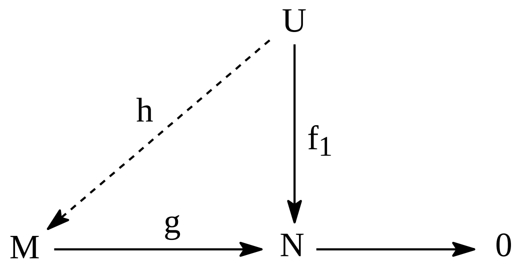
Proposition (1.1) Let U and M be modules, the following are equivalent

1. U is a small pointwise M-projective module;

2. For every small short exact sequence with middle term $0 \longrightarrow K \xrightarrow{f} M \xrightarrow{g} N \longrightarrow 0$, the sequence $0 \longrightarrow \mathrm{Hom}(U,K) \xrightarrow{\mathrm{Hom}(1,f)} \mathrm{Hom}(U,M) \xrightarrow{\mathrm{Hom}(1,g)} \mathrm{Hom}(U,N) \longrightarrow 0$ is short exact;

3. For every small submodule K of M, every homomorphism h:U→ $\frac{M}{k}$, $\frac{M}{k}$ factor through the natural epimorphism π:M→ $\frac{M}{k}$

<u>**Proof:**</u> **(1⇒2)** It is enough to show that, Hom(I,g) is an epimorphism.Let f_1∈Hom(U,N) and consider the following diagram:



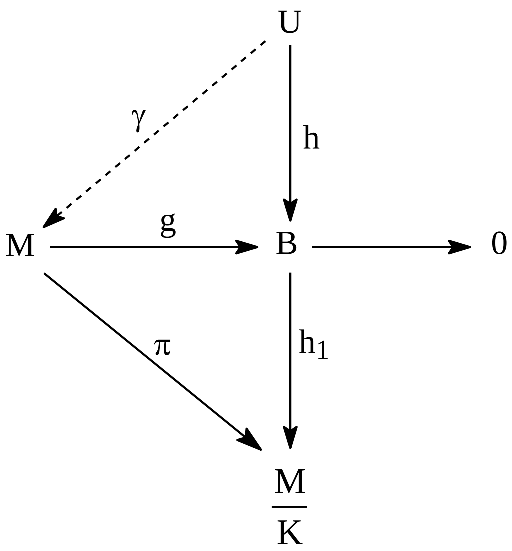
Since g is a small epimorphism and U is a small pointwise M-projective module then for every m∈M there exists a homomorphism h:U→M such that g∘h(m) = f_1(m).Now,(Hom(I,g)(h))(m) = g∘h(m) = f_1(m) and hence Hom(I,g)(h) = g∘h = f_1.

(2⇒3) Let K be a small submodule of M and let h:U→ $\frac{M}{k}$ be an epimorphism. Consider the following small short exact sequence

$$0 \longrightarrow K \xrightarrow{\ i\ } M \xrightarrow{\ \pi\ } \frac{M}{K} \longrightarrow 0$$

where i is the inclusion homomorphism and π is the natural epimorphism.By(2), the homomorphism Hom(I,π):Hom(U,M)→Hom(U,$\frac{M}{k}$) is an epimorphism. This implies, the existence of a homomorphism f∈Hom(U,M) such that h = Hom(I,π)(f) = π∘f.

(3⇒1) Let g:M→B be a small epimorphism and let h:U→B be any homomorphism. Consider the following diagram:

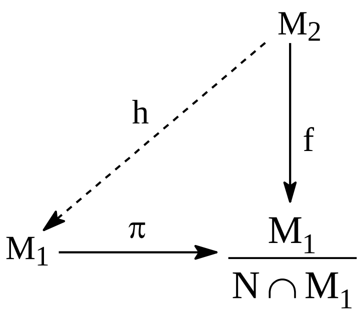


Where K = Ker g , π:M→ $\frac{M}{k}$ is the natural epimorphism and h_1:B→ $\frac{M}{k}$ be the usual isomorphism.By (3), there exists a homomorphism γ:U→M such that π∘γ = h_1∘h.One can check easily that h_1∘g(m) = π(m).Now, h_1∘g∘γ(m) = π∘γ(m) = h_1∘h(m).Thus g∘γ(m) = h(m), since h_1 is an isomorphism.

Proposition (1.2) Let M_1 and M_2 be modules, with M = $M_1 \oplus M_2$, then the following conditions are equivalent:

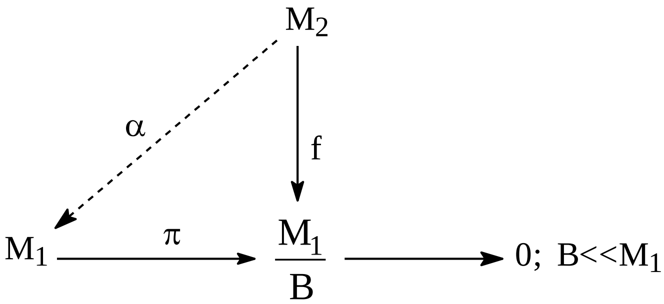
1. M_2 is a small pointwise M_1-projective;
2. For any submodule N of M, such that M_1 is a supplemented of N in M, there exists a submodule N_1 of N such that M = $M_1 \oplus N_1$.

<u>Proof:</u> (1⇒2) Let M_1 be a supplemented of a submodule N of M, then M = N+M_1 with N∩M_1<<M_1.Let π:M_1→ $\frac{M1}{N \cap M1}$ be the natural epimorphism. Define f:M_2→ $\frac{M1}{N \cap M1}$ by f(x) = y+N∩M_1,for all x∈ M_2 ,we have x = y+n, for some y∈M_1 and n∈N.Cleary f is well-defined and a homomorphism. Consider the following diagram:



Since M_2 is a small pointwise M_1-projective, then for every $m\in M_2$ there exists a homomorphism $h:M_2\to M_1$, such that $\pi\circ h(m) = f(m)$.Define $N_1 =\{y-h(y):y\in M_2\}$.We claim that $N_1\leq N$. Let $x\in N_1$,then $x = w-h(w)$,for some $w\in M_2$. Now, $\pi h(w) = f(w)$. Since $M = N+M_1$ and $w\in M_2$,then $w = n+v$ for some $n\in N$ and $v\in M_1$.But $h(w)+N\cap M_1 = f(w) = v+N\cap M_1$.This implies that $h(w)-v\in N$ and thus $w-h(w)\in N$, i.e., $x\in N$.It is easy to show that $M = M_1+N_1$. Let $w\in M_1\cap N_1$,so $w = y-h(y)$ for some $y\in M_2$.Thus $w+h(y) = y = 0$.Therefore $w = 0$.Hence $M = M_1\oplus N_1$.

(2⇒1) Let $\pi:M_1\to \frac{M1}{B}$ be the natural epimorphism, where $B<<M_1$ and $f:M_2\to \frac{M1}{B}$ be any homomorphism.Define $N =\{x-y:f(x) = \pi(y)$,where $x\in M_2,y\in M_1\}$.It is clear that $M = M_1+N$. Claim that $N\cap M_1\leq B$.Let $w\in N\cap M_1$, so $w\in N$ and hence $w = m_2-m_1$,for some $m_2\in M_2,m_1\in M_1$,where $f(m_2) = \pi(m_1)$.Thus $w+m_1 = m_2 = 0$,since $M = M_1\oplus M_2$.Therefore $\pi(m_1) = 0$ which implies that $m_1\in B$ and hence $w\in B$.But $B<<M_1$,thus $N\cap M_1<<M_1$.Thus M_1 is a supplemented of N in M. By (2),there exists a submodule N_1 of N such that $M = M_1\oplus N_1$. Define $\alpha:M_2 \to M_1$ by $\alpha(w) = v$,where $w = n+v$ for some $n\in N_1$ and $v\in M_1$. Clearly α is well-defined and a homomorphism.And make the following diagram commutative:



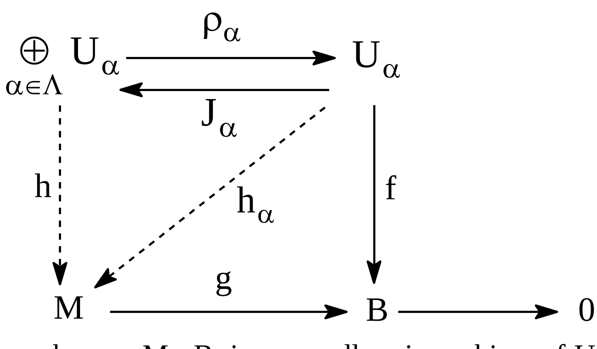
Let $w\in M_2$,then $w = n+v$, where $n\in N_1$ and $v\in M_1$,but $n\in N$,so $n = x-y$,where $f(x) = \pi(y)$.Hence $w = x-y+v$ which implies that $w-x = v-y\in M_1\cap M_2 = 0$. Thus $w = x$ and $v = y$.Therefore $\pi\alpha(w) = \pi(v) = \pi(y) = f(x) = f(w)$. Consequently M_2 is a small pointwise M_1-projective module.

2. Some Properties of Small Pointwise M-Projective Module

In this section, give some basic properties of small pointwise M-projective module. Start with the following:

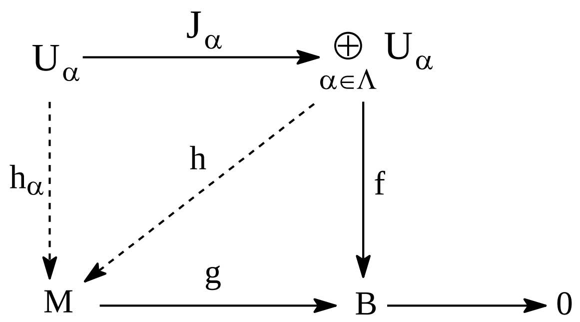
Proposition (2.1) Let M be a module and $\{U_\alpha\}_{\alpha\in\Lambda}$ be an indexed set of modules. Then $\bigoplus_{\alpha\in\Lambda}U_\alpha$ is a small pointwise M-projective if and only if every U_α is a small pointwise M-projective.

Proof:(⇒) Let $\bigoplus_{\alpha\in\Lambda}U_\alpha$ be a small pointwise M-projective and let $\alpha\in\Lambda$. Consider the following diagram:



where $g:M\to B$ is a small epimorphism, $f:U_\alpha\to B$ is any homomorphism, P_α and J_α are the projections and the injection homomorphisms,respectively. Since $\bigoplus_{\alpha\in\Lambda}U_\alpha$ is small pointwise M-projective,then for every $\psi\in\bigoplus_{\alpha\in\Lambda}U_\alpha$, there exists a homomorphism $h:\bigoplus_{\alpha\in\Lambda}U_\alpha\to M$ such that $g\circ h(\psi) = f\circ P_\alpha(\psi)$.Let $h_\alpha= h\circ J_\alpha:U_\alpha\to M$.Now,for every $m\in M$,$g\circ h_\alpha(m) = g\circ h\circ J_\alpha(m) = f\circ\rho_\alpha\circ J_\alpha(m) = f\circ I(m) = f(m)$.

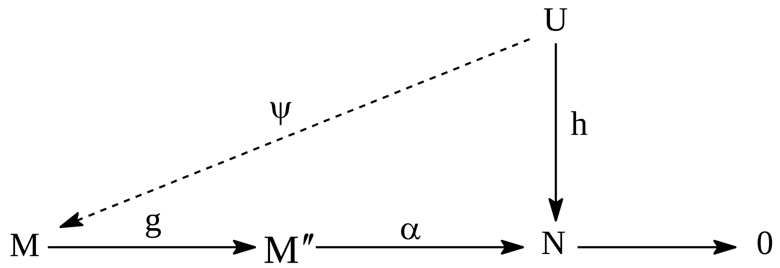
(⇐) Let $g:M\to B$ be a small epimorphism and let $f:\bigoplus_{\alpha\in\Lambda}U_\alpha\to B$ be any homomorphism.For each $\alpha\in\Lambda$, consider the following diagram:



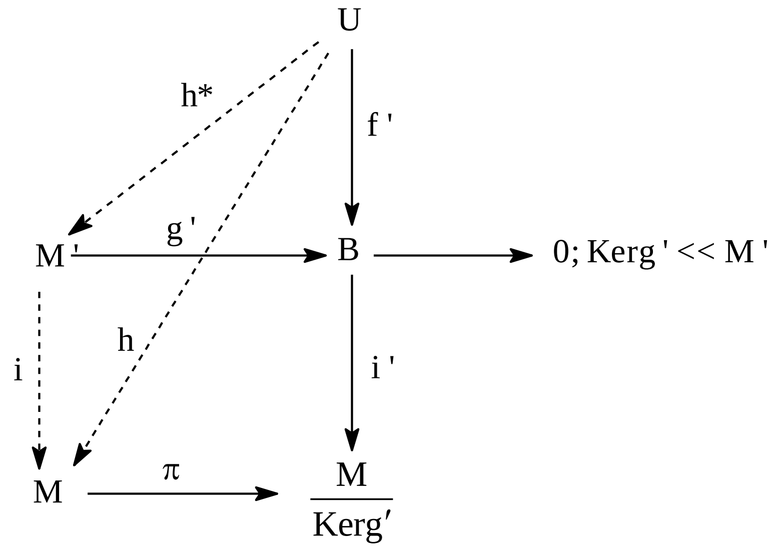
Where $J_\alpha:U_\alpha\to\bigoplus_{\alpha\in\Lambda}U_\alpha$ is the injection homomorphism. Since U_α is small pointwise M-projective, for each $\alpha\in\Lambda$, for every $m\in U_\alpha$, there exists a homomorphism $h_\alpha:U_\alpha\to M$, such that $g\circ h_\alpha(m) = f\circ J_\alpha(m)$; for each $\alpha\in\Lambda$. Define $h: \bigoplus_{\alpha\in\Lambda}U_\alpha \to M$ by $h(\psi) = \sum_{\alpha\in\Lambda} h_\alpha(\psi(\alpha))$.Clearly h is well defined and a homomorphism. Also,$(g\circ h)(\psi) = g(h(\psi)) = g(\sum_{\alpha\in\Lambda} h_\alpha(\psi(\alpha))) = \sum_{\alpha\in\Lambda}(g\circ h_\alpha)(\psi(\alpha)) = \sum_{\alpha\in\Lambda}(f\circ j_\alpha)(\psi(\alpha)) = f(\sum_{\alpha\in\Lambda} j_\alpha(\psi(\alpha))) = f(\psi)$. Hence $\bigoplus_{\alpha\in\Lambda}U_\alpha$ is a small pointwise M-projective module.

Proposition(2.2) Let M and U be modules. If $0\longrightarrow M'\xrightarrow{f}M\xrightarrow{g}M''\longrightarrow 0$ is a small short exact sequence and U is a small pointwise M-projective, then U is small pointwise M' and M''-projective.

Proof: To show that U is a small pointwise M''-projective, let $\alpha:M''\to N$ be an small epimorphism and let $h:U\to N$ be any homomorphism. Now, consider the following diagram:



By [1,proposition(19.3)] $\alpha\circ g$ is a small epimorphism and since U a small pointwise M-projective,for every $u\in U$ there exists a homomorphism $\psi:U\to M$ such that $\alpha\circ g\circ\psi(u) = h(u)$,i.e.,$g\circ\psi$ is the required homomorphism. To show that U is a small pointwise M'-projective, let $g':M'\to B$ be a small epimorphism and let $f':U\to B$ be homomorphism.Consider the following diagram:



where i is the inclusion homomorphism and π is the natural epimorphism. Define $i':B\to \frac{M}{\mathrm{Ker}\, g'}$ by $i'(b) = a+\mathrm{Ker}\, g'$ for all $b\in B$, where $b = g'(a)$,for some $a\in M'$.Since U is a small pointwise M-projective module,for every $u\in U$, there exists a homomorphism $h:U\to M$ such that $\pi\circ h(u) = i'\circ f'(u)$. Claim that $h(U)\le M'$.Let $w\in h(U)$,then there exists $u_1\in U$ such that $w = h(u_1)$.Now, $\pi\circ h(u_1) = i'\circ f'(u_1) = i'\circ g'(a)$ for some $a\in M'$. Hence $\pi h(u_1) = a+\text{Ker } g'$ and therefore $a_1-h(u_1)\in \text{Ker } g'\leq M'$.Thus $h(u_1)\in M'$ and consequently $h(U)\leq M'$. Define $h^*:U\to M'$ by $h(x) = h^*(x)$,for all $x\in U$.Now, $i'\circ g'\circ h^*(u) = \pi\circ i\circ h^*(u) = \pi\circ h^*(u) = \pi\circ h(u) = i'\circ f'(u)$.Since i' is a monomorphism, $g'\circ h^*(u) = f'(u)$. Hence U is a small pointwise M'-projective module.

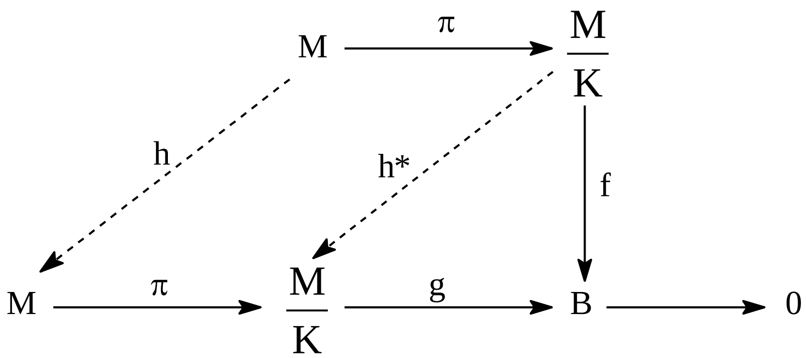
Corollary (2.3) Let $M = M_1\oplus M_2$, where M_1 and M_2 are modules. If M is a small pointwise projective module, then M_1 is a self-small pointwise projective and M_2 is a self-small pointwise projective and also M_1 is a small pointwise M_2-projective and M_2 is a small pointwise M_1-projective.

<u>Proof:</u> Since $M = M_1\oplus M_2$ is a small pointwise projective module,so $M_1\oplus M_2$ is a small pointwise M_1-projective and $M_1\oplus M_2$, is a small pointwise M_2-projective.By proposition (2.1), get the results.

Let M be an R- module. A submodule N of M is called fully invariant, if $f(N)\leq N$, for each $f\in End(M)$ [4,p.172].

Proposition (2.4) Let M be a small pointwise projective module and let K be a fully invariant submodule of M. If $K<<M$, then $\frac{M}{k}$ is a self-small pointwise projective module.

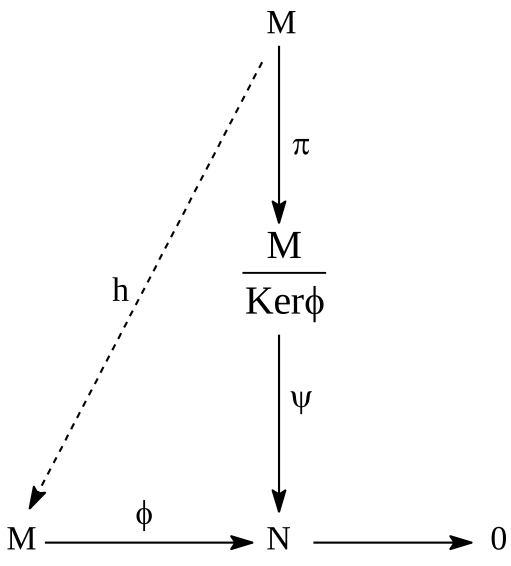
<u>Proof:</u> Consider the following diagram:



where $g:\frac{M}{k}\to B$ is a small epimorphism,$f:\frac{M}{k}\to B$ is any homomorphism and $\pi:M\to\frac{M}{k}$ is the natural epimorphism.By small pointwise projectivity of M, and $g\circ\pi$ is small epimorphism,for every $m\in M$ there exists a homomorphism $h:M\to M$,such that $g\circ\pi\circ h(m) = f\circ\pi(m)$.Define $h^*:\frac{M}{k}\to\frac{M}{k}$ by $h^*(m+K) = h(m)+K$ for all $m\in M$. To show that h^* is well defined.Let $m_1+K = m_2+K$, which implies that $m_1-m_2\in K$ and since K is a fully invariant submodule, thus $h(m_1-m_2)\in h(K)\leq K$. Hence $h(m_1)+K = h(m_2)+K$. Clearly h^* a homomorphism. Now, $g\circ h^*(m+K) = g\circ\pi\circ h(m) = f\circ\pi(m) = f(m+K)$.

Proposition (2.5) Let M be a self-small pointwise projective module and let $\phi:M\to N$ be a small epimorphism, then there exists a homomorphism $h\in End(M)$ such that $\text{Ker}\phi$ is invariant under h.

<u>Proof:</u> Consider the following diagram:

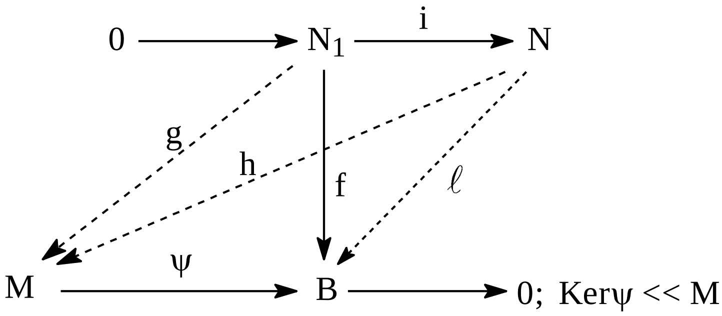


Where $\psi: \frac{M}{Ker\phi} \to N$ is the usual isomorphism defined by $\psi(m+Ker(\phi)) = \phi(m)$ for all $m \in M$ and π is the natural epimorphism. Since M is self-small pointwise projective module,for every $m_1 \in M$. there exists a homomorphism $h:M \to M$ such that $\phi \circ h(m_1) = \psi \circ \pi(m_1)$.Now,to show that $h(Ker\phi) \leq Ker\phi$,let $w \in Ker\phi$,then $\phi h(w) = \psi(w+Ker(\phi)) = \phi(w) = 0$ and hence $h(w) \in Ker\phi$.This implies that $Ker\phi$ is invariant under h.

A submodule K of an R-module M is M-cyclic submodule of M, if it isomorphic to M/X, for some submodule X of M [5]. The following proposition gives a condition under which a module M is N-injective.

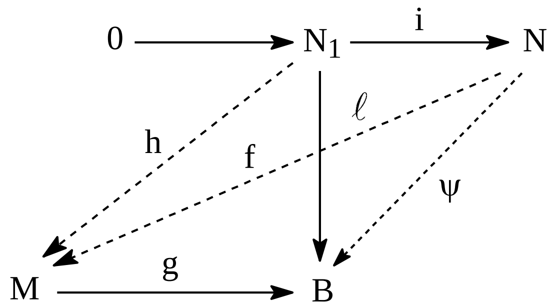
Proposition(2.6) Let M,N be modules.If N is a small pointwise M-projective and every M-cyclic submodule of M is N-injective,then M is N-injective and every submodule of N is a small pointwise M-projective.

Proof: Let N be a small pointwise M-projective and suppose that every M-cyclic submodule is N-injective. Since M is trivially M-cyclic,then M is N-injective. Let $\psi:M \to B$ be a small epimorphism and let $f:N_1 \to B$ be any homomorphism,where N_1 is a submodule of N.Consider the following diagram:



where $i:N_1 \to N$ is the inclusion homomorphism. Since B is M-cyclic module, thus by our hypothesis B is N-injective module. Therefore, there exists a homomorphism $\ell:N \to B$ such that $\ell \circ i = f$. But N is a small pointwise M-projective module, so for every $n \in N$ there exists a homomorphism $h:N \to M$ such that $\psi \circ h(n) = \ell(n)$. Define $g:N_1 \to M$ by $g = h \circ i$. Now, $\psi \circ g(n) = \psi \circ h \circ i(n) = \ell \circ i(n) = f(n)$.
The converse holds if M is hollow.

Suppose that M is N-injective and every submodule of N is a small pointwise M-projective.Thus N is a small pointwise M-projective module. Let B be M-cyclic submodule of M. Consider the following diagram:

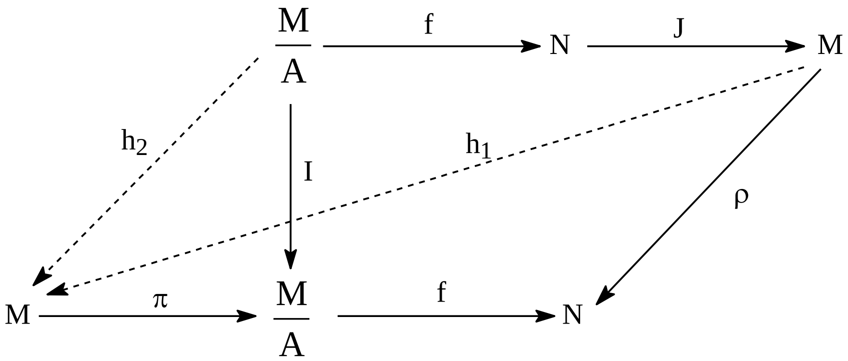


where $i:N_1 \to N$ is the inclusion homomorphism and $f:N_1 \to B$ is any homomorphism and $g:M \to B$ is the required epimorphism onto B, since B is M-cyclic module. In fact g is a small epimorphism, since M is hollow. By our assumption, N_1 is a small pointwise M-projective module. Thus, for every $n_1 \in N_1$, there exists a homomorphism $h:N_1 \to M$ such that $g \circ h(n_1) = f(n_1)$.But M is N-injective so,there exists a homomorphism $\ell:N \to M$ such that $\ell \circ i = h$. Define $\psi:N \to B$ by $g \circ \ell$. Now, $\psi \circ i = g \circ \ell \circ i = g \circ h = f$.

A sufficient condition for self-small pointwise projective module to be S.F, has been provided in the following.

Proposition (2.7) Let M be a self-small pointwise projective module and let $A \leq M$, then $A << M$ and $\frac{M}{A}$ is isomorphic to direct summand of M if and only if $A = 0$.

<u>Proof:</u> **(⇒)** Let $\pi:M\to \frac{M}{A}$ be the natural epimorphism, where $A<<M$ and $\frac{M}{A}$ is isomorphic to a direct summand N of M. Let $f: \frac{M}{A} \to N$ be an isomorphism. Consider the following diagram:



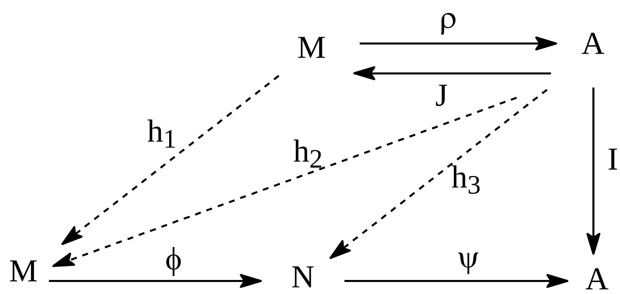
Where J and ρ are the injection homomorphism and the projection homomorphism respectively, and $I:\frac{M}{A} \to \frac{M}{A}$ is the identity.Since M is a self-small pointwise projective module, for every $m\in M$ there exists a homomorphism $h_1:M\to M$ such that $f\circ\pi\circ h_1(m) = \rho(m)$.Define $h_2:\frac{M}{A} \to M$ by $h_2 = h_1\circ J\circ f$.Now, $f\circ\pi\circ f_{2\ gc} = f\circ\pi\circ h_1\circ J\circ f = \rho\circ J\circ f = I\circ f$. Thus $f\circ\pi\circ h_2 = f$,which implies that $\pi\circ h_2 = I$.Since f is isomorphism.Therefore the sequence $\pi:M\to \frac{M}{A} \to 0$ splits and hence $A = 0$.

(⇐) Trivial. A module M is called a small cover for a module N, if there exists a small epimorphism $\phi:M\to N$ [6].

Proposition (2.8) A small cover of a small pointwise projective module is a small pointwise projective.

Proposition (2.9) Let M be a self-small pointwise projective module and let A be a direct summand of M. If M is a small cover of N and N is a small cover of A, then $M\cong A\cong N$.

<u>Proof:</u> Since M is a small cover of N and N is a small cover of A, where A is a direct summand of M, there exists a small epimorphisms $\phi:M\to N$ and $\psi:N \to A$. Consider the following diagram:



Where ρ, J are the projection and the injection homomorphisms respectively and $I:A\to A$ is the identity.Since M is a self-small pointwise projective module and $\psi\circ\phi$ is a small epimorphism, for every $m\in M$ there exists a homomorphism $h_1:M\to M$ such that $\psi\circ\phi\circ h_1(m) = \rho(m)$.Define $h_2:A\to M$ by $h_2 = h_1\circ J$.Also, define $h_3:A\to N$ by$h_3 = \phi\circ h_2$.Now,$\psi\circ h_3 = \psi\circ\phi\circ h_2 = \psi\circ\phi\circ h_1\circ J = \rho\circ J = I$.Thus, the small short exact sequence $N \xrightarrow{\psi} A \longrightarrow 0$ splits and hence $N\cong A$. Also $\psi\circ\phi\circ h_2 = \psi\circ\phi\circ h_1\circ J = \rho\circ J = I$.Hence the small short exact sequence $M \xrightarrow{\psi o\phi} A \longrightarrow 0$ splits and therefore $M\cong A$.Consequently $M\cong A\cong N$.

3. Small Pointwise Projective Cover

In this section, introduce the concept of a small pointwise projective cover and give some properties of this notion.

A pair (P,f) is a small pointwise projective cover for a module M,if there exists a small epimorphism from P onto M, where P is a small pointwise projective module.

A ring R is called Von-Neumann regular if for each $a\in R$, there exists $b\in R$ such that $a = a.b.a$ [1,3.1]

Example (3.1):

1. Z_6 is Von-Neumann regular ring.
2. The ring (P(X), Δ, ∩) is Von-Neumann regular ring, where P(X) is the power set of X and Δ is the symmetric difference and ∩ is the intersection.
3. The ring Z is not Von-Neumann regular

A ring R is called a Boolean ring, if for each a∈R, $a^2 = a$ [1, p.25(9)]

Remark (3.2)

1. Each Boolean ring is commutative.
2. Each Boolean ring is Von-Neumann regular.
3. Every subring and every factor ring of a Boolean ring is a Boolean ring.
4. For any index set Λ, the product $\prod_{\Lambda} R$ is a Boolean ring, where R is a Boolean ring.

A ring is called semisimple,if each module over R is a projective [7,17.4].

Example (3.3)

1. The ring Z_6 is semisimple.
2. The ring Z is not semisimple, since Q as Z-module is not projective module.

A ring R is called cosemisimple if Rad(M) = 0,for each R-module M [2].

The following proposition gives a characterization of cosemisimple ring.

Proposition (3.4) [1, 23.5(2)] A commutative ring is cosemisimple if and only if it is Von-Neumann regular.

Remark (3.5) Every semisimple ring is cosemisimple, but the converse is not true.

Example (3.6) Let R be the direct product of countably infinite many copies of Z_2 Clearly Z_2 is a Boolean ring, thus by remark (3.2), R is a Boolean ring and R is a Von-Neumann regular. By proposition (3.4) implies that R is a cosemisimple ring. Let I be the direct sum of countably infinite many copies of Z_2 inside of R , Claim that R is not semisimple ring. Clearly, $\frac{R}{I}$ is an R-module.Now, assume $\frac{R}{I}$ is a projective module.Consider the following short exact sequence:

$$0 \longrightarrow I \xrightarrow{\ i\ } R \xrightarrow{\ \pi\ } \frac{R}{I} \longrightarrow 0$$

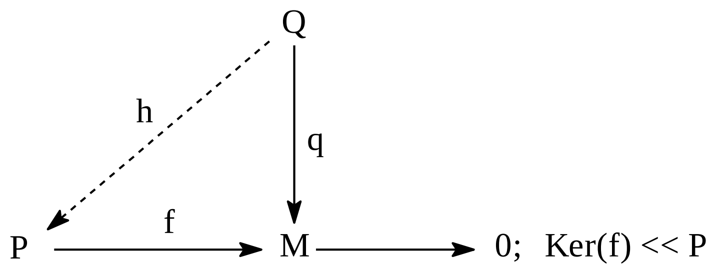
where i is the inclusion homomorphism and π is the natural epimorphism.By [6,17(1,2)(3)], this sequence splits,i.e.,R = I⊕K,where K≤R. Thus $\frac{R}{K} \cong I$. But $\frac{R}{K}$ is cyclic and hence I is cyclic. A contradiction.

Remark If a module has projective cover, then it has a small pointwise projective cover. But the converse is not true in general. See example (3.6).

Now, prove, if a module have a small pointwise projective cover, then it is unique up to isomorphism.

Proposition(3.7) Suppose that a module M has a small pointwise projective cover (P,f). If Q is a small pointwise projective module, with q:Q→M is a small epimorphism, then Q ≅ P.

__**Proof:**__ Let f:P→M be a small epimorphism, where P is a small pointwise projective module and let q:Q→M be a small epimorphism, where Q is a small pointwise projective module. Consider the following diagram:

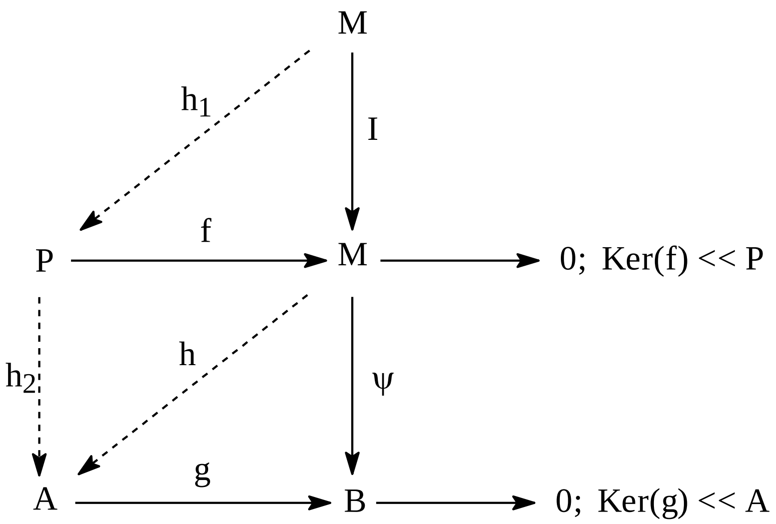


By small pointwise projectivity of Q,for every q∈Q, there exists a homomorphism h:Q→P,such that f∘h(q) = q(q).Claim that h is an epimorphism.To show that h is onto,it is enough to prove that P = Ker(f)+h(Q).Now,let x∈P,then f(x) = q(y) for some y∈Q so,f(x) = f(h(y)) and this implies that x-h(y)∈Kerf,i.e.,P = Ker(f)+h(Q),but Ker(f)<<P,thus h(Q) = P and h is onto.To prove h is

a monomorphism,let $w\in Ker(h)$,then $h(w) = 0$ and thus $fh(w) = q(w) = 0$.Which implies that $w\in Ker(q)$. Consequently $Ker(h)\leq Ker(q)<<Q$.Therefore $Ker(h)<<Q$ and hence $h:Q\to P$ is a small epimorphism. But P is a small pointwise projective module, so the sequence $Q\xrightarrow{h}P\longrightarrow 0$ splits.Thus h is a monomorphism.

Proposition (3.8) Let M be a module and let (P,f) be a small pointwise projective cover for M. If M is a small pointwise P-projective, then M is a small pointwise projective.

__Proof:__ Let $g:A\to B$ be a small epimorphism and $\psi:M\to B$ be any homomorphism. Consider the following diagram:



where $i:M\to M$ is the identity.Since P is a small pointwise projective module, for every $p\in P$, there exists a homomorphism $h_2:P\to A$,such that $g\circ h_2(p) = \psi\circ f(p)$.Also,since M is a small pointwise P-projective for every $m\in M$ there exists a homomorphism $h_1:M\to P$,such that $f\circ h_1(m) = I(m)$.Define $h = h_2\circ h_1$.Now, $g\circ h(m) = g\circ h_2\circ h_1(m) = \psi\circ f\circ h_1(m) = \psi\circ I(m) = \psi(m)$.

Let M be a module and let $S = End(M)$.Let N be a proper submodule of M.N is called an S-prime submodule of M, if whenever $f(m)\in N$, for some $f\in S$ and $m\in M$, then $f(M)\leq N$ or $m\in N$[8].

Proposition (3.9) Let N be an S-prime submodule of a small pointwise projective module M. Assume that, there exists $0\neq f\in S$ such that $f^{2} = f$,$f(N)\leq N$ and $f(N)<< M$. Then $\frac{M}{N}$ has a small pointwise projective cover.

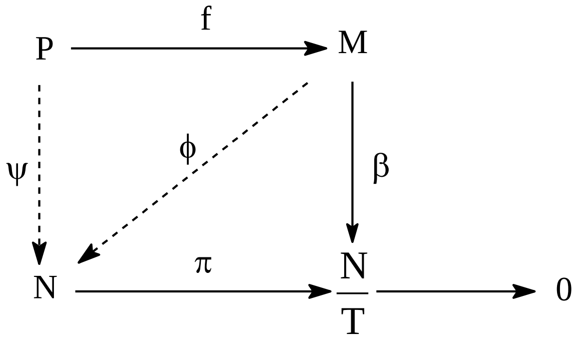
__Proof:__ Let N be an S-prime submodule of a small pointwise projective module M and $0 \neq f\in S$ such that $f^2 = f$, $f(N)\leq N$ and $f(N)<<M$. Since $f(1-f)(m) = 0\in N$ for all $m\in M$ and N is S-prime,so $f(M)\leq N$ or $(1-f)(m)\leq N$. Assume $(1-f)(m)\notin N$,for some $m\in M$,then $f(M)\leq N$ and hence $f(M) = f(N)$. But $M = f(M)\oplus(1-f)(M)$, thus $M = f(N)\oplus(1-f)(M)$ and therefore $(1-f)(M) = M$, since $f(N)<<M$. Hence $f = 0$ a contradiction. Consequently,$(1-f)(m)\in N$ for each $m\in M$. Define $h:f(M)\to \frac{M}{N}$ by $h(f(m)) = m+N$,for all $m\in M$.we have to show that h is well-defined.Let $f(m_1) = f(m_2)$, which implies that $f(m_1-m_2) = 0$, but $m_1-m_2 = (1-f)(m_1-m_2)\in N$. Therefore $m_1+N = m_2+N$. Clearly h is a homomorphism and onto. It is easy to show that $Ker(h) = f(N)<<M$,thus $f(f(N))<<f(M)$.Thus $f(N)<<f(M)$. Since $f(M)$ is a small pointwise projective module and h is a small epimorphism.Therefore $(f(M),h)$ is a small pointwise projective cover for $\frac{M}{N}$.

Now. need the following proposition to obtain a characterization of a module M to be small pointwise N-projective if M has a small pointwise projective cover.

Proposition(3.10) Let M be a small pointwise N-projective module and let (P,f) be a small pointwise cover of M.Then for every homomorphism $\psi:P\to N$, there exists a homomorphism $\phi:M\to N$ such that $\phi\circ f = \psi$.

__Proof:__ Let $f:P\to M$ be a small epimorphism and let $\psi:P\to N$ be any homomorphism. Let $T = \psi(Ker(f))$. By [6,5.18], $\psi(Ker(f))<<N$. Define $\beta:M\to \frac{N}{T}$ by $\beta(m) = \pi(\psi(x))$ for all $m\in M$,where $m = f(x)$ and $\pi:N\to \frac{N}{T}$ is the natural epimorphism.To show that β is well-defined.Suppose that $m = f(x) = f(y)$ for

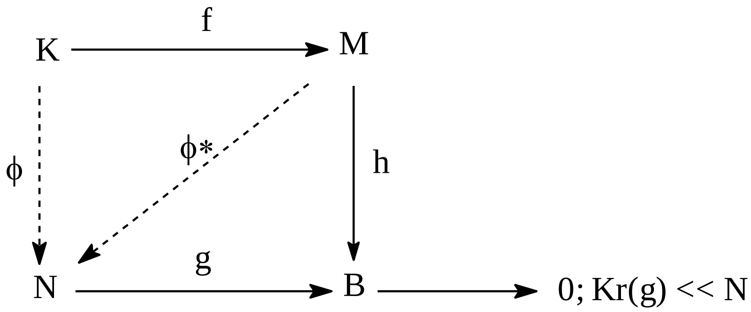
some $x,y\in P$, then $x-y\in Ker(f)$,which implies that $\psi(x-y)\in T$. Hence $\pi(\psi(x)) = \pi(\psi(y))$.It is easy to show that β is a homomorphism.Consider the following diagram:



Since M is a small pointwise N-projective module,for every $m\in M$, there exists a homomorphism $\phi:M\rightarrow N$ such that $\pi\circ\phi(m) = \beta(m)$.Now,let $x\in P$, $(\beta\circ f)(x) = \beta(f(x))$.But $\beta(f(x)) = (\pi\circ\phi)(f(x))$ and hence $(\pi\circ\psi)(x) = (\pi\circ\phi\circ f)(x)$,for all $x\in P$.Therefore $(\psi-\phi\circ f)(P)\leq T$.Let $X = \{w\in P:(\phi\circ f)(w) = \psi(w)\}$. Claim that $X = P$. Clearly $X\leq P$. Let $x\in P$ then $(\psi-(\phi\circ f))(x)\in T$, but $T = \psi(Ker(f))$ and hence$(\psi-(\phi\circ f))(x) = \psi(k)$, for some $k\in Ker(f)$.Thus$\psi(x-k) = (\phi\circ f)(x-k)$,therefore $x-k\in X$ which implies that $P = Ker(f)+X$, but $Ker(f)<<P$.Hence $X = P$.

Proposition (3.11) Let M,N and K be modules, where K is a small pointwise projective and $f:K\rightarrow M$ be an epimorphism. Then M is a small pointwise N-projective if for every homomorphism $\phi:K\rightarrow N$,there exists a homomorphism $\phi^*:M\rightarrow N$, such that $\phi^*\circ f = \phi$.

Proof: Let $g:N\rightarrow B$ be a small epimorphism and $h:M\rightarrow B$ be any homomorphism. Consider the following diagram:



By small pointwise projectivity of K for every $k\in K$, there exists a homomorphism $\phi:K\rightarrow N$, such that $g\circ\phi(k) = h\circ f(k)$. By our hypothesis, there exists a homomorphism $\phi^*:M \rightarrow N$, such that $\phi^*\circ f = \phi$, and so $g\circ\phi^*\circ f = g\circ\phi = h\circ f$.For $m\in M$, we have $(g\circ\phi^*)(m) = g(\phi^*(m)) = g(\phi^*(f(x)))$,here $m = f(x)$,for some $x\in K$.Hence $(g\circ\phi^*)(m) = (g\circ\phi^*\circ f)(x) = (g\circ\phi^*)(f(x)) = (g\circ\phi)(x) = h(f(x)) = h(m)$. Therefore M is a small pointwise N-projective module.

From (3.10) and (3.11) get the following:

Theorem (3.12) Let M and N be modules and assume M has a small pointwise projective cover (P,f), then M is a small pointwise N-projective module if and only if for every homomorphism $\psi:P\rightarrow N$, there exists a homomorphism $\hat{\psi}:M\rightarrow N$, such that $\hat{\psi}\circ f = \psi$.

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