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Pseudo -y- closed - Injective Modules

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Abstract
In this paper, we introduce new concepts of pseudo-y-closed -injective module, and quasi- pseudo- y-closed- injective module. This work which is generalization of pseudo-injective modules and y-closed-injective modules. We have provided some characteristics and descriptions of those concepts. CLS-modules have been characterized in terms of pseudo-y- closed-injective modules. We have shown the relationships of quasi-pseudo-y-closed-injective with other concepts, including a Co-Hopfian, directly finite modules.

Keywords: pseudo-y-closed- injective module ; quasi - pseudo -y-closed-injective module ; fully pseudo yc- stable

مقاسات الاغمارية من النمط –yالكاذبة المغلقة

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الخلاصة
في هذه الورقة، قدمنا مفاهيم جديده ; مقاسات الاغمارية المغلقة من النمط–y الكاذبة ، و مقاسات الاغمارية المغلقة من النمط – y شبه الكاذبة .هذا العمل هو تعميم لمقاسات الاغمارية الكاذبة و المقاسات الاغمارية المغلقة من النمط –y . لقد قدمنا بعض الخصائص والتوصيفات حول تلك المفاهيم . تم وصف مقاسات من النمط – CLS من حيث مقاسات الاغمارية المغلقه من النمط–y الكاذبة .بينا علاقات مقاس الاغمارية المغلقة من النط–y شبة الكاذبة مع المفاهيم الاخرى , بما في ذلك Co–Hopfian ، و وحدات محدودة مباشرة .

1. Introduction

Throughout this paper, R is a ring with identity, and every R-module is a unitary left R-module, $B \subseteq D$ denotes B is a submodule of an R-module D. Hom $_R$(D, K) (Mon $_R$(D,K)) denotes all an R-homomorphism (R-monomorphism) from D to R- module K over ring R. Let D and K be R-modules, D is called **(**a pseudo**)**-K-injective if for any $\beta \in$ Hom $_R$(A, D) **(** Mon $_R$ (A, D) **)** where $A \subseteq K$ there exist $\lambda \in$ Hom $_R$(K, D) with $\lambda i = \beta$, where i be an inclusion map. An R-module D is said to be **(**pseudo**)**-quasi-injective if D is a

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(pseudo**)**-D- injective. Also, we say D is an injective if it is a K-injective for any R-module K (see [1-3]).
A submodule B of an R-module D is called an essential submodule of D denote by ($B \subseteq_e D$) if $B \cap H \neq 0$, $\forall\ 0 \neq H \subseteq D$, and an R-module D is said to be uniform if every submodule of D is an essential submodule of D, see [4]. For a submodule B of an R-module D is said to be closed in D (briefly, $B \subseteq_c D$) if B has no proper essential extension inside D. The submodule Z (B) of D define as Z (B) = {b∈B**:** ann (b) $\subseteq_e$ R} is called singular of D. If Z (D) =D **(**Z (D) = 0**)**, then D is singular **(**nonsingular**)**. A submodule B of an R-module D is called y-closed (briefly, $B \subseteq_{yc} D$) if D/B be a nonsingular. Every y-closed submodule is closed, but the convers is not true, see [4].

An R-module D is called extending (or CS-module) if any closed submodule of D is direct summand. For an R- module D is said to be CLS-module if each y-closed submodule is direct summand. Clearly, every CLS-module is CS- module, see [5].

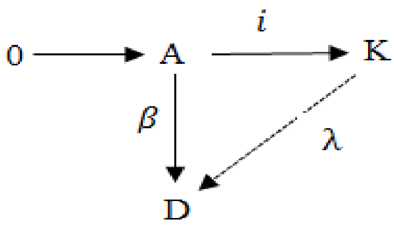
H.S. Lamyaa in [6], introduces the concept of yc- injectivity. Let K be an R-module, an R-module D is called K- y-closed -injective (briefly, D is K-yc-injective), if for any $\beta \in Hom_R(B, D)$, where $B \subseteq_{yc} K$, there exists $\delta \in Hom_R(K, D)$ with $\delta\ i = \beta$. If D is D - yc-injective, then D is said to be self-y-closed-injective or quasi-y-closed- injective (briefly, D is quasi-yc-injective). We say D is yc-injective if it is K- yc -injective, for any R-module K.
In [7] and [8], an R- module D is a pseudo-K-c-injective if for any $\beta \in Mon_R(A,D)$ where $A \subseteq_c K$, there exists $\lambda \in Hom_R(K, D)$ such that $\lambda\ i = \beta$. An R-module D is said to be co-Hopfian (Hopfian) if each surjective (injective) endomorphism : $D \rightarrow D$ is automorphism see [9]. An R- module D is directly finite if it is not isomorphic to a proper direct summand of D, see [1]. A submodule B of R-module D is said to be stable, if for any $\beta \in Hom_R(B, D)$, then $\beta(B) \subseteq B$. We say D is a fully stable if any submodule of D is stable see [10]. An homomorphism β: $B \longrightarrow D$ is called C- homomorphism if β(B) is closed in D see [8].

In this work, we give more characterizations of pseudo -y-closed - injective. Also, we prove that an R-module K is CLS- module iff every module is pseudo -K-yc-injective iff for any y-closed submodule of K is pseudo -K-yc-injective. And a sufficient condition for quasi-pseudo- y-closed-injective to be Co-Hopfian is given.

2. Pseudo-y-closed- Injective Module.

In this section, we introduce the concept of pseudo- y - closed-injective module with some example and the relation with other concepts. This concept is generalization of yc-injective and pseudo-K- injective.

Definition 2.1: Let D and K be R-modules. Then D is pseudo-K-y-closed-injective (briefly D is pseudo-K-yc-injective) if for any $\beta \in Mon_R(A, D)$ where (A is an y-closed submodule of K), there exists $\lambda \in Hom_R(K,D)$ with $\lambda\ i = \beta$. Where i be an inclusion map, i.e, the following diagram:



is commute.
Also, an R- module D is referred pseudo-yc-injective, if D is pseudo-K-yc-injective, for any K be R-module.

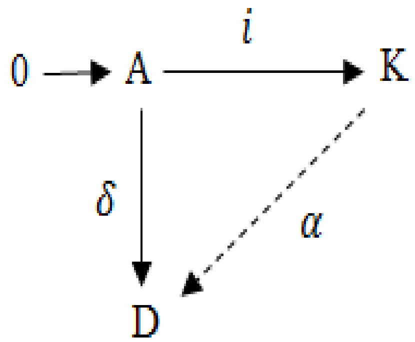
Examples and Remarks 2.2:

1. Clearly, every pseudo -K-injective module is a pseudo-K-yc -injective. The opposite is explained in (6).

2. For an R- module K is simple y-closed R- module, if (0) and K are only y-closed submodule of K. Consider the module Z_2 as Z-module, clear that itis simple, but Z_2 is singular, thus by [6] we get Z_2 is only y-closed submodule of Z_2. Hence, Z_2 is not simple y-closed Z-module. We know that Z as Z-module is not simple, (0) and Z are only y-closed submodules of Z see [6], therefore, Z is simple y-closed Z-module. This means there are no relationship between simple R- module and simple y-closed R- module.

3. If K be a simple y-closed R- module, then each R- module D is pseudo- K-yc-injective.

Proof: Aussme that D be R-module. Let A ⊆yc K and $\delta \in$ Mon $_R$(A, D). Consider the illustration below**:**



Since K be a simple y-closed R-module, we have A=0 or A=K. If A=0, then for any $\delta \in$Mon $_R$(A, D) let δ(a)=0 for all a∈ A, so there exist an R- homomorphism α**:** K⟶D such that α(k)=0 for all k ∈ K, it is follows that αi(a)=α(a)=0=δ(a), a ∈A, hence α is an extension of δ. Now, if A= K. Clearly, D is a pseudo-K- yc- injective.

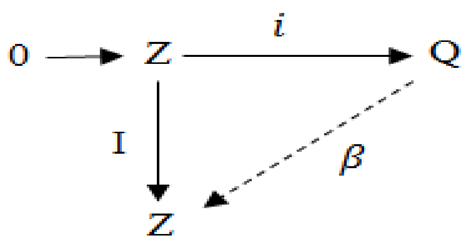
4. If K be a nonsingular uniform R-module. Then any R- module D is a pseudo-K-yc-injective.

Proof: Let D be an R-module. As K is uniform, then it is easy to verify that (0) and K are only closed submodule of K. But K is nonsingular, thus by [4, Proposition 2.4, p.43] we get (0) and K are y-closed submodule of K, let A ⊆yc K where A is not trivial, hence A ⊆c K because every y-closed submodule is closed see [6], this is a contradiction. Therefore, K is a simple y-closed R-module. So, by (**3**) we have D is a pseudo- K-yc-injective.

5. Consider the modules Q and Z as Z- module. By (**4**), it is clear any R-module D is a pseudo Q - yc -injective and pseudo - Z- yc- injective.

6. The converse of (**1**) is not true. A Z-module Z by (**5**) is a pseudo-Q- yc-injective, but Z is not a pseudo- Q- injective.

Proof: Suppose that Z is a pseudo - Q - injective. Consider the illustration below**:**



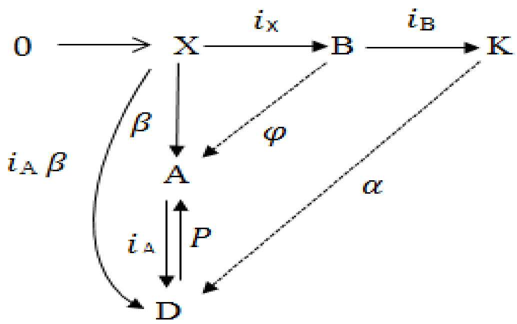
Where I is the identity map. Since Z is a pseudo -Q- injective, there exist $\beta \in$ Hom $_Z$(Q, Z) such that βi= I. But Hom $_Z$(Q, Z) =0 see [11], we have I=0, which is a contradiction. Therefore, Z is not pseudo- Q -injective

7. Clearly, K- yc - injective which is a pseudo -K-yc-injective. By [6], it follows that for a singular R- module K, any R- module D is a pseudo- K- yc- injective.

8. Every pseudo-K-c-injective module is a pseudo-K-yc-injective, the converse case has been discussed in an R-module K = $Z_8 \oplus Z_2$ as Z-module. A submodule B= <(2, 1)> of K is closed but it is not direct summand of K by [12]. Assume that B is a pseudo-K-c-injective. By [7] we get B ⊆⊕ K which is a contradiction. So, B is not a pseudo-K-c-injective. But K is a singular, hence B is a pseudo-K-yc -injective by (**7**).
9. Let K be an R-module. An R-module D is called a K-c-injective if any closed submodule A of K, and any β∈ Hom $_R$(A, D) can be extended to α ∈ Hom $_R$(A,D). Also, we say D is a quasi-c-injective if it is a D-c-injective, see [13]. Clearly, any K-c-injective is a pseudo-K-c-injective. Therefore, by [13] and by Remark 2.2, (8) we have for CS-module H then every R-module is a pseudo-K-yc –injective
♦K-injective⟹ pseudo-K-injective ⟹ pseudo-K-c-injective ⟹ pseudo-K-yc-injective.
In the result below we show that, for an R- module is a nonsingular semi-simple then the concepts of the pseudo- K-injective, pseudo- K-c-injective and pseudo-K-yc- injective are equivalents.

Proposition 2.3: Let K be an R-module. If K is a nonsingular semi-simple, then the following statements are equivalent**:**
1. pseudo-K-yc-injective-module;
2. pseudo-K-c-injective-module;
3. pseudo- K – injective- module.
Proof: (**1**) ⟹ (**2**) Suppose that D is a pseudo-K-yc-injective module. Let A ⊆c K and $\beta \in$ Mon $_R$(A, D). Since K is a nonsingular, thus by [4, Proposition 2.4, p.43] we get A ⊆yc K. By the assumption, there exists $\alpha \in$ Hom $_R$(K, D) such that $\alpha i=\beta$. Hence, D is a pseudo-K-c- injective.
(**2**)⟹ (**3**) Assume that D is a pseudo- K-c-injective module. Let B ⊆ K and δ ∈ Mon $_R$(B, D). As K is a semi-simple, hence B ⊆⊕ K and it follows B ⊆c K. By (**2**), there exist $\varphi \in$Hom $_R$(K, D) such that $\varphi i = \delta$, so D is a pseudo - K – injective.
(**3**) ⟹ (**1**) It is obvious.
Now, we give the properties of direct summand in a pseudo-yc -injective module.

Proposition 2.4: Let D and K be R- modules. If D is a pseudo -K-yc-injective and A is a direct summand of D and B is y-closed submodule of K, then
1. A is a pseudo- B -yc-injective module.
2. A is a pseudo-K-yc- injective module.
Proof: **(1)**. Assume that X ⊆yc B and β ∈ Mon $_R$(X, A). Since A ⊆⊕ K, there exists a submodule U of D such that K= A⊕U. Consider the illustration below**:**



Where i_X:X⟶B, i_B:B⟶K, i_A:A⟶D be inclusion maps of X in B, B in K and A in D respectively, since X ⊆yc B and B ⊆yc K, thus by [6] we get X ⊆yc K, but D is a pseudo-K-yc -injective and $i_A\, \beta$: X⟶D be an R-monomorphism, there exists α∈Hom $_R$(K, D) such that α i_B i_K = i_A β. Putφ=Pαi_B, where P is a projection map from D to A. Clearly,φ is an R-homomorphism, φ i_X = P α i_B i_K = P i_A β =β. Hence A is pseudo - B - yc -injective.

2. Since K $\trianglelefteq$yc K, see [6]. By (**1**), we have A is pseudo- K- yc -injective module.

Corollary 2.5: Let D and L be two R- modules. Then D is a pseudo-L-yc-injective if and only if D is pseudo- H- yc -injective, for any H is y-closed submodule of L.

Proof: Suppose that D is a pseudo-L- yc-injective. Let H $\subseteq$yc L. Clearly, D is a direct summand from itself and so, by Proposition 2.4, (1) we have D is a pseudo-H-yc-injective. Conversely, since L $\subseteq$yc L, hence D is a pseudo- L- yc - injective.

Corollary 2.6: Let D be an R-module and K be a nonsingular R- module. If D is a pseudo- K-yc-injective, X $\subseteq\oplus$ D and Y $\subseteq\oplus$ K, then X is a pseudo-Y-yc- injective.

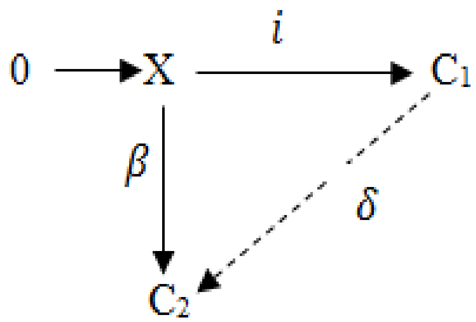
Proof: Assume that D is a pseudo-K- yc- injective and X $\subseteq\oplus$ D, Y $\subseteq\oplus$ K. It follows that Y $\subseteq$c K, as K is a nonsingular, thus by [4, Proposition 2.4, p.43] we have Y $\subseteq$yc K, therefore, X is a pseudo- Y-yc-injective by Proposition 2.4, (1).

Proposition 2.7: Let C_1 and C_2 be two R-modules and D =$C_1 \oplus C_2$. Then the following are equivalent:

1. C_2 is a pseudo -C_1- yc -injective module;
2. for any K $\subseteq$ D, K $\cap C_2 = 0$ and π_1(K) $\subseteq$yc C_1 where (π_1 the natural projection of D into C_1), there exists submodule K_1 of D such that K $\trianglelefteq K_1$ and D = $K_1 \oplus C_2$.

Proof: (**1**) $\Longrightarrow$ (**2**) Suppose that K$\subseteq$ D such that K$\cap C_2 = 0$ and let π_1: D $\longrightarrow C_1$, π_2: D $\longrightarrow C_2$ are the projection mapping with π_1(K) $\subseteq$yc C_1. Define β**:** π_1(K) $\longrightarrow C_2$ as follows for all k =a+b (a $\in C_1$, b $\in C_2$), β(a) = b. Clear that, β is well- define and R-monomorphism. So, by (**1**), there exist $\delta\in$ Hom $_R(C_1, C_2)$ such that $\delta i=\beta$. Define K_1={a+δ(a): a$\in C_1$}, and claim D= $K_1 \oplus C_2$. Let d $\in$ D, we get d= d_1+d_2 where $d_1 \in C_1$ and $d_2 \in C_2$, thus d =d_1+d_2=**(**$d_1+\delta(d_1)+d_2-\delta(d_1)$**)**$\in K_1 \oplus C_2$. Now, suppose that d $\in K_1 \cap C_2$, therefore, d = $d_2=d_1+\delta(d_1)$ such that $d_1\in C_1$, $d_2 \in C_2$. Hence $d_1=d_2-\delta(d_1) \in C_1 \cap C_2$=0. So $K_1 \cap C_2$ =0. Thus D = $K_1 \oplus C_2$. Let k = a+b $\in$K, where a $\in C_1$, b $\in C_2$. Since π_1(k) = a, π_2(k) = b, we have k = $\pi_1(k)+\pi_2(k) = \pi_1(k)+\delta\, \pi_1(k) \in K_1$, since $\delta\, \pi_1(k) = \beta\, \pi_1(k) = \pi_2(k)$, hence K $\subseteq K_1$.

(**2**)$\Longrightarrow$ (**1**). Let X $\subseteq$yc C_1 and $\beta\in$ Mon $_R(X, C_2)$. Define K={x$-\beta$(x): x $\in$X}, clear that K $\subseteq$D. To show that K $\cap C_2$=0, let h$\in$K $\cap C_2$, then h$\in C_2$ and h = x$-\beta$(x), x $\in$X, hence x= h+β(x) $\in C_1 \cap C_2$, therefore, we get x= 0, then K $\cap C_2$=0. It's easy to prove X = π_1(k), so π_1(k) $\subseteq$yc C_1, then by (2), there exists K_1 be submodule of D such that K $\subseteq K_1$, D= $K_1 \oplus C_2$. Let π_2: D $\longrightarrow C_2$ denote the projection with Ker $\pi_2=K_1$ and let δ: $C_1 \longrightarrow C_2$ be the restriction of π to C_1. Consider the following diagram**:**

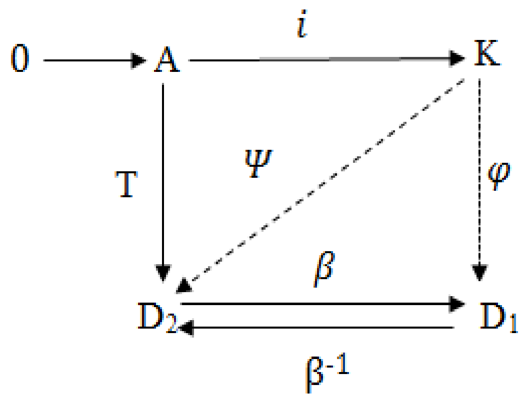


For every x $\in$ X, $\delta(x) =\pi(x) = [(x) - \beta(x) +\beta(x)] = \pi[x - \beta(x)]+\beta(x)= \beta(x)$, hence δ extends β, therefore, C_2 is a pseudo-C_1- yc- injective.

Now, we provide some basic properties of a pseudo- yc- injective module.

Proposition 2.8: Let D_1, D_2 and K be R- modules, $D_1 \cong D_2$. If D_1 is a pseudo-K- yc- injective then D_2 is a pseudo-K- yc -injective.

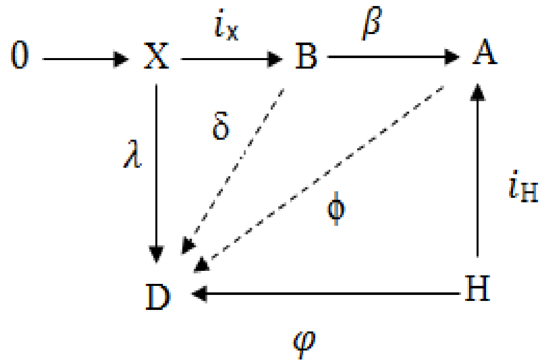
Proof: Assume that D_1 is pseudo-K- yc-injective. To show that D_2 is a pseudo-K-yc-injective. Let A $\subseteq$yc K and T $\in$ Mon $_R(A, D_2)$. Consider the illustration below**:**



Since $D_1 \cong D_2$, there exists β: $D_2 \longrightarrow D_1$ be an R- isomorphism, therefore $\exists \beta^{-1}$:$D_1 \longrightarrow D_2$ also be an R- isomorphism. Clearly, βT is an R-monomorphism from A to D_1. As D_1 is a pseudo-D-yc-injective, there exists φ :K $\longrightarrow D_1$be an R-homomorphism such that $\varphi i = \beta$ T. Define Ψ :K $\longrightarrow D_2$ by$\Psi = \beta^{-1}\varphi$, hence Ψ is R-homomorphism and $\Psi i = \beta^{-1} \varphi i = \beta^{-1} \beta$T = T. Therefore, D_2 is a pseudo-K- yc -injective module.

Proposition 2.9: Let A, B and D be an R-modules such that A≅B and for any y-closed submodule X of B, Ker $\beta \subseteq$ X with $\beta \in$ Hom $_R$(A, B). If D is a pseudo-A -yc-injective, then D is a pseudo-B- yc injective.

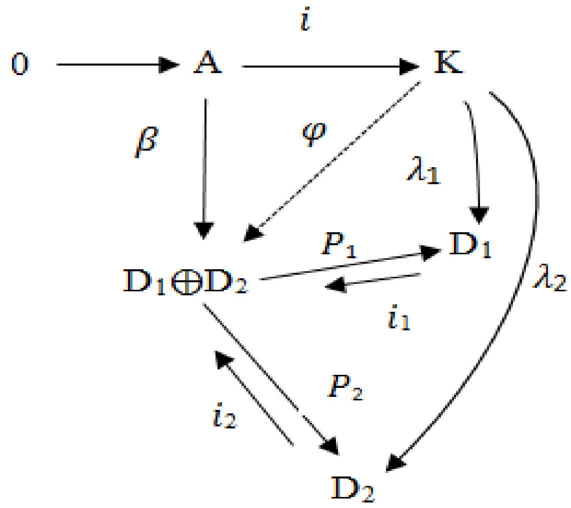
Proof: Let X ⊆yc B and λ:X ⟶D be an R-monomorphism. Since A≅B, there exist β: B⟶A be an R-isomorphism. Put H = β(X), as Ker $\beta \subseteq$ X, thus by [6] we get H ⊆yc A Consider the illustration below**:**



Where i_X, i_H are the inclusion maps. Define φ: H⟶D by $\varphi(\beta(x)) = \lambda(x)$, x ∈ X. It is clear that φ is an R- monomorphism. As D is a pseudo-A-yc-injective, there exists ϕ:A⟶D such that $\phi\, i_H = \varphi$. Put $\delta = \phi\, \beta$. Clearly, δ be R- homomorphism, therefore, $\lambda(x) = \varphi(\beta(x)) = \phi i_H (\beta(x)) = \phi(i_H\ (\beta(x))) = \phi\ (\beta(x)) = \phi\beta(x) = \phi\beta(i_X(x)) = \delta(i_X(x)) = \delta\ i_X(x)$. Hence D is a pseudo-B-$yc$-injective.

Proposition 2.10: Let D_1, D_2 and K be R-modules. If D_1 and D_2 are pseudo -K-yc- injective modules, then $D_1 \oplus D_2$ is a pseudo -K-yc- injective.

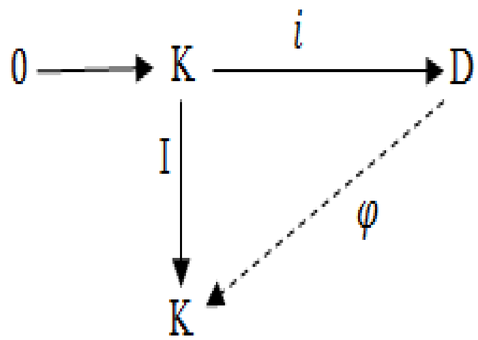
Proof: Assume that D_1 and D_2 are pseudo -K-yc- injective modules. Let A ⊆yc K and let β ∈Mon $_R$(A, $D_1 \oplus D_2$).Consider the illustration below:



Where i_1, i_2 are the inclusion maps, P_1, P_2 are the projection map. As D_1 and D_2 are pseudo -K-yc- injective, there exists $\lambda_1 \in$ Hom $_R(K, D_1)$and $\lambda_2 \in$ Hom $_R(K, D_2)$ such that $\lambda_1 i = P_1\beta$ and $\lambda_2\, i = P_2\, \beta$. Define: $K \longrightarrow D_1 \oplus D_2$ by $\varphi(k) = (\lambda_1(k), \lambda_2(k))$, for all k $\in$K. We prove that $\varphi i = \beta$. Let a $\in$ A, then $\beta(a) = (d_1, d_2)$, where $d_1 \in D_1$ and $d_2 \in D_2$. $\varphi i(a) = \varphi(i(a)) = (\lambda_1(i(a)), \lambda_2(i(a))) = (P_1\beta(a), P_2\beta(a)) = (d_1, d_2)$. Therefore, $D_1 \oplus D_2$ is pseudo -K-yc- injective modules.

Proposition 2.11: Let D be an R- module and K $\subseteq$yc D. If K is a pseudo-D- yc-injective, then K $\subseteq\oplus$ D

Proof: Suppose K is pseudo- D- yc -injective. Let I: K $\longrightarrow$K be the identity map. Consider the illustration below**:**



Since K is a pseudo-D- yc- injective, there exists $\varphi \in$ Hom $_R$(D, K) such that I= φi. To show that D = Ker$\varphi \oplus$K, since Kerφ and K $\subseteq$ D, we have Ker φ +K $\subseteq$M, let d $\in$ D clearly, d $-\varphi(d) \in$ Ker φ, therefore, d= (d $-\varphi(d)) + \varphi(d) \in$ Ker φ +K. Hence Ker φ + K = D. Now, we to show that Ker $\varphi \cap$K= 0, let $a \in$Ker φ $\cap$K, hence $\varphi(a)$= 0 but $\varphi(a) = \varphi i(a)$= I(a) = a we have a=0. Therefore, K $\subseteq\oplus$ D.

We introduce concept prior to the following outcome.

Definition 2.12: A homomorphism **(**monomorphism**)** β:A$\longrightarrow$ B is called yc-homomorphism **(**yc - monomorphism**)** if β(A) $\subseteq$yc B.

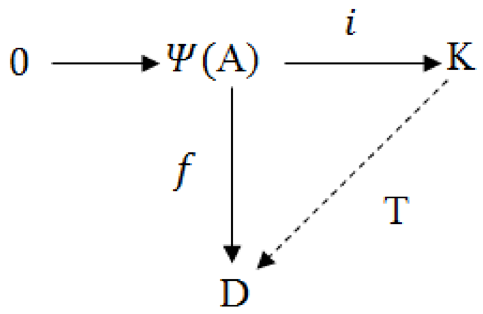
Example: If we take Z_4 and Z_2 as Z-module, let β: $Z_4 \longrightarrow Z_2$such that $\beta(0) = \beta(2)$ =0, $\beta(1) = \beta(3)$ =1. It is easy to prove that β is homomorphism and $\beta(Z_4) = Z_2$, hence β is yc-homomorphism. Clear that any yc-homomorphism is a C- homomorphism. The converse is not true, let f: $Z_2 \longrightarrow Z_2$ defined as follows $f(Z_2)$ = 0, therefore, f is C -homomorphism, but is not yc-homomorphism since 0 is not y-closed of Z_2.

In the following proportion, we give a characteristics of a pseudo -yc -injective.

Proposition 2.13: Let D and K be two an R-modules, then the following are equivalence.

1. D is a pseudo -K- yc -injective module;
2. For any R- module A, any yc -monomorphism Ψ:A⟶K and for any $\lambda\in$ Mon $_R$(A, D), there exists T ∈ Hom $_R$(K,D) with λ= T Ψ.

Proof :(1) ⟹ (2) Let A be an R- module, Ψ:A⟶K be an yc -monomorphism and $\lambda\in$ Mon $_R$(A, D). Since Ψ: A⟶K is an yc- monomorphism, we have Ψ(A) ⊆yc K. Defined f:Ψ(A) ⟶D by $f(\Psi(a)) = \lambda(a)$ for all a∈A. Consider the illustration below**:**

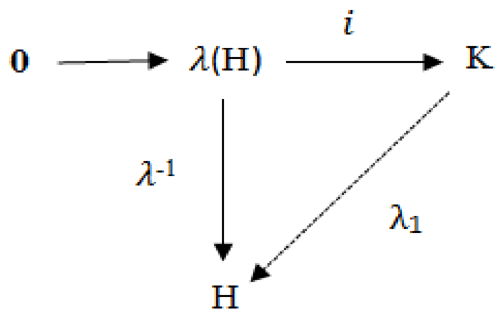


Clearly, f is R-monomorphism. As D is a pseudo-K-yc-injective, there exists T∈ Hom $_R$(K, D), such that T i =f. Therefore, we have T Ψ(a) =T (Ψ(a)) = T $i(\Psi(a)) = f(\Psi(a)) = \lambda(a)$. Hence λ =T Ψ.

(2) ⟹ **(1)** Let H ⊆yc K and g: H⟶D be a monomorphism. It is clear that the inclusion map i is yc- monomorphism. By (2), then there exists T∈ Hom $_R$(K , D) such that T i= g. Hence D is a pseudo-K-yc-injective module.

Proposition 2.14: If H is a pseudo- K-yc-injective module, then any yc-monomorphism from H to K is splits.

Proof: Let λ:H⟶K is yc-monomorphism, we get λ(H) ⊆yc K. Consider the illustration below:

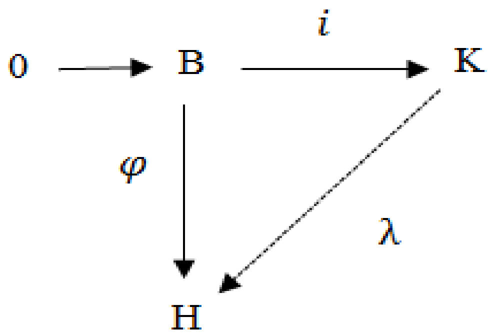


Define λ^{-1}: λ(H) ⟶H such that $\lambda^{-1}\lambda(H) = H$. As H is a pseudo-K- yc-injective, there exists λ_1∈Hom $_R$(K, H), where $\lambda_1 i = \lambda^{-1}$. So, for all h∈H we have$\lambda_1\lambda(h) = \lambda_1(i(\lambda(h))) = \lambda_1 i(\lambda(h)) = \lambda^{-1}(\lambda(h)) = \lambda^{-1}\lambda(h)$ =h= I_H(h). Therefore, λ is splits by [10].

The CLS-module is described in terms of pseudo- yc -injective modules in the proposition that follows.

Proposition 2.15: Let K be an R-module, then the next are equivalent.
1. K is CLS- module;
2. Every module is pseudo-K-yc- injective module;
3. For any S, S ⊆yc K then S is pseudo-K- yc - injective module.

Proof: (1) ⟹ (2). Let H be any R-module. We show H is a pseudo- K- yc-injective module, let B ⊆yc K and $\varphi\in$ Mon $_R$(A, H). Consider the illustration below**:**



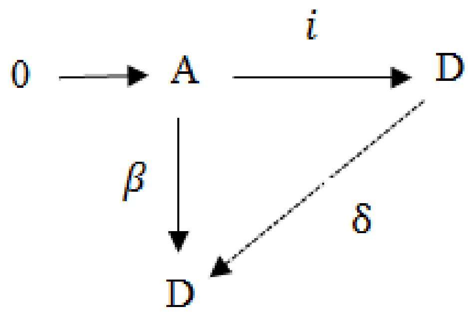
By (1), we have B ⊆⊕ K. So, ∃ B_1 ⊆ K, where K= B⊕B_1. Define λ: K⟶Hby $\lambda(b+b_1)$ = φ(b), if $b_1 = 0$ and λ $(b+b_1)$ = 0otherwise, b∈Band b_1∈B_1.Therefore, λ extends to.
(2) ⟹ **(3)** It is clear.
(3) ⟹ **(1)** Let S ⊆yc D, by (3), then S is a pseudo-K- yc- injective module, therefore, S ⊆⊕ D by Proposition 2.11.
If for every intersection of two direct summand in R- module D is direct summand , then D is has the summand intersection property (SIP) see [14] .

Proposition 2.16: Let D be a nonsingular R- module, if for any y-closed submodule of D is a pseudo -D-yc-injective module, then D has SIP.
Proof: Let A_1 and A_2 be two direct summand of D. To show A_1∩A_2 ⊆⊕ D, since D be a nonsingular, we have A_i ⊆yc D, i =1, 2. So, A_1∩A_2 ⊆yc D by [6]. Thus by hypothesis A_1∩A_2 is pseudo-D-yc-injective. Hence A_1∩A_2 ⊆⊕ D by Proposition.2.11.

3. Quasi- pseudo -y-closed- Injective Module.

In this section, we introduce the concept of quasi -pseudo- y - closed-injective module which is a proper generalization of pseudo - injective.

Definition 3.1: An R- module D is called a quasi -pseudo- y-closed -injective module (briefly, D is a quasi- p -yc- injective). If for each A ⊆yc D and β∈Mon $_R$(A, D), there exists $\delta \in$ End$_R$(D) such that $\beta = \delta i$, i.e., the following diagram**:**

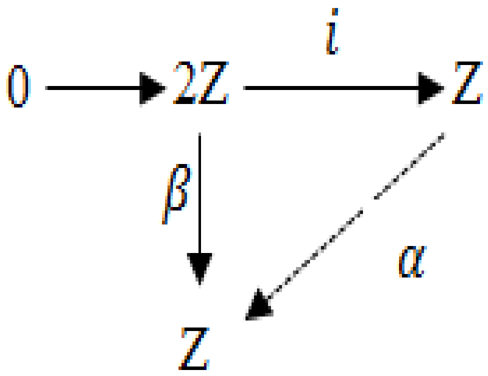


is commute.
A ring R is referred self - pseudo- yc- injective module, if R is pseudo -R_R -yc - injective module.
Note that: An R-module D is a quasi- p-yc-injective, if it is a pseudo -D- yc- injective. Therefore, every pseudo-yc -injective is quasi - p-yc- injective. If D is quasi - injective module, then D is quasi-yc-injective and clearly D is quasi-p-yc-injective, the opposite is not true in general.

Examples and Remarks 3.2:
1. A Z-module D= Z_p ⊕Q. By [13], we get D is quasi-c-injective. Clearly, D is a quasi-yc-injective. Hence, D is a quasi- p -yc-injective.

2. Any pseudo-injective is quasi- p-yc- injective. The opposite is not true, by Remark.2.2, (5) Z as Z-module is a quasi- p - yc-injective. But Z is not pseudo- injective

Proof: Assume that Z is a pseudo-injective and $\beta \in$ Mon $_R$(2Z, Z), β(2n)= n for each n$\in$ Z. Consider the illustration below:

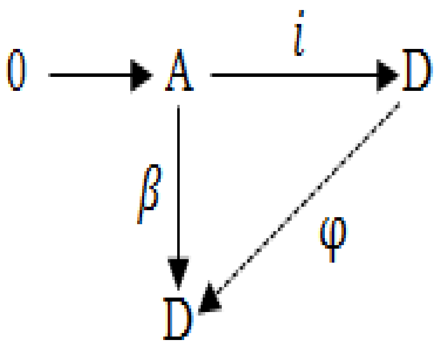


Since Z is a pseudo- injective, there exist $\alpha \in$Hom $_R$(Z) such that $\alpha i = \beta$. For each n$\in$Z, we get n = β(2n) = $\alpha(i$(2n)) = α(2n) = 2n α(1) we have α(1) = 1 ⁄ 2 $\notin$ Z, which is a contradiction. Therefore, Z is not a pseudo injective.

In the next result we discuss the relationship between a quasi- p -yc- injective module and a quasi-yc- injective module.

Proposition 3.3: Every nonsingular uniform quasi-p-yc-injective module is a quasi - yc-injective.

Proof: Assume that D is nonsingular uniform quasi- p-yc-injective module. Let A ⊆yc D and $\beta \in$ Hom $_R$(A, D). Consider the illustration below:



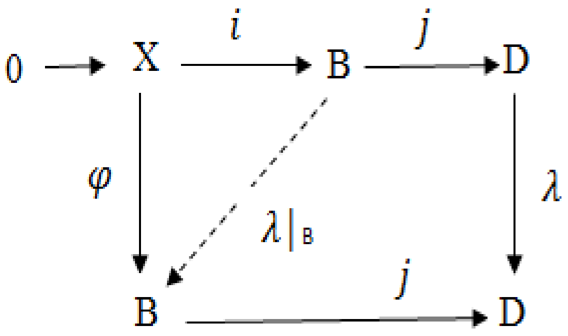
Ker β ⊆ A, then Ker β=0 or Ker β ≠0. If Kerβ =0, thus β is an R-monomorphism. Then β is extendable to an R-homomorphism φ: D⟶D, because D is a quasi- p-yc- injective, hence D is a quasi-yc-injective. If Ker β≠0. As D is a nonsingular, then Ker β ⊆c A by [15], since any submodule in uniform is uniform, thus A is uniform submodule, hence Ker β =A. In this case β can be extended to a homomorphism of D to D. Therefore, D is a quasi-yc injective.

A submodule H of R- module D is referred a fully invariant if for each $\beta \in$ End $_R$(H), then β(H) ⊆ H, see [10].

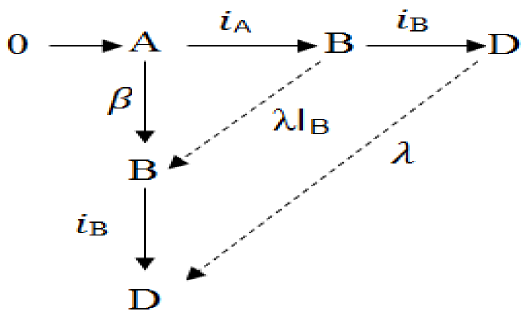
An R- module D is said to be a multiplication, if for all S be submodule of D, then S =I D for some I is ideal of R see [16].

The next result, we give a condition under which an y-closed submodule of a quasi - p- yc-injective module is a quasi -p - yc- injective.

Proposition 3.4: Let D be a quasi-p- yc-injective module and B ⊆yc D, then the next statements hold**:**
1. If B is fully invariant of D, then B is a quasi- p- yc- injective.
2. If D is multiplication, then B is a quasi- p- yc- injective.
Proof :(**1**) Suppose that B is fully invariant of D. Let X ⊆yc B and $\varphi \in$ Mon $_R$(X, B). Consider the illustration below**:**



Since X ⊆yc Band B ⊆yc D, we have X ⊆yc D by [6]. As D is a quasi-p -yc-injective, there exists $\lambda \in$End $_R$(D) such that $j\varphi = \lambda j i$. Since B is fully invariant, then λ(B) ⊆ B and $\lambda|_B \in$ End $_R$(B). Hence, φ extends $\lambda|_B$.
Proof: (2) Assume that D is a multiplication. Let A ⊆yc B and $\beta \in$ Mon $_R$(A, B). Since B ⊆yc D. It follows that by [5], A ⊆yc D. Now, consider the illustration below**:**

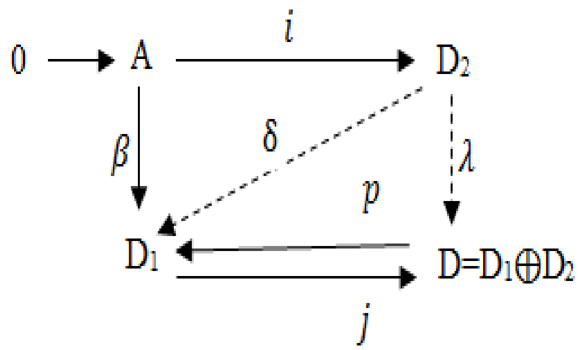


Since D is a quasi-p - yc-injective, there exist $\lambda \in$ End $_R$(D) such that $\lambda i_B i_A = i_B \beta$. Since D is multiplication, we get B = I D for some ideal I of R. Therefore, $\lambda|_B = \lambda(B) = \lambda$ (I D) = I λ(D) ⊆ I D =B. Hence, β extends $\lambda|_B$.
The R- modules D and L are referred relatively injective module, if D is L-injective and L is D- injective see [15].

In the following definition we introduce the concept of the relatively pseudo-yc-injective module:

Definition 3.5: Let B_1 and B_2 be R- modules. B_1 and B_2 are called relatively pseudo-yc-injective module, if $B_1$is pseudo -B_2 - yc -injective and B_2 is pseudo-B_2 - yc-injective.

Theorem 3.6: Let D = $D_1 \oplus D_2$ be a quasi- p -yc- injective module and nonsingular, then D_1 and $D_2$are relatively pseudo-yc-injective module.
Proof: Let D be a quasi -p-yc -injective module and nonsingular. To show that D_1 is a pseudo-D_2- yc- injective. Let A ⊆yc D_2, β: A⟶D_1 be any R- monomorphism, j: D_1⟶D be an injection homomorphism, and p:D⟶D_1 be a projection homomorphism. Define α:A⟶ D by α (a) = ((a), a) for each a ∈ A. Consider the illustration below**:**



Clearly, α is an R-monomorphism, since $D_2 \subseteq^{\oplus}$ D, then D_2 ⊆yc D, this is because D is a nonsingular, as D is a quasi- p-yc- injective, this means D is a pseudo-D-yc-injective, which implies D is a pseudo D_2-yc-injective by Corollary 2.5. Then there exists $\lambda \in$Hom $_R(D_2$, D) such that λi = α, putδ= $p\lambda$. Therefore, δi(a) = $p\lambda i$(a) = $p\alpha$(a) = p (α(a)) = p (β (a), a) = β (a). Hence, D_1 is a pseudo- D_2 yc -injective.

Corollary 3.7: Let D= $\oplus_{i\in I}D_i$ be an R- modules, where I= {1, 2, ..., n} and n ∈ Z^+. If D be a quasi -p-yc-injective module and nonsingular, then K_i and K_j are relatively pseudo-yc-injective module for all i, j ∈ I where i ≠ j.
Proof: By Theorem 3.8.

Lemma 3.8: [15] An R-module D is directly finite if and only if $\beta\lambda$ = I implies that $\lambda\beta$ = I for each β, λ ∈ U = End $_R$(D) such that I is an identity map of D .

The following result provides a necessary condition for the quasi - p - yc - injective module to satisfy the Hopfian condition.

Proposition 3.9: A quasi-p- yc-injective module D is co-Hopfian if and only if it is directly finite
Proof: Let β: D⟶D be an R- monomorphism and I:D⟶D be the identity map. As D is a quasi p-yc -injective, there exist λ ∈End $_R$(D) such that $\lambda\beta$ =I. By Lemma 3.10, we get $\beta\lambda$ =I, this means that β is an isomorphism. Hence, D is co -Hopfian. Conversely, suppose that D is a co-Hopfian. Letβ, λ∈ U= End $_R$(D) such that $\lambda\beta$ = I. Then β is an R – monomorphism and β^{-1} exists. Therefore, λ = $\lambda\beta\beta^{-1}$=I β^{-1}=β^{-1}. Hence, $\beta\lambda$= $\beta\beta^{-1}$ = I.

Corollary 3.10: If D is an indecomposable quasi-p -yc-injective module, then D isco-Hopfian
Proof: As every indecomposable module is directly finite, thus by Proposition 3.9, we get D is co-Hopfian.

Corollary 3.11: If D is a Hofian module and quasi-p-yc-injective, then M is co-Hopfian.
A submodule A of R-module D is pseudo- stable, if β (A) ⊆ A for each β∈ Mon $_R$(A, D). An R-module D is referred fully pseudo-stable if each submodule of D is a pseudo-stable, see [10].
In the next, we give fully pseudo yc-stable module as a proper generalization of fully pseudo-stable.

Definition 3.12: An R- module D is referred fully pseudo yc-stable, if for any y-closed submodule of D is a pseudo-stable.
Note that, any pseudo-stable submodule is a pseudo yc-stable submodule, as well as each fully pseudo stable R- module is fully pseudoyc-stable. But the opposite is not true, for

example Z as Z -module is fully pseudo yc-stable, because (0) and Z only is y-closed submodule of Z. But, Z is not fully pseudo stable by [10].

The following result gives a characterization of fully pseudo yc-stable module.

Proposition 3.13: If D is fully pseudo yc-stable R-module, then every yc-monomorphism β: D⟶D is an R- epimorphism.

Proof: Let β: D⟶D is yc- monomorphism, this means that β(D) ⊆yc D. Define λ: β (D) ⟶D as follows $\lambda(\beta$(d)) = d for each d∈ D. Clearly, λ is well-defined and an R-isomorphism but D is fully pseudo yc- stable, hence $\lambda(\beta$(D)) ⊆ β(D). As λ is an R-epimorphism, then $\lambda(\beta$(D)) = D this means D ⊆ β(D). Therefore, β is an R-epimorphism.

Proposition 3.14: Let D be a multiplication R- module. If D is a quasi-p -yc-injective, then D is fully pseudo yc -stable.

Proof: Suppose D is a quasi-p- yc-injective. Let A ⊆yc D and β∈Mon $_R$(A,D), since D is multiplication, then A= I D where I is an ideal of R. Since D is a quasi-p -yc- injective, there exist λ ∈ End $_R$(D) such that $\lambda i = \beta$, where i be a map of inclusion. Hence β(A) = λi(A) = λ(A) = λ(I D) =I λ(D) ⊆ I D=A.

4. Conclusions

Through this paper, we reached the following conclusions: Any pseudo-K- c- injective is a pseudo-K-yc- injective, we give an example of a pseudo-K- yc-injective which is no pseudo-K- c- injective. And the direct summand of a pseudo-K-y-closed -injective is a pseudo-B-y-closed -injective for any B is y-closed submodule of K. And the direct sum of pseudo-K- yc -injective is a pseudo-K- yc - injective.

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