DOI: 10.24996/ijs.2024.65.5.25

ISSN: 0067-2904

Solving Inverse System of Time-Fractional Burgers Equation

Majid Erfanian [1]*, Hamed Zeidabadi [2], Hessam Hosseinnezhad[3]

[1]*Department of Science, School of Mathematical Sciences, University of Zabol, Zabol, Iran*

[2]*Department of Applied Mathematics, Faculty of Mathematics and Computer Sciences, Hakim Sabzevari University,Sabzevar, Iran*

[3]*Department of Basic Sciences, Shahid Sattari Aeronautical University of Science and Technology, Tehran 15916-34311, Iran*

Received: 13/2/2023 Accepted: 11/6/2023 Published: 30/5/2024

Abstract

Applying the finite difference method for the time discretization. We present a numerical method to approximate the inverse system of Burgers equations (ISBE) with the time-fractional. By using the finite difference formula and extended cubic B-splines collocation method (EXCBSM), we determine the approximated solution of this inverse system problem. The convergence analysis is investigated and the order of convergence is obtained. The advantage of this study is comparing it with the other method such as the cubic B-spline collocation method. Also, to clarify the presented method, figures and comparisons of the approximate solutions with the exact value have been presented. Finally, the diagrams of errors for our methods are shown in the figures.

Keywords: System of time fractional Burgers equations, Inverse problems, Noisy data, Finite difference method, Cubic B-splines collocation method, Time-fractional derivatives, Convergence analysis

1 Introduction

In the world of mathematical problems, there exist models of problems that cannot be answered directly, such as reconstruction and identification problems. These kinds of problems, which have a description of the parameters that we cannot directly observe, are called inverse problems (IPs). In this problem, aside from the main function, the unknown functions include some of the functions in formulating the direct problem, that nominated the solution to the inverse problem. These equations have been extensively discussed in a wide spectrum of applications in mathematics and physics, such as the Burgers' equation, which is a fundamental partial differential equation (PDE) that appears in various branches of engineering and physics [1], such as heat equation [2] [3] [4] fluid mechanics, nonlinear acoustics, gas dynamics, modeling of turbulence [5], boundary layer behavior, shock wave formation [6], mass transport, electrohydrodynamic (EMHD) model in plasma physics [7], parabolic equation [8] and traffic flow. This equation describes the integrated process of convection-diffusion in physics [9]. This equation was first introduced by Bateman in 1915 [10] and later developed by Burgers in 1948 [11]. The study of the motion of particles in a fluid goes back to Brown. If the effect of gravity upon the particles is

*Email: erfaniyan@uoz.ac.ir

considered, these particles will be heavier than the surrounding fluid, and the resulting motion is called sedimentation. In the dilute limit, in which the volume fraction of the particles is much less than one, the velocity resembles very closely the famous Burgers equation of one-dimensional. The time-dependent evolution of particle motion is predicted by the Burgers equation. Also, the one-dimensional coupled Burgers' equation can be taken as a simple model of sedimentation and evolution of scaled volume concentrations of two kinds of particles in fluid suspensions and colloids under the effect of gravity [12]. The extended model of coupled Burgers' equation was first derived by Esipov to study the model of polydispersive sedimentation [13]. To solve the coupled system of Burgers' equations with time-fractional derivative numerically, various approaches have been studied by many authors [14]. Space and time-fractional Burgers' differential equation was first treated by Momani [1]. The coupled system of time-fractional derivatives of non-homogeneous Burgers' equations is solved by the fractional homotopy analysis transform method [15], and the 1D time-fractional coupled Burger equation is solved analytically via fractional complex transform [16]. In the polydispersive case, Esipove [13], introduced a system of the coupled Burgers equations. The coupled Burgers equations predict an interesting phenomenon, which was termed phase shifts. This phenomenon is observable in a bidisperse system. The particle size distribution function evolves in an interesting way near the interface. In the Episov model, he applied the continuity equation which describes the conservation of species with concentration $c(s,r)$ and the particle flux $\boldsymbol{J}(x.\ t)$. In the case of very small particles, they experience Brownian motion at this equation [8]

$$\frac{\partial c}{\partial t}+div\ \boldsymbol{J}=0,\ \text{ that }\quad \boldsymbol{J}=\boldsymbol{V}(c)c-D(c)\nabla c. \tag{1}$$

The $D(c)$ distribution and the $\boldsymbol{V}(c)$ velocity is known from Brownian motion. It was recognized that Eq. 1 with c-dependent velocity resembles very closely the famous Burgers equation of one-dimensional compressible flow.

Let $\vartheta_1(s,r)$, and $\vartheta_2(s,r)$ be the concentrations of particles with a bimodal distribution of particle sizes, both obeying continuity equations. In simplified form, at small concentrations, the two coupled Burgers equations are introduced [7] for $s,r\in[0,1]$,

$$\frac{\partial\vartheta_1\ (s,r)}{\partial r}=\frac{\partial^2\vartheta_1(s,r)}{\partial s^2}+2\vartheta_1(s,r)\frac{\partial\vartheta_1(s,r)}{\partial s}-\frac{\partial\big(\vartheta_1(s,r)\vartheta_2(s,r)\big)}{\partial s}+F(s,r), \tag{2}$$

and

$$\frac{\partial\vartheta_2\ (s,r)}{\partial r}=\frac{\partial^2\vartheta_2(s,r)}{\partial s^2}+2\vartheta_2(s,r)\frac{\partial\vartheta_2(s,r)}{\partial s}-\frac{\partial\big(\vartheta_1(s,r)\vartheta_2(s,r)\big)}{\partial s}+G(s,r). \tag{3}$$

In equations (2) and (3) s,r are spatial coordinates and temporal coordinates respectively. $\frac{\partial\vartheta_1}{\partial r}$, and $\frac{\partial\vartheta_2}{\partial r}$ are unsteady terms, $\vartheta_1\frac{\partial\vartheta_1}{\partial s}$ and $\vartheta_2\frac{\partial\vartheta_2}{\partial s}$ are the nonlinear convection term. Also, $\frac{\partial^2\vartheta_1}{\partial s^2}$ and $\frac{\partial^2\vartheta_2}{\partial s^2}$ are the diffusion term, finally $G(s,r)$ and $F(s,r)$ are the source term.

It is well known that the integer-order differential operators and the integer-order integral operators are local but the fractional-order differential operators and the fractional-order integral operators are nonlocal. To solve the coupled system of Burgers' equations with time-fractional derivative numerically, various approaches have been studied by many authors [14] [17] [18] [19]. Space and time-fractional Burgers' differential equation was first treated by Momani [20]. The coupled system of time-fractional derivatives of non-homogeneous Burgers' equations is solved by the fractional homotopy analysis transform method [15], at the Gegenbauer wavelets are used to present two numerical methods for solving the coupled system of Burgers' equations with a time-fractional derivative [14]. Consider one dimensional coupled nonlinear Burgers' time fractional Caputo derivatives equation in the generalized form [14], with $s, r \in [0,1]$ and $0 < \alpha, \beta \leq 1$ as follow

$$\begin{cases} \dfrac{\partial^{\alpha} \vartheta_1(s,r)}{\partial r^{\alpha}} = \dfrac{\partial^2 \vartheta_1(s,r)}{\partial s^2} + 2\vartheta_1(s,r)\dfrac{\partial \vartheta_1(s,r)}{\partial s} - \dfrac{\partial\left(\vartheta_1(s,r)\vartheta_2(s,r)\right)}{\partial s} + F(s,r), \\ \\ \dfrac{\partial^{\beta} \vartheta_2(s,r)}{\partial r^{\beta}} = \dfrac{\partial^2 \vartheta_2(s,r)}{\partial s^2} + 2\vartheta_2(s,r)\dfrac{\partial \vartheta_2(s,r)}{\partial s} - \dfrac{\partial\left(\vartheta_1(s,r)\vartheta_2(s,r)\right)}{\partial s} + G(s,r). \end{cases} \tag{4}$$

Also, the initial conditions and the Dirichlet boundary conditions for $s \in (0,1)$ and $r \in [0, T]$, respectively are

$$\vartheta_1(s,0) = f_1(s), \quad \vartheta_2(s,0) = f_2(s), \tag{5}$$

and

$$\vartheta_1(0,r) = p_1(r), \quad \vartheta_2(0,r) = p_2(r), \vartheta_1(1,r) = q_1(r), \quad \vartheta_2(1,r) = q_2(r), \tag{6}$$

over specified condition, for $a \in (0,1)$, and $r \in [0, T]$ is

$$\vartheta_1(a,r) = w_1(r), \quad \vartheta_2(a,r) = w_2(r), \tag{7}$$

which T represents the final time. Furthermore, the functions $\vartheta_1(s,r)$ and $\vartheta_2(s,r)$ and the boundary conditions $p_1(r), p_2(r)$ are unknown and must be determined from over-specified data. One of the best subjects in many branches of engineering and science is inverse problems. In 1911, the first examples of inverse problems have been published by German mathematician Hermann Weyl; see [21]. Inverse problems are applied in remote sensing, geophysics, the heat capacity of solids, natural language processing, heat transfer, signal processing, oceanography, thermal conductivity, and many other fields; see [22] [23] [24] [25].

The fractional differential equations are employed in mathematics such as dynamical systems and control systems; in physics and engineering, for example, electrochemistry, physical phenomena, and fluid mechanics; and in economics, biological population models and social science such as food supplements; see [26] [27].

In this work, we solve one system of time-fractional inverse parabolic problems by using finite difference formula for time discretization and the extended cubic B-splines for spatial variables [28] [29]. In July 2006, Momani has published an article on the subject of the physical processes of acoustic waves through a gas-filled pipe, which is one of the first works in the fractional Burgers equation. Also, this equation is shock waves or a class of physical flows and the evolution of the scaled volume effect of gravity [30]. Furthermore, in [31], solved by using a A hybrid scheme a time fractional inverse parabolic problem were solved.

This article is organized into four sections. In section 2, a description of the extended cubic B-splines functions and procedure for implementation of the present method is illustrated . Also, we obtain numerical solutions for these problems. In section 3, we prove the convergence of our method. Also, we calculate the order of the method. In section 4, we consider two examples and solve them by finite difference method (FDM) and extended cubic B-splines collocation method (EXCBSM), cubic B-splines method (CBSM), trigonometric cubic B-spline method (TCBSM) and radial basis function method (RBFM). The conclusion of the work is shown in Section 5.

2 Description of the EXCBS functions

In this section, we define the basic function and discretization of problem 4, for this purpose, we have used the EXCBS that defined the interval [0,1], as [32]

$$E_i(s) = \frac{1}{24h^4}\begin{cases} 4h(1-\lambda)(s-s_{i-2})^3 + 3\lambda(s-s_{i-2})^4, & s \in [s_{i-2}, s_{i-1}], \\ (4-\lambda)h^4 + 12h^3(s-s_{i-1}) + 6h^2(2+\lambda)(s-s_{i-1})^2 & \\ -12h(s-s_{i-1})^3 - 3\lambda(s-s_{i-1})^4, & s \in [s_{i-1}, s_i], \\ (4-\lambda)h^4 + 12h^3(s_{i+1}-s) + 6h^2(2+\lambda)(s_{i+1}-s)^2 & \\ -12h(s_{i+1}-s)^3 - 3\lambda(s_{i+1}-s)^4, & s \in [s_i, s_{i+1}], \\ 4h(1-\lambda)(s_{i+2}-s)^3 + 3\lambda(s_{i+2}-s)^4, & s \in [s_{i+1}, s_{i+2}], \\ 0, & otherwise, \end{cases} \tag{8}$$

that $\lambda \in \mathbb{R}$. In this definition, we introduce a uniformly distributed set of nodes as $0 = s_0 < s_1 < \cdots < s_N = 1$ over the spatial domain [0,1], that the step length is $h = s_{i+1} - s_i$, for $i = 0,1,\dots,N-1$. Also, for using these bases we should extend the set of nodal points to $s_{-3} < s_{-2} < s_{-1} < s_0$, and $s_N < s_{N+1} < s_{N+2} < s_{N+3}$ [32].

In addition, the derivatives of $E_i(s)$ at the nodes s_i's are obtained as the following formulas [32]

$$E_m(s_i) = \begin{cases} \frac{8+\lambda}{12}, & if\ m = i, \\ \frac{4-\lambda}{24}, & if\ |m-i| = 1, \\ 0, & if\ |m-i| \geq 2, \end{cases} \tag{9}$$

$$E'_m(s_i)=\begin{cases} 0, & if\ m=i, \\ \dfrac{1}{2h}, & if\ m=i-1, \\ -\dfrac{1}{2h}, & if\ m=i+1, \\ 0, & if\ |m-i|\geq 2, \end{cases} \tag{10}$$

$$E''_m(s_i)=\begin{cases} -\dfrac{2+\lambda}{h^2}, & if\ m=i, \\ \dfrac{2+\lambda}{2h^2}, & if\ |m-i|=1, \\ 0, & if\ |m-i|\geq 2. \end{cases} \tag{11}$$

Definition 1. The Caputo fractional derivatives of order α concerning time t, is defined as [33]:

$$\frac{\partial^\alpha \vartheta_1(s,r)}{\partial r^\alpha}=\begin{cases} \dfrac{1}{\Gamma(\lambda-\alpha)}\displaystyle\int_0^r (r-x)^{\lambda-\alpha-1}\frac{\partial^\lambda \vartheta_1(s,x)}{\partial x^\lambda}dx, & \lambda-1<\alpha<\lambda, \\ \dfrac{\partial^\lambda \vartheta_1(s,r)}{\partial r^\lambda}, & \alpha=\lambda\in \mathrm{N}. \end{cases}$$

Where $h=\frac{1}{N}$ and $\tau=\frac{T}{M}$ are the step size in s and r axes respectively. We assume the discretization of the time-fractional derivative as follows [34]:

$$D_r^\alpha \vartheta_1(s_i,r_{k+1})\approx \frac{\tau^{-\alpha}}{\Gamma(2-\alpha)}[\vartheta_{1}{}_i^{k+1}-\vartheta_{1}{}_i^{k}+\sum_{l=1}^{k} a_l^\alpha(\vartheta_{1}{}_i^{k+1-l}-\vartheta_{1}{}_i^{k-l})], \tag{12}$$

that $\vartheta_{1}{}_i^k$ is a numerical approximation of $\vartheta_1(s_i,r_k)$, and $a_k^\alpha=(k+1)^{1-\alpha}-(k)^{1-\alpha}$ with $s_i=i\,h$ and $r_k=k\,\tau$ for $i=0,1,\dots,N$, and $k=0,1,\dots,M$, respectively. Also,

$$D_r^\alpha \vartheta_1(s,r)=\frac{\partial^\alpha \vartheta_1(s,r)}{\partial r^\alpha},$$

$$\frac{\partial^2}{\partial s^2}\vartheta_1(s_i,r_{k+1})\approx \frac{\vartheta_{1}{}_{i+1}^{k+1}-2\vartheta_{1}{}_i^{k+1}+\vartheta_{1}{}_{i-1}^{k+1}}{h^2},$$

$$\frac{\partial}{\partial s}\vartheta_1(s_i,r_{k+1})\approx \frac{\vartheta_{1}{}_{i+1}^{k+1}-\vartheta_{1}{}_{i-1}^{k+1}}{2h}.$$

First, for applying the proposed method (EXCBSM), we expressing $\vartheta_1(s,r)$ and $\vartheta_2(s,r)$, as

$$\nu_1(s,r)=\sum_{i=-1}^{N+1} C_i(r)E_i(s), \qquad \nu_2(s,r)=\sum_{i=-1}^{N+1} D_i(r)E_i(s). \tag{13}$$

We know that C_i and D_i are unknown time-dependent quantities which must be determined. Thus, by using conditions (4)-(7), we approximate solutions to the boundary value problem. For this purpose by extended cubic B-spline functions and their derivatives in (11), and substituting (13) into (4) at the points $s=s_m$, we have

$$(e_1C_{m-1}^{\alpha}+e_2C_m^{\alpha}+e_1C_{m+1}^{\alpha})=\{(e_4C_{m-1}+e_5C_m+e_4C_{m+1})+ 2e_3(e_1C_{m-1}+e_2C_m+e_1C_{m+1})(-C_{m+1}+C_{m-1})-e_3(e_1C_{m-1}+e_2C_m+e_1C_{m+1})(-D_{m+1}+D_{m-1})-e_3(e_1D_{m-1}+e_2D_m+e_1D_{m+1})(-C_{m+1}+C_{m-1})\}+F(s_m,r), \tag{14}$$

and

$$(e_1D_{m-1}^{\beta}+e_2D_m^{\beta}+e_1D_{m+1}^{\beta})=\{(e_4D_{m-1}+e_5D_m+e_4D_{m+1})+ 2e_3(e_1D_{m-1}+e_2D_m+e_1D_{m+1})(-D_{m+1}+D_{m-1})-e_3(e_1C_{m-1}+e_2C_m^{n}+ e_1C_{m+1})(-D_{m+1}+D_{m-1})-e_3(e_1D_{m-1}+e_2D_m+e_1D_{m+1})(-C_{m+1}+C_{m-1})\}+G(s_m,r), \tag{15}$$

where

$$e_1=\frac{4-\lambda}{24},\quad e_2=\frac{8+\lambda}{12},\quad e_3=\frac{1}{2h},\quad e_4=\frac{2+\lambda}{2h^2},$$

$$e_5=-\frac{2+\lambda}{h^2},\quad C_m^{\alpha}=\frac{\partial^{\alpha}C_m(r)}{\partial r^{\alpha}},\quad D_m^{\beta}=\frac{\partial^{\alpha}D_m(r)}{\partial r^{\beta}}. \tag{16}$$

Using the Caputo derivative formulation for $C_m^{\alpha},, D_m^{\beta}$, we discretize the time-fractional derivative as (12) at $r=r_{k+1}$. After simplifying, we have

$$(e_1C_{m-1}^{k+1}+e_2C_m^{k+1}+e_1C_{m+1}^{k+1})=\gamma_{\alpha}\{(e_4C_{m-1}^{k}+e_5C_m^{k}+e_4C_{m+1}^{k})+ 2e_3(e_1C_{m-1}^{k}+e_2C_m^{k}+e_1C_{m+1}^{k})(-C_{m+1}^{k}+C_{m-1}^{k})-e_3(e_1C_{m-1}^{k}+e_2C_m^{k}+e_1C_{m+1}^{k})(-D_{m+1}^{k}+D_{m-1}^{k})-e_3(e_1D_{m-1}^{k}+e_2D_m^{k}+e_1D_{m+1}^{k})(-C_{m+1}^{k}+C_{m-1}^{k})\}+\gamma_{\alpha}F(s_m,r_{k+1})+(1-a_1^{\alpha})(e_1C_{m-1}^{k}+e_2C_m^{k}+e_1C_{m+1}^{k})+a_k^{\alpha}\vartheta_1(s_m,0)+\sum_{j=1}^{k-1}(a_j^{\alpha}-a_{j+1}^{\alpha})(e_1C_{m-1}^{k-j}+e_2C_m^{k-j}+ e_1C_{m+1}^{k-j}) \tag{17}$$

and

$$(e_1D_{m-1}^{k+1}+e_2D_m^{k+1}+e_1D_{m+1}^{k+1})=\gamma_{\beta}\{(e_4D_{m-1}^{k}+e_5D_m^{k}+e_4D_{m+1}^{k})+ 2e_3(e_1D_{m-1}^{k}+e_2D_m^{k}+e_1D_{m+1}^{k})(-D_{m+1}^{k}+D_{m-1}^{k})-e_3(e_1D_{m-1}^{k}+e_2D_m^{k}+e_1D_{m+1}^{k})(-C_{m+1}^{k}+C_{m-1}^{k})-e_3(e_1C_{m-1}^{k}+e_2C_m^{k}+e_1C_{m+1}^{k})(-D_{m+1}^{k}+D_{m-1}^{k})\}+\gamma_{\beta}G(s_m,r_{k+1})+(1-a_1^{\beta})(e_1D_{m-1}^{k}+e_2D_m^{k}+e_1D_{m+1}^{k})+a_k^{\beta}\vartheta_2(s_m,0)+\sum_{j=1}^{k-1}(a_j^{\beta}-a_{j+1}^{\beta})(e_1D_{m-1}^{k-j}+e_2D_m^{k-j}+ e_1D_{m+1}^{k-j}). \tag{18}$$

Where

$$\gamma_{\alpha}=\Gamma(2-\alpha)\tau^{\alpha},,\gamma_{\beta}=\Gamma(2-\beta)\tau^{\beta}.$$

For simplifying, we set

$$X_m^{k}=\gamma_{\alpha}\{(e_4C_{m-1}^{k}+e_5C_m^{k}+e_4C_{m+1}^{k})+2,e_3(e_1C_{m-1}^{k}+e_2C_m^{k}+e_1C_{m+1}^{k})(-C_{m+1}^{k}+C_{m-1}^{k})-e_3(e_1C_{m-1}^{k}+e_2C_m^{k}+e_1C_{m+1}^{k})(-D_{m+1}^{k}+D_{m-1}^{k})-e_3(e_1D_{m-1}^{k}+e_2D_m^{k}+ e_1D_{m+1}^{k})(-C_{m+1}^{k}+C_{m-1}^{k})\}+\gamma_{\alpha}F(s_m,r_{k+1})+(1-a_1^{\alpha})(e_1C_{m-1}^{k}+e_2C_m^{k}+e_1C_{m+1}^{k})+ a_k^{\alpha}\vartheta_1(s_m,0)+\sum_{j=1}^{k-1}(a_j^{\alpha}-a_{j+1}^{\alpha})(e_1C_{m-1}^{k-j}+e_2C_m^{k-j}+e_1C_{m+1}^{k-j}),$$

and

$$Y_m^k = \gamma_\beta\{(e_4 D_{m-1}^k + e_5 D_m^k + e_4 D_{m+1}^k) + 2e_3(e_1 D_{m-1}^k + e_2 D_m^k + e_1 D_{m+1}^k)(-D_{m+1}^k + D_{m-1}^k) - e_3(e_1 D_{m-1}^k + e_2 D_m^k + e_1 D_{m+1}^k)(-C_{m+1}^k + C_{m-1}^k) - e_3(e_1 C_{m-1}^k + e_2 C_m^k + e_1 C_{m+1}^k)(-D_{m+1}^k + D_{m-1}^k)\} + \gamma_\beta G(s_m, r_{k+1}) + (1 - a_1^\beta)(e_1 D_{m-1}^k + e_2 D_m^k + e_1 D_{m+1}^k) + a_k^\beta \vartheta_2(s_m, 0) + \sum_{j=1}^{k-1} (a_j^\beta - a_{j+1}^\beta)(e_1 D_{m-1}^{k-j} + e_2 D_m^{k-j} + e_1 D_{m+1}^{k-j}).$$

Thus,

$$\left(e_1 C_{m-1}^{k+1} + e_2 C_m^{k+1} + e_1 C_{m+1}^{k+1}\right) = \mathbf{X}_m^k,$$

$$(e_1 D_{m-1}^{k+1} + e_2 D_m^{k+1} + e_1 D_{m+1}^{k+1}) = \mathbf{Y}_m^k. \tag{19}$$

But, to obtain a unique solution of the system of (19), consisting of $2(N+1)$ equations in $2(N+3)$ unknown coefficients, we need four additional constraints. Thus, by imposing the boundary conditions (6) and the specified condition (7), so, for $s_z = zh = a, , 1 \le z \le N - 1$, we have

$$\begin{gathered}
\vartheta_1(a, r_{k+1}) = \nu_1(a, r_{k+1}) = e_1 C_{z-1}^{k+1} + e_2 C_z^{k+1} + e_1 C_{z+1}^{k+1} = w_1(r_{k+1}), \\
\vartheta_1(s_N, r_{k+1}) = \nu_1(s_N, r_{k+1}) = e_1 C_{N-1}^{k+1} + e_2 C_N^{k+1} + e_1 C_{N+1}^{k+1} = q_1(r_{k+1}), \\
\vartheta_2(a, r_{k+1}) = \nu_2(a, r_{k+1}) = e_1 D_{z-1}^{k+1} + e_2 D_z^{k+1} + e_1 D_{z+1}^{k+1} = w_2(r_{k+1}), \\
\vartheta_2(s_N, r_{k+1}) = \nu_2(s_N, r_{k+1}) = e_1 D_{N-1}^{k+1} + e_2 D_N^{k+1} + e_1 D_{N+1}^{k+1} = q_2(r_{k+1}),
\end{gathered}$$

or

$$AX = B, \tag{20}$$

where

$$X = [C_{-1}^{k+1}, C_0^{k+1}, C_1^{k+1}, \dots, C_{N+1}^{k+1}, D_{-1}^{k+1}, D_0^{k+1}, D_1^{k+1}, \dots, D_{N+1}^{k+1}]^T,$$

$$B = [\mathbf{X}_{-1}^k, \mathbf{X}_0^k, \mathbf{X}_1^k, \dots, \mathbf{X}_{N+1}^k, \mathbf{Y}_{-1}^k, \mathbf{Y}_0^k, \mathbf{Y}_1^k \dots, \mathbf{Y}_{N+1}^k]^T,$$

where

$$\mathbf{X}_{-1}^k = \omega_1(r_{k+1}), \ \ \mathbf{X}_{N+1}^k = q_1(r_{k+1}), \ \ \mathbf{Y}_{-1}^k = \omega_2(r_{k+1}), \ \ \mathbf{Y}_{N+1}^k = q_2(r_{k+1}),$$

and the matrix of $A_{2(N+3)\times 2(N+3)}$ is

$$A = \begin{pmatrix} M & | & O \\ - & - & - \\ O & | & M \end{pmatrix}. \tag{21}$$

The matrix O is a zero matrix, and the matrix of $M_{(N+3)\times(N+3)}$ is defined as follows

$$M = \begin{pmatrix}
0 & \dots & 0 & e_1 & e_2 & e_1 & 0 & \dots & 0 \\
e_1 & e_2 & e_1 & 0 & 0 & 0 & 0 & \dots & 0 \\
0 & e_1 & e_2 & e_1 & 0 & 0 & 0 & \dots & 0 \\
\vdots & & \dots & \dots & \dots & \dots & \dots & \dots & \vdots \\
\vdots & & & \dots & \dots & \dots & & & \vdots \\
\vdots & & & & \dots & \dots & \dots & & \vdots \\
\vdots & & & & & \dots & \dots & \dots & \vdots \\
0 & 0 & 0 & 0 & \dots & 0 & e_1 & e_2 & e_1 \\
0 & 0 & 0 & 0 & \dots & 0 & e_1 & e_2 & e_1
\end{pmatrix},$$

that

$$M[1, z+1] = e_1, , M[1, z+2] = e_2, , M[1, z+3] = e_1.$$

With solving (20) by the Tikhonov regularization method, the coefficients C_j and D_j are obtained and with these coefficients, we can obtain the approximation solutions.

$$p_1(r^k) = e_1 C_{-1}^k + e_2 C_0^k + e_1 C_1^k, \qquad k = 0, 1, \dots,$$

$$p_2(r^k) = e_1 D_{-1}^k + e_2 D_0^k + e_1 D_1^k, \qquad k = 0, 1, \dots,$$

$$\nu_1(s_j, r^k) = e_1 C_{j-1}^k + e_1 C_j^k + e_1 C_{j+1}^k, \qquad k = 0, 1, \dots, \qquad j = 0, 1, \dots, , N,$$

$$\nu_2(s_j, r^k) = e_1 D_{j-1}^k + e_2 D_j^k + e_1 D_{j+1}^k, \qquad k = 0, 1, \dots, \qquad j = 0, 1, \dots, , N.$$

By using condition (5) the initial vectors C^0 and D^0 are calculated. Also, the following expressions the boundary and over specified conditions such as (6) and (7) by the following expressions are found.

$$\nu_1(a, 0) = e_1 C_{z-1}^0 + e_2 C_z^0 + e_1 C_{z+1}^0 = w_1(0),$$

$$\nu_2(a, 0) = e_1 D_{z-1}^0 + e_2 D_z^0 + e_1 D_{z+1}^0 = w_2(0),$$

$$\nu_1(s_j, 0) = e_1 C_{j-1}^0 + e_2 C_j^0 + e_1 C_{j+1}^0 = f_1(s_j), \qquad , 0 \le j \le N,$$

$$\nu_2(s_j, 0) = e_1 D_{j-1}^0 + e_2 D_j^0 + e_1 D_{j+1}^0 = f_2(s_j), \qquad , 0 \le j \le N,$$

$$\nu_1(s_N, 0) = e_1 C_{N-1}^0 + e_2 C_N^0 + e_1 C_{N+1}^0 = q_1(0),$$

$$\nu_2(s_N, 0) = e_1 D_{N-1}^0 + e_2 D_N^0 + e_1 D_{N+1}^0 = q_2(0),$$

or

$$A^* X^0 = B^*, \tag{22}$$

where the matrix of singular and ill-posed A^* is equal with A and

$$X^0 = [C_{-1}^0, C_0^0, C_1^0, \dots, C_{N+1}^0, D_{-1}^0, D_0^0, D_1^0, \dots, D_{N+1}^0]^T,$$
$$B^* = [\omega_1(0), f_1(s_0), f_1(s_1), \dots, f_1(s_N), q_1(0), \omega_2(0), f_2(s_0), f_2(s_1), \dots, f_2(s_N), q_2(0)]^T.$$

To solve and obtain the solution of the linear algebraic equations (20) and (22) and then to estimate of X^0 we can use the Tikhonov regularization method as [35]

$$F_\sigma(X) = \| A\, X - B \|_2^2 + \sigma \| R^{(z)}, X, \|_2^2,$$

$$F_\sigma(X^0) = \| A\, X^0 - B^* \|_2^2 + \sigma \| R^{(z)}, X^0 \|_2^2.$$

If the order of the Tikhonov regularization method is one or two, thus the matrix $R^{(z)}$, for $z = 1, 2$, and $M = 2(N+3)$ is given by

$$R^{(1)}=\begin{pmatrix}-1 & 1 & 0 & \dots & 0 & 0 & 0\\ 0 & -1 & 1 & 0 & \dots & 0 & 0\\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots\\ 0 & 0 & \dots & 0 & -1 & 1 & 0\\ 0 & 0 & 0 & \dots & 0 & -1 & 1\end{pmatrix}\in\mathbb{R}^{(M-1)\times(M)},$$

$$R^{(2)}=\begin{pmatrix}1 & -2 & 1 & \dots & 0 & 0 & 0\\ 0 & 1 & -2 & 1 & \dots & 0 & 0\\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots\\ 0 & 0 & \dots & 1 & -2 & 1 & 0\\ 0 & 0 & 0 & \dots & 1 & -2 & 1\end{pmatrix}\in\mathbb{R}^{(M-2)\times(M)}.$$

Therefore, by using the generalized cross-validation (GCV) scheme we determine a suitable value of σ, [36] [37] [38] .

The Tikhonov regularized solutions of the systems (20) and (22) are given by

$$X_\sigma=[A^T\,A+\sigma(R^{(z)})^T,R^{(z)}]^{-1}A^TB, \tag{23}$$

$$X_\sigma^0=[A^T\,A+\sigma(R^{(z)})^T,R^{(z)}]^{-1}A^TB^*. \tag{24}$$

3 Convergence analysis

In this section, we have proved the convergence and calculated the order of the method. For this purpose, we prove three theorems.

Theorem 1. The collocation approximations $\nu_1(s,r_k)$ and $\nu_2(s,r_k)$ for the solutions $\vartheta_1(s,r_k)$ and $\vartheta_2(s,r_k)$ of the inverse problem $(4)-(47)$ satisfy the following error estimate

$$\|\,(\vartheta_1(s,r_k)-\nu_1(s,r_k),\vartheta_2(s,r_k)-\nu_2(s,r_k))\,\|_\infty\leq\mu h^2, \tag{25}$$

for sufficiently small h (i.e. for sufficiently large N) where μ is a positive constant.

Proof. Let $\vartheta_1(s,r)$ and $\vartheta_2(s,r)$ be the exact solutions to the problem (4) with the boundary conditions, initial conditions, over specific conditions, and also

$$\nu_1(s,r)=\sum_{i=-1}^{N+1}C_i(r)E_i(s),\quad \nu_2(s,r)=\sum_{i=-1}^{N+1}D_i(r)E_i(t),$$

be the extended B-spline collocation approximations to $\vartheta_1(s,r)$ and $\vartheta_2(s,r)$. Due to round-off errors in computations, we assume that $\hat{\nu}_1(s,r)$ and $\hat{\nu}_2(s,r)$ be the computed splines for $\nu_1(s,r)$ and $\nu_2(s,r)$ so that

$$\hat{\nu}_1(s,r)=\sum_{j=-1}^{N+1}\hat{C}_j(r)E_j(s),\quad \hat{\nu}_2(s,r)=\sum_{j=-1}^{N+1}\widehat{D}_j(r)E_j(s).$$

To estimate the errors

$$\|\,\vartheta_1(s,r)-\nu_1(s,r)\,\|_\infty,\quad \|\,\vartheta_2(s,r)-\nu_2(s,r)\,\|_\infty,$$

we must estimate the errors

$$\|\,\vartheta_1(s,r)-\hat{\nu}_1(s,r)\,\|_\infty,\quad \|\,\hat{\nu}_1(s,r)-\nu_1(s,r)\,\|_\infty,$$

and

$$\parallel \vartheta_2(s,r) - \hat{v}_2(s,r) \parallel_\infty, \quad \parallel \hat{v}_2(s,r) - v_2(s,r) \parallel_\infty,$$

from (20) for $\hat{v}_1$ and $\hat{v}_2$ we have

$$A\hat{X} = \hat{B}, \tag{26}$$

where

$$\hat{X} = [\hat{C}_{-1}^{k+1}, \hat{C}_0^{k+1}, \hat{C}_1^{k+1}, \dots, \hat{C}_{N+1}^{k+1}, \hat{v}_{-1}^{k+1}, \hat{v}_0^{k+1}, \hat{v}_1^{k+1}, \dots, \hat{v}_{N+1}^{k+1}]^T,$$

$$\hat{B} = [\mathbf{X}_{-1}^k, \hat{\mathbf{X}}_0^k, \hat{\mathbf{X}}_1^k, \dots, \hat{\mathbf{X}}_N^k, \mathbf{X}_{N+1}^k, \mathbf{Y}_{-1}^k, \hat{\mathbf{Y}}_0^k, \hat{\mathbf{Y}}_1^k, \dots, \hat{\mathbf{Y}}_N^k, \mathbf{Y}_{N+1}^k]^T,$$

with

$$\mathbf{X}_{-1}^k = w_1(r_k), \ \ \mathbf{X}_{N+1}^k = q_1(r_k), \ \ \mathbf{Y}_{-1}^k = w_2(r_k), \ \ \mathbf{Y}_{N+1}^k = q_2(r_k).$$

By subtracting (20) from (26), we have

$$A(X - \hat{X}) = (B - \hat{B}), \tag{27}$$

where

$$B - \hat{B} = [0, \mathbf{X}_0^k - \hat{\mathbf{X}}_0^k, \mathbf{X}_1^k - \hat{\mathbf{X}}_1^k, \dots, \mathbf{X}_N^k - \hat{\mathbf{X}}_N^k, 0, 0, \mathbf{Y}_0^k - \hat{\mathbf{Y}}_0^k, \mathbf{Y}_1^k - \hat{\mathbf{Y}}_1^k, \dots, \mathbf{Y}_N^k - \hat{\mathbf{Y}}_N^k, 0], \tag{28}$$

and for every $0 \le m \le N$,

$$\mathbf{X}_m^k = \gamma_\alpha[v_{1''}(s_m,r_k) + 2v_1(s_m,r_k)v_{1'}(s_m,r_k) - v_1(s_m,r_k)v_2(s_m,r_k))_s\,]$$
$$+a_k^\alpha v_1(s_m,0) + (1 - a_1^\alpha)v_1(s_m,r_k) + \sum_{j=1}^{k-1} (a_j^\alpha - a_{j+1}^\alpha)\vartheta_1(s_m,r_{k-j}),$$

and

$$\hat{\mathbf{X}}_m^k = \gamma_\alpha\big[\hat{v}_1''(s_m,r_k) + 2\hat{v}_1(s_m,r_k)\widehat{v'}_1(s_m,r_k) - \hat{v}_1(s_m,r_k)\hat{v}_2(s_m,r_k))_x\,\big]$$
$$+a_k^\alpha \hat{\vartheta}_1(s_m,0) + (1 - a_1^\alpha)\hat{v}_1(s_m,r_k) + \sum_{j=1}^{k-1} (a_j^\alpha - a_{j+1}^\alpha)\hat{v}_1(s_m,r_{k-j}),$$

and

$$\mathbf{Y}_m^k = \gamma_\beta[v''_2(s_m,r_k) + 2v_2(s_m,r_k)d'_2(s_m,r_k) - (v_1(s_m,r_k)v_2(s_m,r_k))_s\,]$$
$$+ a_k^\beta \vartheta_2(s_m,0) + (1 - a_1^\beta)v_2(s_m,r_k) + \sum_{j=1}^{k-1} (a_j^\beta - a_{j+1}^\beta)v_2(s_m,r_{k-j}),$$

and

$$\hat{\mathbf{Y}}_m^k = \gamma_\beta\big[\widehat{v'}'_2(s_m,r_k) + 2\hat{v}_2(s_m,r_k)\widehat{v'}_2(s_m,r_k) - \hat{v}_1(s_m,r_k)\hat{v}_2(s_m,r_k))_x\,\big] +$$
$$a_k^\beta \hat{v}_2(s_m,0) + (1 - a_1^\beta)\hat{v}_2(s_m,r_k) + \textstyle\sum_{j=1}^{k-1} (a_j^\beta - a_{j+1}^\beta)\hat{v}_2(s_m,r_{k-j}).$$

So

$$|\mathbf{X}_m^k - \hat{\mathbf{X}}_m^k| = |\gamma_\alpha[(v''_1(s_m,r_k) - \widehat{v}''_1(s_m,r_k)) - 2v_1(s_m,r_k)v'_1(s_m,r_k) + 2\hat{v}_1(s_m,r_k)\widehat{v'}_1(s_m,r_k) - (v_1(s_m,r_k)v_2(s_m,r_k))_s + (\hat{v}_1(s_m,r_k)\hat{v}_2(s_m,r_k))_s] + a_k^\alpha(v_1(s_m,0) - \hat{v}_1(s_m,0)) + (1 - a_1^\beta)(v_1(s_m,r_k) - \hat{v}_1(s_m,r_k)) + \textstyle\sum_{j=1}^{k-1} (a_j^\alpha - a_{j+1}^\alpha)(v_1(s_m,r_{k-j}) - \hat{v}_1(s_m,r_{k-j}))|.$$

By using the Cauchy-Schwarz inequality, we have

$$|\mathbf{X}_m^k - \hat{\mathbf{X}}_m^k| \le \gamma_\alpha|(v''_1(s_m,r_k) - \hat{v}''_1(s_m,r_k))| + |a_k^\alpha||(v_1(s_m,0) -$$

$\hat{\nu}_1(s_m,0))| + |1-a_1^\alpha||\nu_1(s_m,r_k)-\hat{\nu}_1(s_m,r_k)| + \sum_{j=1}^{k-1} |a_j^\alpha - a_{j+1}^\alpha||\nu_1(s_m,r_{k-j}) - \hat{\nu}_1(s_m,r_{k-j})| + \gamma_\alpha(\mathrm{I}),$

that

$$(\mathrm{I}) = |-2\nu_1(s_m,r_k)\nu'_1(s_m,r_k) + 2\hat{\nu}_1(s_m,r_k)\widehat{\nu'}_1(s_m,r_k) - (\nu_1(s_m,r_k)\nu_2(s_m,r_k))_x + (\hat{\nu}_1(s_m,r_k)\hat{\nu}_2(s_m,r_k))_x|,$$

$$(I) = |[(\nu_1^2(s_m,r_k)-\hat{\nu}_1^2(s_m,r_k)) + (\nu_1(s_m,r_k)\nu_2(s_m,r_k)) - (\hat{\nu}_1(s_m,r_k)\hat{\nu}_2(s_m,r_k))]_s| = |[(\nu_1(s_m,r_k))(\nu_1(s_m,r_k)+\nu_2(s_m,r_k)) - (\hat{\nu}_1(s_m,r_k))(\hat{\nu}_1(s_m,r_k)+\hat{\nu}_2(s_m,r_k))]_s| = |[(\nu_1(s_m,r_k))(\nu_1(s_m,r_k)-\hat{\nu}_1(s_m,r_k)) + (\nu_1(s_m,r_k))(\nu_2(s_m,r_k)-\hat{\nu}_2(s_m,r_k)) + (\nu_1(s_m,r_k)-\hat{\nu}_1(s_m,r_k))(\hat{\nu}_1(s_m,r_k)+\hat{\nu}_2(s_m,r_k))]_s| = |[2(\nu_1(s_m,r_k))(\nu_1(s_m,r_k)-\hat{\nu}_1(s_m,r_k)) + (\nu_1(s_m,r_k))(\nu_2(s_m,r_k)-\hat{\nu}_2(s_m,r_k)) - (\nu_1(s_m,r_k)-\hat{\nu}_1(s_m,r_k))^2 + (\nu_2(s_m,r_k))(\nu_1(s_m,r_k)-\hat{\nu}_1(s_m,r_k)) - (\nu_2(s_m,r_k)-\hat{\nu}_2(s_m,r_k))(\nu_1(s_m,r_k)-\hat{\nu}_1(s_m,r_k))]_s|,$$

then, after simplifying, we have

$$|\mathbf{X}_m^k - \widehat{\mathbf{X}}_m^k| \le \gamma_\alpha|(\nu''_1(s_m,r_k)-\widehat{\nu''}_1(s_m,r_k))| + |a_k^\alpha||(\nu_1(s_m,0)-\hat{\nu}_1(s_m,0))| + |1-a_1^\alpha||\nu_1(s_m,r_k)-\hat{\nu}_1(s_m,r_k)| + \sum_{j=1}^{k-1}|a_j^\alpha - a_{j+1}^\alpha||\nu_1(s_m,r_{k-j})-\hat{\nu}_1(s_m,r_{k-j})| + \gamma_\alpha[2|\nu'_1(s_m,r_k)||\nu_1(s_m,r_k)-\hat{\nu}_1(s_m,r_k)| + 2|\nu_1(s_m,r_k)||\nu'_1(s_m,r_k)-\widehat{\nu'}_1(s_m,r_k)| + |\nu'_1(s_m,r_k)||\nu_2(s_m,r_k)-\hat{\nu}_2(s_m,r_k)| + |\nu_1(s_m,r_k)||\nu'_2(s_m,r_k)-\hat{\nu}'_2(s_m,r_k)| + 2|\nu_1(s_m,r_k)-\hat{\nu}_1(s_m,r_k)||\nu'_1(x_m,r_k)-\hat{\nu}'_1(x_m,r_k)| + |\nu'_2(s_m,r_k)||\nu_1(s_m,r_k)-\hat{\nu}_1(s_m,r_k)| + |\nu_2(s_m,r_k)||\nu'_1(s_m,r_k)-\hat{\nu}'_1(s_m,r_k)| + |\nu'_2(s_m,r_k)-\hat{\nu}'_2(s_m,r_k)||\nu_1(s_m,r_k)-\hat{\nu}_1(s_m,r_k)| + |\nu_2(s_m,r_k)-\hat{\nu}_2(s_m,r_k)||\nu'_1(s_m,r_k)-\hat{\nu}'_1(s_m,r_k)|].$$

By using the theorem (2.4.3.3) by Stoer and Bulirsch [39], we have

$$|\mathbf{X}_m^k - \widehat{\mathbf{X}}_m^k| \le \gamma_\alpha(\lambda_2, L_{\vartheta_1}, h^2) + (1-a_1^\alpha)\lambda_0, L_{\vartheta_1}, h^4 + a_0^\alpha\lambda_0, L_{\vartheta_1}, h^4 + (\lambda_0, L_{\vartheta_1}, h^4)\sum_{j=1}^{n-1}|(a_j^\alpha - a_{j+1}^\alpha)| + \gamma_\alpha((2M_{\vartheta_1}\lambda_0, L_{\vartheta_1}, h^4) + 2M_{\vartheta_1}(\lambda_1, L_{\vartheta_1}, h^3) + M_{\vartheta_1}(\lambda_0, L_{\vartheta_2}, h^4) + M_{\vartheta_1}(\lambda_1, L_{\vartheta_2}, h^3) + 2(\lambda_0, L_{\vartheta_1}, h^4)(\lambda_1, L_{\vartheta_2}, h^3) + M_{\vartheta_2}(\lambda_0, L_{\vartheta_1}, h^4) + M_{\vartheta_2}(\lambda_1, L_{\vartheta_1}, h^3) + (\lambda_1, L_{\vartheta_2}, h^3), (\lambda_0, L_{\vartheta_1}, h^4) + (\lambda_0, L_{\vartheta_2}, h^4), (\lambda_1, L_{\vartheta_1}, h^3)).$$

After simplifying, we get

$$|\mathbf{X}_m^k - \widehat{\mathbf{X}}_m^k| \le h^2(\gamma_\alpha(\lambda_2, L_{\vartheta_1},) + (1-a_1^\alpha)\lambda_0, L_{\vartheta_1}, h^2 + a_0^\alpha\lambda_0, L_{\vartheta_1}, h^2 + (\lambda_0, L_{\vartheta_1}, h^2)\sum_{j=1}^{n-1}|(a_j^\alpha - a_{j+1}^\alpha)| + \gamma_\alpha((2M_{\vartheta_1}\lambda_0, L_{\vartheta_1}, h^2) + 2M_{\vartheta_1}(\lambda_1, L_{\vartheta_1}, h^2) + M_{\vartheta_1}(\lambda_0, L_{\vartheta_2}, h^2) + M_{\vartheta_1}(\lambda_1, L_{\vartheta_2}, h^1) + 2h^5(\lambda_0, L_{\vartheta_1})(\lambda_1, L_{\vartheta_2}) + M_{\vartheta_2}(\lambda_0, L_{\vartheta_1}, h^2) + M_{\vartheta_2}(\lambda_1, L_{\vartheta_1}, h^1) + h^5(\lambda_1, L_{\vartheta_2}), (\lambda_0, L_{\vartheta_1}) + h^5(\lambda_0, L_{\vartheta_2}), (\lambda_1, L_{\vartheta_1}))). \quad (29)$$

We can rewrite (29) as follows

$$|\mathbf{X}_m^k - \widehat{\mathbf{X}}_m^k| \le h^2 M_1, \quad (30)$$

where

$$M_1 = (\gamma_\alpha(\lambda_2, L_{\vartheta_1},) + (1-a_1^\alpha)\lambda_0, L_{\vartheta_1}, h^2 + a_0^\alpha\lambda_0, L_{\vartheta_1}, h^2 + (\lambda_0, L_{\vartheta_1}, h^2)\sum_{j=1}^{n-1}|(a_j^\alpha - a_{j+1}^\alpha)| + \gamma_\alpha((2M_{\vartheta_1}\lambda_0, L_{\vartheta_1}, h^2) + 2M_{\vartheta_1}(\lambda_1, L_{\vartheta_1}, h^2) + M_{\vartheta_1}(\lambda_0, L_{\vartheta_2}, h^2) + M_{\vartheta_1}(\lambda_1, L_{\vartheta_2}, h^1) + 2h^5(\lambda_0, L_{\vartheta_1})(\lambda_1, L_{\vartheta_2}) + M_{\vartheta_2}(\lambda_0, L_{\vartheta_1}, h^2) + M_{\vartheta_2}(\lambda_1, L_{\vartheta_1}, h^1) + h^5(\lambda_1, L_{\vartheta_2}), (\lambda_0, L_{\vartheta_1}) + h^5(\lambda_0, L_{\vartheta_2}), (\lambda_1, L_{\vartheta_1}))).$$

Similar results can be obtained for $\mathbf{Y}_m^k - \widehat{\mathbf{Y}}_m^k$, i.e.

$$|\mathbf{Y}_m^k - \widehat{\mathbf{Y}}_m^k| \leq h^2 M_2. \tag{31}$$

Setting $\mathbf{M} = \max\{M_1, M_2\}$, we have

$$|\mathbf{X}_m^k - \widehat{\mathbf{X}}_m^k| \leq \mathbf{M}, h^2, \tag{32}$$

$$|\mathbf{Y}_m^k - \widehat{\mathbf{Y}}_m^k| \leq \mathbf{M}\, h^2. \tag{33}$$

From (28), (32), and (33), it is deduced that

$$\| B - \hat{B} \|_\infty \leq \mathbf{M}\, h^2. \tag{34}$$

Since, the matrix A in (27) is an ill-posed matrix, from the Tikhonov regularized solution (23), we get

$$(X - \hat{X}) = [A^T A + \sigma (R^{(z)})^T, R^{(z)}]^{-1} A^T, (B - \hat{B}).$$

Using the relation (34) and taking the infinity norm, we find

$$\| X - \hat{X} \|_\infty \leq \| (A^T A + \sigma (R^{(z)})^T, R^{(z)})^{-1} A^T \|_\infty, \| B - \hat{B} \|_\infty \leq \| (A^T A + \sigma (R^{(z)})^T, R^{(z)})^{-1} A^T \|_\infty, \mathbf{M}, h^2 \leq \mathbf{M}_1 h^2, \tag{35}$$

where

$$\mathbf{M}_1 = \| (A^T A + \sigma (R^{(z)})^T, R^{(z)})^{-1} A^T \|_\infty, \mathbf{M}.$$

Now, we compute $\| (\vartheta_1(s, r_k) - \nu_1(s, r_k), \vartheta_2(s, r_k) - \nu_2(s, r_k)) \|_\infty$ as the following

$$\| (\vartheta_1(s, r_k) - \nu_1(s, r_k), \vartheta_2(s, r_k) - \nu_2(s, r_k)) \|_\infty = \| \vartheta_1(s, r_k) - \nu_1(s, r_k) \|_\infty + \| \vartheta_2(s, r_k) - \nu_2(s, r_k) \|_\infty \leq \| \vartheta_1(s, r_k) - \hat{\nu}_1(s, r_k) \|_\infty + \| \hat{\nu}_1(s, r_k) - \vartheta_1(s, r_k) \|_\infty + \| \vartheta_2(s, r_k) - \hat{\nu}_2(s, r_k) \|_\infty + \| \hat{\nu}_2(s, r_k) - \vartheta_2(s, r_k) \|_\infty .$$

such that $\nu_1(s, r_k) - \hat{\nu}_1(s, r_k) = \sum_{i=-1}^{N+1} (C_i^k - \hat{C}_i^k) E_i(s)$,

and

$$|\nu_1(s_m, r_k) - \hat{\nu}_1(s_m, r_k)| \leq \max_{-1 \leq i \leq N+1} \{|C_i^k - \hat{C}_i^k|\} \sum_{i=-1}^{N+1} |E_i(s_m)|,\ 0 \leq m \leq N,$$

and

$$\nu_2(s, r_k) - \hat{\nu}_2(s, r_k) = \sum_{i=-1}^{N+1} (D_i^k - \widehat{D}_i^k) E_i(s),$$

thus

$$|\nu_2(s_m, r_k) - \hat{\nu}_2(s_m, r_k)| \leq \max_{-1 \leq i \leq N+1} \{|D_i^k - \widehat{D}_i^k|\} \sum_{i=-1}^{N+1} |E_i(s_m)|,\ 0 \leq m \leq N.$$

By using the values of $E_i(s_m)$'s given in Section 2, one can easily see that

$$\sum_{i=-1}^{N+1} |E_i(s_m)| \leq \frac{7}{4}, \quad 0 \leq m \leq N,$$

therefore

$$\| \nu_1(s_m, r_k) - \hat{\nu}_1(s_m, r_k) \|_\infty \leq \frac{7}{4} \mathbf{M}_1 h^2,\ \| \nu_2(s_m, r_k) - \hat{\nu}_2(s_m, r_k) \|_\infty \leq \frac{7}{4} \mathbf{M}_1 h^2. \tag{36}$$

So, according to (25) and (36), we obtain

$$\| (\vartheta_1(s,r_k) - \nu_1(s,r_k), \vartheta_2(s,r_k) - \nu_2(s,r_k)) \|_\infty \le 2\lambda_0 L h^4 + \frac{7}{4}\mathbf{M}_1 h^2 + \frac{7}{4}\mathbf{M}_1 h^2$$

since $h \in (0,1)$ then $h^4 \le h^2$ then

$$\| (\vartheta_1(s,r_k) - \nu_1(s,r_k), \vartheta_2(s,r_k) - \nu_2(s,r_k)) \|_\infty \le 2\lambda_0 L h^4 + \frac{7}{4}\mathbf{M}_1 h^2 + \frac{7}{4}\mathbf{M}_1 h^2 \le 2\lambda_0 L h^2 + \frac{7}{2}\mathbf{M}_1 h^2 = h^2(2\lambda_0 L + \frac{7}{2}\mathbf{M}_1)،$$

Setting $\mu = 2\,\lambda_0 L, + \frac{7}{2}\mathbf{M}_1,$

thus

$$\| \left(\vartheta_1(s,r_k) - \nu_1(s,r_k), \vartheta_2(s,r_k) - \nu_2(s,r_k)\right) \|_\infty \le \mu, h^2. \quad \blacksquare$$

In addition, we calculate the time discretization process of Eq. (19). For this purpose, we discretize the system of (4) in the time variable

$$D_r^\alpha \vartheta_1(s_i,r_n) = \frac{1}{\Gamma(1-\alpha)}\int_0^{r_n} \frac{\partial u(s_i,t)}{\partial t}(s_n - t)^{-\alpha}dt = \frac{1}{\Gamma(1-\alpha)}\sum_{j=1}^{n}\int_{(j-1)\tau}^{j\tau}\left[\frac{\vartheta_{1_i}^{\,j} - \vartheta_{1_i}^{\,j-1}}{\tau} + O(\tau)\right](n\tau - t)^{-\alpha}dt = \frac{1}{\Gamma(1-\alpha)}\sum_{j=1}^{n}\{[\frac{\vartheta_{1_i}^{\,j} - \vartheta_{1_i}^{\,j-1}}{\tau r} + O(\tau r)][(n-j+1)^{1-\alpha} - (n-j)^{1-\alpha}]\}\frac{\tau^{1-\alpha}}{1-\alpha} = \frac{\tau^{-\alpha}}{\Gamma(2-\alpha)}\sum_{j=1}^{n}(\vartheta_{1_i}^{\,j} - \vartheta_{1_i}^{\,j-1})((n-j+1)^{1-\alpha} - (n-j)^{1-\alpha}) + \frac{1}{\Gamma(2-\alpha)}\sum_{j=1}^{n}((n-j+1)^{1-\alpha} - (n-j)^{1-\alpha})O(\tau^{2-\alpha}). \quad (37)$$

Theorem 2. Let $\vartheta_1(s,r)$ and $\vartheta_2(s,r)$ be the solutions of the initial boundary value problem (4)-(7). Also, suppose that $\nu_1(s,r_k)$ and $\nu_2(s,r_k)$ are the collocation approximation to the solutions $\vartheta_1(s,r_k)$ and $\vartheta_2(s,r_k)$ after the temporal discretization stage. Then the error estimate of the discrete scheme is given by

$$\| (\vartheta_1(s,r_k) - \nu_1(s,r_k), \vartheta_2(s,r_k) - \nu_2(s,r_k)) \|_\infty \le \eta(\tau^{2-\alpha} + h^2),$$

where η is some finite constant.

Proof. The time discretization process (19) that we use to discretize the system (4)-(7) in the time variable has one-order convergence. So, according to Theorem 3, we have

$$\| (\vartheta_1(s,r_k) - \nu_1(s,r_k), \vartheta_2(s,r_k) - \nu_2(s,r_k)) \|_\infty \le \eta(\tau^{2-\alpha} + h^2),$$

where η is some finite constant. Thus the order of convergence of our process is $O(\tau^{2-\alpha} + h^2)$. $\blacksquare$

4 Numerical examples

In real world applications, there are many components that affect data quality, such as data source, the sampling period and how the information is collected. Some studies estimate that even in controlled environments there are at least 5% of errors in a data set, we named of nosey data. In many of works, researchers are using this technique for cleaning a data set. We distinguish the noisy data two types of noise, in predictive attributes and in the target attribute. In this work we using data noisy for nearest of data to exact solution. We set a arbitrary little number near zero $a = \frac{1}{10}$ for my examples.

In this section, we consider two examples to show the utility of the FDM in solving the inverse system of Burgers equations. For this purpose, we obtain the solution of the linear algebraic equations by applying the extended cubic B-splines collocation method

(EXCBSM). Also, we have compared all of the examples with other methods such as the extended cubic B-splines collocation method (EXCBSM), CBSM, TCBSM, and RBFM. Finally, we have shown in Tables 1 and 2 the comparison between exact and numerical solutions of $\vartheta_1(0,r)$ and $\vartheta_2(0,r)$ in $s=0$ in which $\alpha=0.75$ and $\beta=0.75$.

Example 1. Consider the problem (4)–(6) with $a=\frac{1}{10}$, and

$$\vartheta_1(s,0)=1, \vartheta_2(s,0)=1, 0\le s\le 1,$$
$$\vartheta_1(a,r)=r^3\sin(\mathrm{e}^{-a})+1, \vartheta_2(a,r)=r^3\sin(\mathrm{e}^{-a})+1, 0\le r\le T,$$
$$\vartheta_1(1,r)=r^3\sin(\mathrm{e}^{-1})+1, \vartheta_2(1,r)=r^3\sin(\mathrm{e}^{-1})+1, 0\le r\le T,$$

the exact solution is $\vartheta_1(s,r)=\vartheta_2(s,r)=r^3\sin(\mathrm{e}^{-s})+1$, and

$$F(s,r)=6\,\frac{\sin(\mathrm{e}^{-s})r^{3-\alpha}}{\Gamma(4-\alpha)}+r^3\mathrm{e}^{-2s}\sin(\mathrm{e}^{-s})-r^3\mathrm{e}^{-s}\cos(\mathrm{e}^{-s}),$$
$$G(s,r)=6\,\frac{\sin(\mathrm{e}^{-s})r^{3-\beta}}{\Gamma(4-\beta)}+r^3\mathrm{e}^{-2s}\sin(\mathrm{e}^{-s})-r^3\mathrm{e}^{-s}\cos(\mathrm{e}^{-s}).$$

First, for applying the proposed method (EXCBSM), we expressing $\vartheta_1(s,r)$ and $\vartheta_2(s,r)$, as

$$v_1(s,r)=\sum_{i=-1}^{N+1}C_i(r)E_i(s), \qquad v_2(s,r)=\sum_{i=-1}^{N+1}D_i(r)E_i(s).$$

Then we approximate solutions of the boundary value problem. we extended cubic B-spline functions and their derivatives in (11), and substituting (13) into my example, and by using the Caputo derivative formulation we discretize the time-fractional derivative.

In addition, we explain the proposed method in equation 19 and 20 and solve the linear algebraic equations (20), (22) and generalized cross-validation (GCV) scheme we solve the problem. In Table 1, and Table 2, we have shown the comparison between exact and numerical solutions of $\vartheta_1(0,r)$ and $\vartheta_1(0,r)$ respectively at $s=0$ with noisy data as 0.00001 and $\alpha=0.75$ and $\beta=0.75$.

Table 1: Approximate result of $\vartheta_1(0,r)$ for Example 4 with noisy data as 0.00001

time	Exact	EXCBSM	CBSM	TCBSM	RBFM
r	$p_1(r)$				
0.1	1.000841	1.000872	1.000848	1.000869	1.000886
0.2	1.006731	1.006781	1.006753	1.006766	1.006938
0.3	1.022719	1.022764	1.022808	1.022778	1.023200
0.4	1.053854	1.054065	1.054086	1.054081	1.054700
0.5	1.105183	1.105697	1.105669	1.105713	1.106471
0.6	1.181757	1.182747	1.182642	1.182764	1.183546
0.7	1.288624	1.290307	1.290094	1.290326	1.290985
0.8	1.430833	1.433474	1.433115	1.433495	1.433835
0.9	1.613432	1.617352	1.616804	1.617377	1.617191
1	1.841470	1.847056	1.846270	1.847085	1.846166
RMS		2.0130×10^{-3}	1.7378×10^{-3}	2.0275×10^{-3}	2.1184×10^{-3}

The approximation of error for Example 4 with EXCBSM at $|p_1(r)-p_1^*(r)|$ and $|p_2(r)-p_2^*(r)|$ is shown in Figure 4.

Table 2: Approximate result of $\vartheta_2(0,r)$ for Example 4.

time	Exact	EXCBSM	CBSM	TCBSM	RBFM
r	$p_2(r)$				
0.1	1.000841	1.000934	1.000863	1.000889	1.001052
0.2	1.006731	1.006950	1.006884	1.006954	1.008030
0.3	1.022719	1.023398	1.023220	1.023392	1.026402
0.4	1.053854	1.055355	1.055020	1.055348	1.061484
0.5	1.105183	1.108016	1.107425	1.108009	1.118480
0.6	1.181757	1.186557	1.185561	1.186550	1.202446
0.7	1.288624	1.296151	1.294528	1.296145	1.318446
0.8	1.430833	1.441967	1.439393	1.441961	1.471242
0.9	1.613432	1.629154	1.625180	1.629149	1.665721
1	1.841470	1.862829	1.856870	1.862825	1.906655
RMS		8.1902×10^{-3}	6.1446×10^{-3}	8.1864×10^{-3}	2.8183×10^{-2}

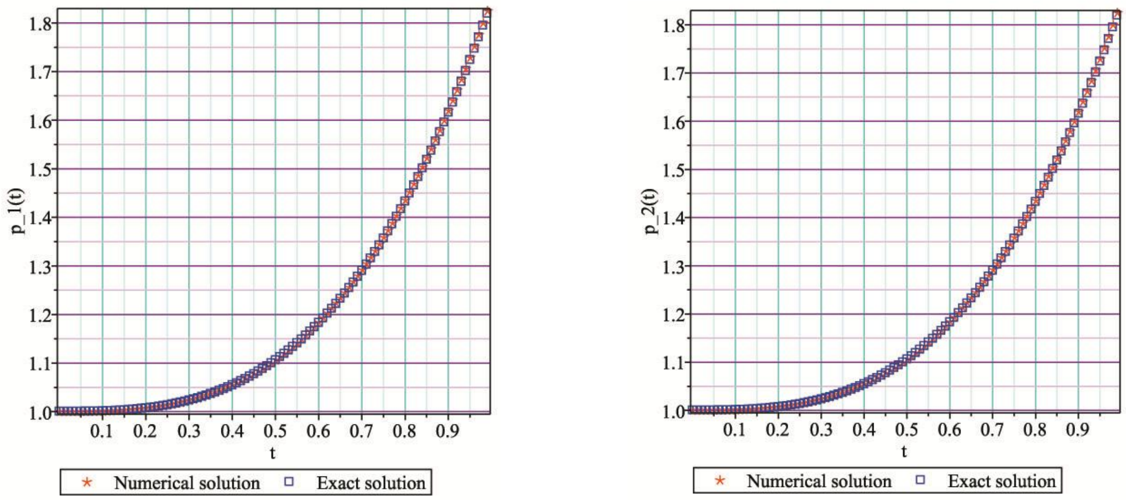

Figure 1: Comparison between the exact and numerical solutions $\vartheta_1(0,r)$ and $\vartheta_2(0,r)$ using the EXCBSM.

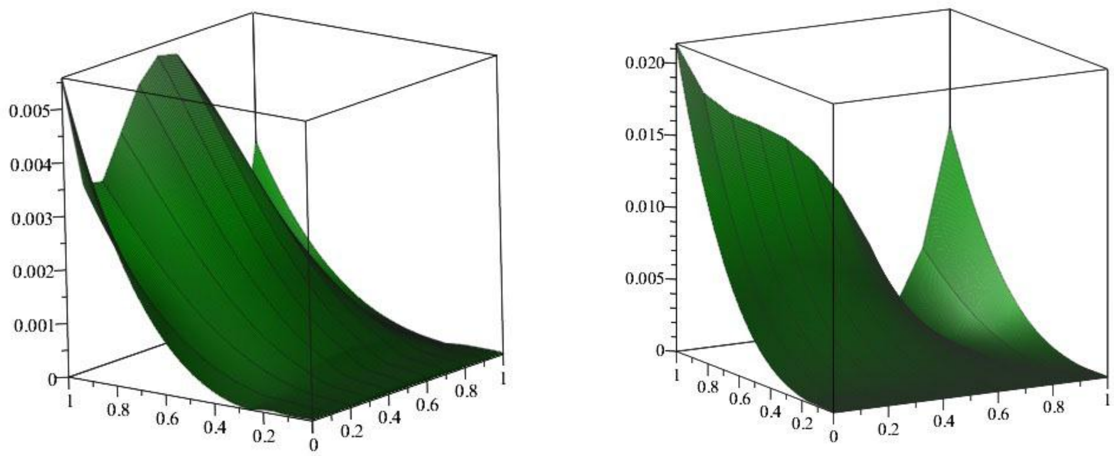

Figure 2: Diagrams of error for Example 4 by using the EXCBSM

Example 2: Consider the problem (4)–(6) with the exact solution

$$\vartheta_1(s,r) = \vartheta_2(s,r) = \frac{r^3}{e^{-s}+2},$$

where the terms $F(s,r)$ and $G(s,r)$ are

$$f(s,r) = 6\ \frac{r^{3-\alpha}}{(e^{-s}+2)\Gamma(4-\alpha)} - 2\ \frac{r^3 e^{-2s}}{(e^{-s}+2)^3} + \frac{r^3 e^{-s}}{(e^{-s}+2)^2},$$

$$G(s,r) = 6\ \frac{r^{3-\beta}}{(e^{-s}+2)\Gamma(4-\beta)} - 2\ \frac{r^3 e^{-2s}}{(e^{-s}+2)^3} + \frac{r^3 e^{-s}}{(e^{-s}+2)^2},$$

and the initial conditions are

$$\vartheta_1(s,0) = 0, \qquad \vartheta_2(s,0) = 0,$$

with the following boundary conditions

$$\vartheta_1(a,r) = \frac{r^3}{e^{-a}+2}, \qquad \vartheta_2(a,r) = \frac{r^3}{e^{-a}+2}, \qquad 0 \le r \le T,$$

$$\vartheta_1(1,r) = \frac{r^3}{e^{-1}+2}, \qquad \vartheta_2(1,r) = \frac{r^3}{e^{-1}+2}, \qquad 0 \le r \le T,$$

in which $a = \frac{1}{10}$. In Tables 3, and 4, we have shown the comparison between exact and numerical solutions of $\vartheta_1(0,r)$ and $\vartheta_2(0,r)$ at $s = 0$, $\alpha = 0.25$ and $\beta = 0.5$. Furthermore, to clarify the accuracy of the present method, the corresponding graphical illustrations are presented in Figure 3.

Table 3: Approximate result of $\vartheta_1(0,r)$ for Example 2

time	Exact	EXCBSM	CBSM	TCBSM	RBFM
r	$p_1(r)$				
0.1	0.000333	0.003843	0.000350	0.000384	0.000801
0.2	0.002666	0.002831	0.002795	0.002831	0.006334
0.3	0.009000	0.009469	0.009428	0.009467	0.021136
0.4	0.021333	0.022399	0.022344	0.022396	0.049658
0.5	0.041666	0.043732	0.043636	0.043725	0.096291
0.6	0.072000	0.075586	0.075396	0.075575	0.165388
0.7	0.114333	0.120094	0.119709	0.120076	0.261273
0.8	0.170666	0.179402	0.178657	0.179374	0.388239
0.9	0.243000	0.255681	0.254320	0.255640	0.550561
1	0.333333	0.351124	0.348780	0.351067	0.752492
RMS		6.5731×10^{-3}	5.8880×10^{-3}	6.5522×10^{-3}	1.6018×10^{-1}

Table 4: Approximate result of $\vartheta_2(0,r)$ for Example 2

time	Exact	EXCBSM	CBSM	TCBSM	RBFM
r	$p_2(r)$				
0.1	0.000333	0.000360	0.000337	0.000360	0.000404
0.2	0.002666	0.002709	0.002689	0.002709	0.003178
0.3	0.009000	0.009083	0.009065	0.009083	0.010639
0.4	0.021333	0.021494	0.021477	0.021494	0.025102
0.5	0.041666	0.041952	0.041938	0.041952	0.048876
0.6	0.072000	0.072469	0.072461	0.072468	0.084273
0.7	0.114333	0.115056	0.115061	0.115055	0.133601
0.8	0.170666	0.171723	0.171750	0.171722	0.199171
0.9	0.243000	0.244481	0.244546	0.244479	0.283292
1	0.333333	0.335340	0.335462	0.335337	0.388272
RMS		7.7591×10^{-4}	8.0462×10^{-4}	7.7452×10^{-2}	2.0995×10^{-2}

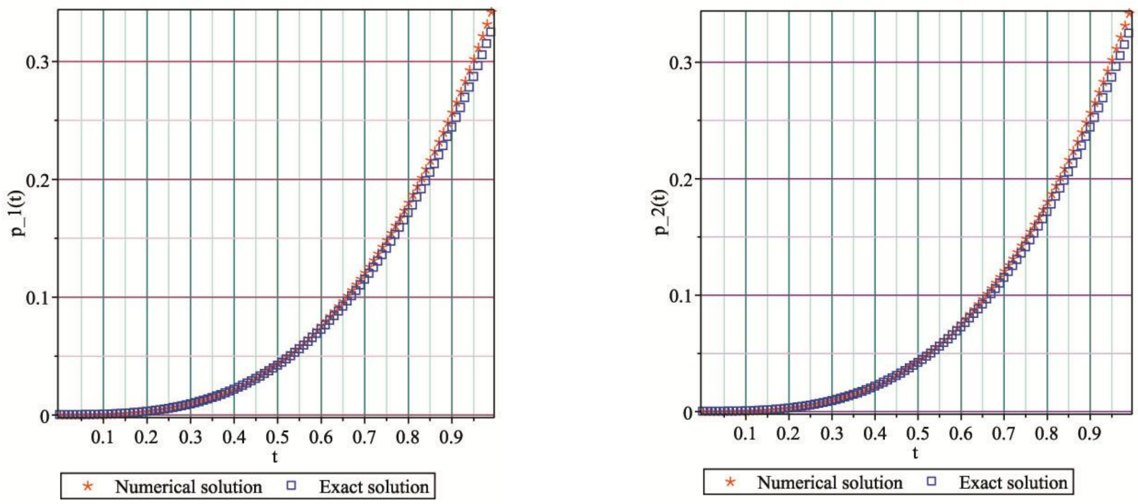

Figure 3: Comparison between the exact and numerical solutions $\vartheta_1(0,r)$ and $\vartheta_2(0,r)$ using the EXCBSM.

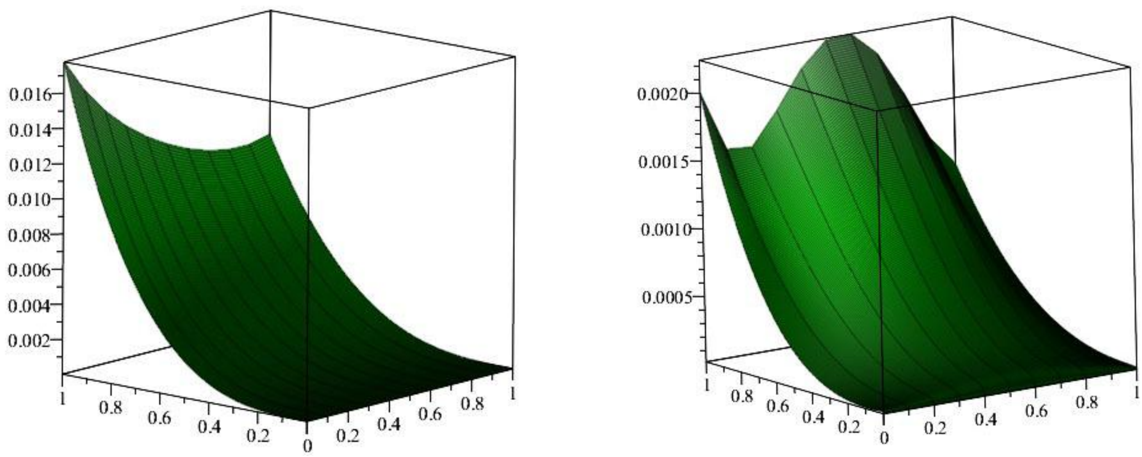

Figure 4: Diagrams of error for Example 2 by using the EXCBSM.

5 Conclusion

It is a fact that the calculated solutions of the time-fractional inverse problem are generally complicated. Hence mathematicians often attempt to achieve approximate solutions. Some of the most efficient and suitable methods of solving this equation are the FDM, EXCBSM, CBSM, TCBSM, and RBFM. In this work, we present a numerical method to approximate the ISB with the time-fractional. For this purpose, we have used the EXCBSM. These techniques do not need to simplify the equation and do not require extra effort to deal with the nonlinear terms, which are the advantages of this study in comparison to the previous methods. First, for applying the proposed method, we extended the cubic B-spline functions. Also, in this study, we proved the convergence of our method and calculated the order of the method. A comparison of the approximation result for Examples 4 and 2 with EXCBS, CBS, TCBS, and RBF, have been demonstrated in Tables 1-4. The approximation of error for examples with extended cubic B-splines collocation method (EXCBSM) at $|p_1(r) - p_1^*(r)|$ and $|p_2(r) - p_2^*(r)|$ are shown in Figures 1 and 2.

Acknowledgments

This work was funded by the University of Zabol, Iran (grant no: UOZ-GR-8409). The authors would like to express their gratitude to the Vice Chancellery for Research and Technology, University of Zabol, for funding this study.

References

[1] M. Landajuela, Burgers equation. BCAM Internship report, Basque Center for Applied Mathematics, 2011.

[2] M. J. Huntu and M. S. Hussein, "Simultaneous Identification of Thermal Conductivity and Heat Source in the Heat Equation," *Iraqi Journal of Science,* pp. 1968-1978., 2021.

[3] M. S. Hussein, Coefficient identification problems in heat transfer, Diss. University of Leeds, 2016.

[4] Q. Mohammed and M. S. Hussein, "Numerical Solution to Recover Time-dependent Coefficient and Free Boundary from Nonlocal and Stefan Type Overdetermination Conditions in Heat Equation.," *Iraqi Journal of Science,* pp. 950-960, 2021.

[5] D. Jeng and W. Meecham, "Solution of forced Burgers equation," *The Physics of Fluids,* vol. 15, pp. 504-506, 1972.

[6] G. Webb and J. McKenzie, "Burgers’ equation and cosmic ray shocks," *Journal of plasma physics,* vol. 31, pp. 337-368, 1984.

[7] E. Moreau and O. Vallée, "The Burgers equation as electrohydrodynamic model in plasma physics," *arXiv preprint physics/0501019,* 2005.

[8] M. S. Hussein and D. Lesnic, "Simultaneous determination of time and space-dependent coefficients in a parabolic equation," *Communications in Nonlinear Science and Numerical Simulation,* vol. 33, pp. 194-217, 2016.

[9] M. Erfanian, "Solving an inverse problem for a time-fractional advection-diffusion equation with variable coefficients by rationalized Haar wavelet method," *Journal of Computational Science,* vol. 64, p. 101869, 2022.

[10] H. Bateman, "Some recent researches on the motion of fluids," *Monthly Weather Review,* vol. 43, pp. 163-170, 1915.

[11] J. M. Burgers, "A mathematical model illustrating the theory of turbulence," *Advances in applied mechanics,* pp. 171-199, 1948.

[12] V. K. Srivastava and et al..., "One-dimensional coupled Burgers’ equation and its numerical solution by an implicit logarithmic finite-difference method," *Aip Advances,* vol. 4, p. 037119, 2014.

[13] S. E. Esipov, "Coupled Burgers equations : a model of polydispersive sedimentation," *Physical*

Review E, vol. 52, p. 3711, 1995.

[14] N. Ozdemir, A. Secer and M. Bayram, "The Gegenbauer Wavelets-Based Computational Methods for the Coupled System of Burgers’ Equations with Time-Fractional Derivative," *Mathematics,* vol. 7, p. 486, 2019.

[15] S. R. Munjam, "Fractional transform methods for coupled system of time fractional derivatives of non-homogeneous Burgers’ equations arise in diffusive effects.," *Computational and Applied Mathematics,* vol. 38, pp. 1-20, 2019.

[16] S. Edeki, G. Akinlabi and I. Ezeki, "Analytical solutions of a 1D time-fractional coupled Burger equation via fractional complex transform," *WSEAS Transactions on Mathematics,* vol. 17, pp. 229-236, 2018.

[17] B. Albuohimad and A. Hojatollah, "The Chebyshev collocation solution of the time fractional coupled Burgers ’ equation," *J . Math . Comput . Sci,* vol. 17, pp. 179-193, 2017.

[18] B. Albuohimad and A. Hojatollah, "On a hybrid spectral exponential Chebyshev method for time-fractional coupled Burgers equations on a semi-infinite domain," *Adv . Differ . Equ,* vol. 85, 2017.

[19] A. Prakash, K. Manoj and K. S. Kapil, "Numerical method for solving fractional coupled Burgers equations," *Appl . Math . Comput,* vol. 260, pp. 314-320, 2015.

[20] A. El-Ajou, O. A. Arqub and S. Moma, "Approximate analytical solution of the nonlinear fractional KdV–Burgers equation : a new iterative algorithm," *Journal of Computational Physics. ,* vol. 293, pp. 81-95, 2015.

[21] W. Hermann, "Aber die asymptotische Verteilung der Eigenwerte," *Nachrichten der Kniglichen Gesellschaft der Wissenschaften zu Gttingen,* pp. 110-117, 1911.

[22] D. Lesnic and L. Elliott, "The decomposition approach to inverse heat conduction," *Journal of Mathematical Analysis and Applications,* vol. 232, pp. 82-98, 1999.

[23] R. Pourgholi, S. H. Tabasi and H. Zeidabadi, "Numerical techniques for solving system of nonlinear inverse problem," *Engineering with Computers,* vol. 34, p. 487–502, 2018.

[24] S. Foadian, R. Pourgholi and S. H. Tabasi, "SOLVING AN INVERSE PROBLEM FOR A GENERALIZED TIME-DELAYED BURGERS-FISHER EQUATION BY HAAR WAVELET METHOD," *Journal of Applied Analysis & Computation,* vol. 10, no. 2, pp. 391-410, 2020.

[25] H. Zeidabadi, R. Pourgholi and S. H. Tabasi, "Solving a nonlinear inverse system of Burgers equations," *International Journal of Nonlinear Analysis and Applications,* vol. 10, no. 1, pp. 35-54, 2019.

[26] G. A. Anastassiou, dvances on Fractional Inequalities, Springer, 2011.

[27] N. Sugimoto, "Burgers equation with a fractional derivative; Hereditary effects on non-linear acoustic waves," *J . Fluid . Mech,* vol. 225, p. 63153, 1991.

[28] M. Erfanian and H. Zeidabadi, "Approximate solution of linear Volterra integrodifferential equation by using cubic B-spline finite element method in the complex plane," *Advances in Difference Equations,* vol. 62, no. 1, 2019.

[29] M. Erfanian, H. Zeidabadi and M. Rashki, "Solving a nonlinear fractional Schrodinger equation using cubic B-splines," *Advances in Difference Equations,* vol. 344, 2020.

[30] J. Nee and J. Duan, "Limit set of trajectories of the coupled viscous Burgers equations," *Applied Mathematics Letters,* vol. 11, no. 1, pp. 57-61, 1998.

[31] H. Zeidabadi, R. Pourgholi and S. H. Tabasi, "A hybrid scheme for time fractional inverse parabolic problem," *Waves in Random and Complex Media,* vol. 30, no. 2, pp. 354-368.

[32] X. L. Han and S. J. Liu, "An extension of the cubic uniform B-spline curves," *J . Comput . Aided . Des . Comput . Graph.,* vol. 15, no. 5, pp. 576-578, 2003.

[33] I. Podlubny, Fractional Differential Equations, New York , NY , USA: Academic, 1999.

[34] Ibraheem, W. Qutaiba and M. S. Hussein, "Determination of time-dependent coefficient in time fractional heat equation," *Partial Differential Equations in Applied Mathematics,* vol. 7, p. 100492, 2023.

[35] L. Martin, L. Elliott, P. J. Heggs , D. B. Ingham, D. Lesnic and X. Wen, "Dual reciprocity boundary element method solution of the Cauchy problem for Helmholtz-type equations with variable coefficients," *j. Sound Vib*, vol. 297, pp. 89-105, 2006.

[36] R. C. Aster, B. Borchers and C. Thurber, Parameter estimation and inverse problems, New Mexico: Tech, 2003.

[37] L. Elden, "A note on the computation of the generalized cross-validation function for ill-conditioned least squares problems," *BIT,* vol. 24, p. 467–472, 1984.

[38] G. H. Golub, M. Heath and G. Wahba, "Generalized cross-validation as a method for choosing a good ridge parameter," *Technometrics,* vol. 21, no. 2, pp. 215-223, 1979.

[39] J. Stoer and R. Bulrisch, An introduction to numerical analysis,, Springer-Verlag, 1991.