



ISSN: 0067-2904
GIF: 0.851

The Spectrum and the Numerical Range of the Product of Finite Numbers of Automorphic Composition Operators on Hardy Space H^2

Eiman H. Abood^{1*}, Samira N. Kadhim², Sarah M. Khalil²

¹Department of mathematics, College of Science, University of Baghdad, Baghdad, Iraq

²Department of Mathematics, College of Science for women, University of Baghdad, Baghdad, Iraq

Abstract

Throughout this paper we study the properties of the composition operator $C_{\alpha_{p_1} \circ \alpha_{p_2} \circ \dots \circ \alpha_{p_n}}$ induced by the composition of finite numbers of special automorphisms of U ,

$$\alpha_{p_i}(z) = \frac{p_i - z}{1 - \overline{p_i}z}$$

Such that $p_i \in U$, $i = 1, 2, \dots, n$, and discuss the relation between the product of finite numbers of automorphic composition operators on Hardy space H^2 and some classes of operators.

Keywords: Composition operators, Conformal automorphisms, Hardy spaces.

الطيف والمدى العددي للمؤثر التركيبي المكون من حاصل ضرب عدد منتهى من المؤثرات التركيبية الأتومورفية المعرفة على فضاء هاردي H^2

ايمان حسن عبود^{1*}، سميره ناجي كاظم²، ساره محمود خليل²

¹قسم الرياضيات، كلية العلوم، جامعه بغداد، بغداد، العراق.

²قسم الرياضيات، كلية العلوم للبنات، جامعه بغداد، بغداد، العراق.

الخلاصة

درسنا في هذه البحث المؤثر التركيبي المحتث من تركيب عدد منتهى من الدوال التالية:

$$\alpha_{p_i}(z) = \frac{p_i - z}{1 - \overline{p_i}z}.$$

حيث ان $i = 1, 2, \dots, n$, $p_i \in U$ ، عندما n عدد منتهى صحيح غير سالب. ودرسنا العلاقة بين خواص

الدالة $\alpha_{p_1} \circ \alpha_{p_2} \circ \dots \circ \alpha_{p_n}$ والطيف والمدى العددي للمؤثر التركيبي $C_{\alpha_{p_1}} C_{\alpha_{p_{n-1}}} C_{\alpha_{p_n}}$

المحتث منها المعروف على فضاء هاردي H^2 .

Introduction

Let U denote the unit ball in the complex plane, the Hardy space H^2 is the collection of functions $f(z) = \sum_{n=0}^{\infty} \hat{f}(n)z^n$, which holomorphic on U such that $\sum_{n=0}^{\infty} |\hat{f}(n)|^2 < \infty$ with $\hat{f}(n)$ denoting the n -th Taylor coefficient of f , and the norm of f is defined by:

*Email: eimanmath@yahoo.com

$$\|f\|^2 = \sum_{n=0}^{\infty} |\hat{f}(n)|^2.$$

The particular importance of H^2 is due to the fact that it is a Hilbert space. Let φ be a holomorphic self-map of U , the composition operator C_φ induced by φ is defined on H^2 by the equation $C_\varphi f = f \circ \varphi$, for every $f \in H^2$ (see [1]). A conformal automorphism of U is a univalent holomorphic mapping of U onto itself. Each such map is a linear fractional, and can be represented as product $w \cdot \alpha_p$, where:

$$\alpha_p(z) = \frac{p-z}{1-\bar{p}z}, \quad (z \in U)$$

For some $p \in U$ and $w \in \partial U$ (see [2]).

The map α_p is called special automorphism of U interchanges the point p and the origin and it is self-inverse map. Let φ holomorphic self-map of U , φ is called an inner function if $|\varphi(z)| = 1$ almost every where on ∂U (see [3]). Clearly every conformal automorphism of U is an inner function.

It is well known that these are all linear fractional transformations, and they come in three flavors (see, e.g., [1, chapter 0]):

- **Elliptic:** If it has one interior fixed point in U and one outside \bar{U} . These automorphisms having derivative < 1 at the interior fixed points.
- **Hyperbolic:** If it has two distinct fixed points on ∂U . These automorphisms having derivative < 1 at the boundary fixed points.
- **Parabolic:** If it has one fixed point of multiplicity 2 on ∂U . These automorphisms having derivative = 1 at a boundary fixed point.

The eigenvalue equation for a composition operator $f \circ \varphi = \lambda f$ is called Schröder's equation. A functional equation has been around 1870. The numerical range of C_φ is the set of all complex numbers of the form $\langle C_\varphi f, f \rangle$, where $f \in H^2$ and $\|f\| = 1$, it is denoted by $W(C_\varphi)$, [4].

In this paper, we try to give a comprehensive picture to the spectrum and the numerical range of the product of finite numbers of automorphic composition operators on Hardy space H^2 . This paper consists of two sections. In section one, we give the Schröder's equation and try to give complete picture of the spectrum of the product of a finite numbers of automorphic composition operators on Hardy space H^2 . In section two, we describe the numerical range of it.

The study of composition operators on Hardy space H^2 provides a rich area in which to explore the connection between operator theory and classical function theory. In [5] Eiman H. A., Samira N. K. and Sara M. K. studied the composition operator $C_{\alpha_{p_1} \circ \alpha_{p_2} \circ \dots \circ \alpha_{p_n}}$ induced by the composition of finite numbers of special automorphisms of U :

$$\alpha_{p_i}(z) = \frac{p_i - z}{1 - \bar{p}_i z}.$$

Such that $p_i \in U$, $i = 1, 2, \dots, n$ and n is a fixed positive integer number and discussed how the change of p_1, p_2, \dots, p_n affects on the properties of the operator $C_{\alpha_{p_n}} C_{\alpha_{p_{n-1}}} \dots C_{\alpha_{p_1}}$. We proved that the composition of finite numbers of special automorphisms of U is a conformal automorphism of U .

Theorem 1 [5, Theorem 1]:

For all $n \in \mathbb{Z}^+$, $\alpha_{p_1} \circ \alpha_{p_2} \circ \dots \circ \alpha_{p_n}(z) = w_m \alpha_{h_m}(z)$, where $p_i \in U$, $h_m \in U$ and $w_m \in \partial U$, for $m \in \mathbb{Z}^+$.

1. The Spectrum of the Product of Finite Numbers of Automorphic Composition Operators on Hardy Space H^2 .

If φ is a holomorphic self-map of U , the eigenvalue equation for the composition operator is $C_\varphi f = \lambda f$ or $f \circ \varphi = \lambda f$. This equation is called **the Schröder's equation** [6].

Recall that **the point spectrum** of an operator T on a Hilbert space H , denoted by $\sigma_p(T)$ is the set of all eigenvalues of T , and **the spectrum** of T denoted by $\sigma(T)$ is the set of all complex numbers λ for which $T - \lambda I$ is not invertible.

In this section we give the spectral information about the product of finite numbers of automorphic composition operators on Hardy space H^2 . We start by the following lemma which is appeared in [7].

Lemma 1.1:

Let ϕ be a holomorphic self-map of U and $p \in U$ is a fixed point of ϕ . Let $\psi = \alpha_p \circ \phi \circ \alpha_p^{-1}$, then $\psi(0) = 0$, $\psi'(0) = \phi'(0)$, $\sigma_p(C_\psi) = \sigma_p(C_\phi)$ and $\sigma(C_\psi) = \sigma(C_\phi)$.

By lemma (1.1) it is not restriction to assume that a fixed point p of ϕ in U is $p = 0$.

Corollary 1.2:

Let $p_i \in U$, $i = 1, 2, \dots, n$; such that $\alpha_{p_1} \circ \alpha_{p_2} \circ \dots \circ \alpha_{p_n}$ is an elliptic automorphism of U , then $\sigma_p(C_{\alpha_{p_n}} \circ C_{\alpha_{p_{n-1}}} \circ \dots \circ C_{\alpha_{p_1}}) = \{-w_m^n, w_m^n\}$.

Proof:

Since $\psi = \alpha_{p_1} \circ \alpha_{p_2} \circ \dots \circ \alpha_{p_n}$ is elliptic automorphism of U , then ψ has a fixed point $p \in U$. This implies by lemma (1.1) that $p = 0$. But by theorem (1) $\psi(z) = w_m \alpha_{h_m}(z)$ then it is clear that $\psi(z) = -w_m z$. Now, set $e_n(z) = z^n$, $n = 0, 1, \dots$. Therefore:

$$C_\psi e_n(z) = e_n(\psi(z)) = (\psi(z))^n = (-w_m z)^n = (-w_m)^n z^n = ((-1)^n w_m^n) e_n(z).$$

Therefore, $(-1)^m w_m^m$ is an eigenvalue of C_ψ for some nonnegative integer m . Thus it is clear that, $\sigma_p(C_\psi) = \{-w_m^n, w_m^n\}$, as desired. ■

Lemma 1.3 [7]:

Suppose that $\phi(z) = \lambda z$, $|\lambda| = 1$. If λ is a root of unitary, then for some nonnegative integer m :

$$\sigma(C_\phi) = \{1, \lambda, \lambda^2, \dots, \lambda^{n-1}\}.$$

Moreover, if λ is not a root of unitary, then:

$$\sigma(C_\phi) = \{\lambda : |\lambda| = 1\} = \partial U.$$

Corollary 1.4:

Let $p_i \in U$, $i = 1, 2, \dots, n$; such that $\alpha_{p_1} \circ \alpha_{p_2} \circ \dots \circ \alpha_{p_n}$ is an elliptic automorphism of U , then either for some nonnegative integer n :

$$\sigma(C_{\alpha_{p_n}} \circ C_{\alpha_{p_{n-1}}} \circ \dots \circ C_{\alpha_{p_1}}) = \{1, \lambda, \lambda^2, \dots, \lambda^{n-1}\},$$

or

$$\sigma(C_{\alpha_{p_n}} \circ C_{\alpha_{p_{n-1}}} \circ \dots \circ C_{\alpha_{p_1}}) = \{\lambda : |\lambda| = 1\} = \partial U.$$

Proof:

Since $\psi = \alpha_{p_1} \circ \alpha_{p_2} \circ \dots \circ \alpha_{p_n}$ is an elliptic automorphism of U , then ψ has a fixed point $p \in U$ and ψ has a form $\psi = \alpha_p \circ \phi \circ \alpha_p^{-1}$, where $\phi(z) = \lambda z$, $|\lambda| = 1$.

Hence, the one can get the result immediately by lemma (1.2) and lemma (1.3). ■

Theorem 1.5 [8]:

Let ϕ be an inner function, which is a linear fractional has a Denjoy-Wolff point $p \in \partial U$, then:

$$\sigma(C_\phi) = \{\lambda : |\phi'(p)|^{1/2} \leq |\lambda| \leq |\phi'(p)|^{-1/2}\}.$$

Corollary 1.6:

Let $p_i \in U$, $i = 1, 2, \dots, n$; such that $\alpha_{p_1} \circ \alpha_{p_2} \circ \dots \circ \alpha_{p_n}$ is a parabolic automorphism of U , then:

$$\sigma(C_{\alpha_{p_n}} \circ C_{\alpha_{p_{n-1}}} \circ \dots \circ C_{\alpha_{p_1}}) = \{\lambda : |\lambda| = 1\} = \partial U.$$

Proof:

Since $\psi = \alpha_{p_1} \circ \alpha_{p_2} \circ \dots \circ \alpha_{p_n}$ is parabolic, then ψ has only one fixed point $p \in \partial U$ such that $|\psi'(p)| = 1$. But ψ is an inner function, then by theorem (1.5), we have $\sigma(C_\psi) = \{\lambda : 1 \leq |\lambda| \leq 1\} = \{\lambda : |\lambda| = 1\} = U$. ■

Corollary 1.7:

Let $p_i \in U$, $i = 1, 2, \dots, n$; such that $\psi = \alpha_{p_1} \circ \alpha_{p_2} \circ \dots \circ \alpha_{p_n}$ is hyperbolic automorphism of U , with Denjoy-Wolff point $p \in \partial U$, then:

$$\sigma(C_{\alpha_{p_n}} C_{\alpha_{p_{n-1}}} \dots C_{\alpha_{p_1}}) = \{\lambda : |\psi'(p)|^{1/2} \leq |\lambda| \leq |\psi'(p)|^{-1/2}\}.$$

Proof:

Since $\psi = \alpha_{p_1} \circ \alpha_{p_2} \circ \dots \circ \alpha_{p_n}$ is hyperbolic, then ψ has a Denjoy-Wolff point $p \in \partial U$, such that $|\psi'(p)| < 1$. Since ψ is an inner function, then the proof follows immediately from theorem (1.5). ■

Now, we give the complete description of the spectrum of the product of finite numbers of automorphic composition operators on Hardy space H^2 .

Recall that [9] a Hilbert space operator T that satisfies an equation $T^2 + \lambda T + \mu I = 0$, where λ, μ are two complex numbers is called **quadratic operator**.

Proposition 1.8:

Let $p_i \in U, i = 1, 2, \dots, n$; then

$$\sigma(C_{\alpha_{p_n}} C_{\alpha_{p_{n-1}}} \dots C_{\alpha_{p_1}}) = \{-\sqrt{w_m}, \sqrt{w_m}\}.$$

Proof:

By [corollary (22)] C_ψ is a quadratic composition operator on H^2 , such that $C_\psi^2 = w_m I$, where $\psi(z) = w_m \alpha_{h_m}(z)$, such that $w_m \in \partial U$ and $h_m \in U$ (see theorem (1)). Hence, by the spectral mapping theorem [11, Chapter 6], we have $(\sigma(C_\psi))^2 = \sigma(C_\psi^2) = \sigma(w_m I) = \{w_m\}$.

Therefore, $\sigma(C_\psi) = \{-\sqrt{w_m}, \sqrt{w_m}\}$. ■

Therefore, by proposition (1.8) one can conclude that the spectrum of a quadratic operator can consists of at least two points.

2. The Numerical Range of the Product of Finite Numbers of Automorphic Composition Operators on Hardy Space H^2 .

Recall that **the numerical range** of an operator T on a Hilbert space H is the set of complex numbers,

$$W(T) = \{\langle Tf, f \rangle : f \in H, \|f\| = 1\}, \quad [10].$$

The following proposition collects some properties of the numerical range of an operator, for more details we refer the reader to [10].

Proposition 2.1[10]:

1. $W(T)$ lies in the disc of center 0 and radius $\|T\|$.
2. $W(T)$ contains every eigenvalues of T .
3. $\sigma(T) \subseteq \overline{W(T)}$. ($\overline{W(T)}$ denotes the closure of $W(T)$).
4. If T is normal operator, then $\text{Conv } \sigma(T) = \overline{W(T)}$.
($\text{Conv } \sigma(T)$ denotes the convex hull of $\sigma(T)$).
5. $W(T)$ is convex set of C .
6. If T is the identity, then $W(T) = \{1\}$. More generally, if α and β are complex numbers, then $W(\alpha T + \beta) = \alpha W(T) + \beta$.

In [4] the shapes of the numerical range for composition operators on H^2 induced by some conformal automorphism of U , specially parabolic and hyperbolic are investigated. The authors proved other things in the following results.

Theorem 2.2 [4]:

If ϕ is a conformal automorphism of U is either parabolic or hyperbolic, then $W(C_\phi)$ is a disc centered at the origin.

Theorem 2.3 [4]:

If ϕ is a hyperbolic automorphism of U with antipodal fixed points and it is conformally conjugate to a positive dilation $\phi_r(z) = rz, (0 < r < 1)$, then $W(C_\phi)$ is the open disc of radius $1/\sqrt{r}$ centered at the origin.

Theorem 2.4 [4]:

If φ is elliptic automorphism of U and conformally conjugate to a rotation $\varphi(z)=\lambda z$ ($|\lambda| = 1$) and λ is not a root of unity, then $\overline{W(C_\varphi)}$ is a disc centered at the origin.

Theorem 2.5 [4]:

If φ is elliptic automorphism of U with multiplier -1 , then $\overline{W(C_\varphi)}$ is a (possibly degenerate) ellipse with foci ± 1 . The degenerate case occurs if and only if $\varphi(0) = 0$, in which case $\varphi(z) = z$.

The closure of $W(C_{\alpha_p})$ was described by the authors of [4]. They showed that it is a closed elliptical disc with foci ± 1 . That disc is reduced to its focal axis if and only if $W(C_{\alpha_p}) = [-1, 1]$ if and only if $p = 0$ (see theorem (2.5)). The authors [4] gave a formula for the length of the major axis of the disc. That formula is very hard to use in practical problems. Therefore, recently, the author of [11] found the following practical formula for the length of the aforementioned major axis.

Theorem 2.6 [11]:

For each $p \in U$, then $\overline{W(C_{\alpha_p})}$ is an ellipse with foci at ± 1 and major axis $\frac{2}{\sqrt{1-|p|^2}}$.

We investigate the shape of the numerical range of the product of finite numbers of automorphic composition operators on Hardy space H^2 . For this goal, we give the following preliminaries. Recall that [9] the quantity $\omega(T) = \sup \{ |\langle Tf, f \rangle| : \|f\| = 1 \}$ is called **the numerical radius** of the operator T . The statement that T **attains its norm**, respectively **attains its numerical radius**, means that there is some $f \in H$, such that $\|Tf\| = \|T\|$, respectively $|\langle Tf, f \rangle| = \omega(T)$.

Matache [9] was described the composition operators that attains its norm.

Proposition 2.7 [9]:

A composition operator having inner symbol φ attains its norm if and only if $\varphi(0) = 0$.

A straightforward consequence of proposition (2.7) is the following:

Corollary 2.8:

Let $p_i \in U$, $i = 1, 2, \dots, n$. Then $C_{\alpha_{p_n}} C_{\alpha_{p_{n-1}}} \dots C_{\alpha_{p_1}}$ attains its norm if and only if $p_1 = p_2 = \dots = p_n = 0$.

Proposition 2.9:

Let $p_i \in U$, $i = 1, 2, \dots, n$. If $p_i = 0$, for all $i = 1, 2, \dots, n$. Then $\overline{W(C_{\alpha_{p_n}} C_{\alpha_{p_{n-1}}} \dots C_{\alpha_{p_1}})}$ is a straight line containing the end points $-\sqrt{w_m}$ and $\sqrt{w_m}$.

Proof:

Since $\psi(z) = w_m \alpha_{h_m}(z)$ and $p_i = 0$, for all $i = 1, 2, \dots, n$. Then by theorem (1) we have $h_m = 0$, for m . Hence,

$$\psi(z) = \alpha_{p_1} \circ \alpha_{p_2} \circ \dots \circ \alpha_{p_n}(z)$$

$$= -w_m z$$

Therefore, 0 is a fixed point for ψ , this implies that $C_{\alpha_{p_n}} C_{\alpha_{p_{n-1}}} \dots C_{\alpha_{p_1}}$ is elliptic automorphism of U . Thus by corollary (1.8)

$$\sigma(C_\psi) = \sigma(C_{\alpha_{p_n}} C_{\alpha_{p_{n-1}}} \dots C_{\alpha_{p_1}}) = \left\{ -\sqrt{w_m}, \sqrt{w_m} \right\}.$$

But by [5, Theorem (2)], C_ψ is normal operator, then by proposition (2.1)(4)

$$\text{Conv } \sigma(C_\psi) = \overline{W(C_\psi)}. \text{ This implies that,}$$

$$\overline{W(C_\psi)} = \text{Conv} \left\{ -\sqrt{w_m}, \sqrt{w_m} \right\}.$$

Thus it is clear that $\overline{W(C_\psi)}$ is a straight line containing the end points $-\sqrt{w_m}$ and $\sqrt{w_m}$ ■

Theorem 2.10 [9]:

The Numerical range of a quadratic operator having spectrum consisting of two distinct points a and b is an open or a closed elliptical disc, possibly, degenerate (that is, reduced to its focal axis). The major axis of the disc has length $\|T - aI\|$ and the length of the minor axis is $\sqrt{\|T - aI\|^2 - |a - b|^2}$.

The elliptical disc is closed if and only if T attains its norm or equivalently, if and only if it attains its numerical radius. Since the numerical range of the operator that attains norm is closed, then by corollary (2.8), proposition (2.9) and theorem (2.10) one can get the directly result.

Corollary 2.11:

Let $p_i \in \mathbb{U}$, $i = 1, 2, \dots, n$ such that $p_i = 0$, for all $i = 1, 2, \dots, n$; then $W(C_{\alpha_{p_n}} C_{\alpha_{p_{n-1}}} \dots C_{\alpha_{p_1}})$ is a straight line containing the end points $-\sqrt{w_m}$ and $\sqrt{w_m}$.

Corollary 2.12:

Let $p_i \in \mathbb{U}$, $i = 1, 2, \dots, n$ such that $p_j \neq 0$, for some $j = 1, 2, \dots, n$, then $W(C_{\alpha_{p_n}} C_{\alpha_{p_{n-1}}} \dots C_{\alpha_{p_1}})$ is an open elliptical disc with major axis of length $\|C_{\alpha_{p_n}} C_{\alpha_{p_{n-1}}} \dots C_{\alpha_{p_1}} - \sqrt{w_m} I\|$ and the length of minor axis is

$$\sqrt{\|C_{\alpha_{p_n}} C_{\alpha_{p_{n-1}}} \dots C_{\alpha_{p_1}} - \sqrt{w_m} I\|^2 - 4}.$$

Proof:

Since C_ψ is quadratic operator, where $\psi = C_{\alpha_{p_n}} C_{\alpha_{p_{n-1}}} \dots C_{\alpha_{p_1}}$ and $p_j \neq 0$, for some $j = 1, 2, \dots, n$; then by corollary (2.8) C_ψ is not attains norm. Hence one can get the result directly by corollary (1.8) and theorem (2.10). ■

References

1. Shapiro, J. H. **1993**. *Composition operators and classical function theory*, Springer-Verlage, New York.
2. Rudin, W. **1987**. *Real and complex analysis*, McGraw-Hill Pub. Com. Lid., New York.
3. Duren, P. L. **1970**. *Theory of H^p Space*, Academic Press, New York,.
4. Bourdon, P.S. & Shapiro, J. H. **2000**. The numerical ranges of automorphic composition operator, *J. Math. Ana. and App.*, 251:839-854.
5. Abood E. H., Kadhim S. N., Khalil M. S. **2015**. The product of finite numbers of automorphic composition operators on Hardy space H^2 , to appear in *mathematical theory and modeling*.
6. Shapiro, J. H. **1998**. Composition operators and Schroder's functional equation, *Contemporary Math.*, 213:213-228.
7. Kamowitz, H. **1973**. The spectra of endomorphism of the disc algebra, *Pacific J. Math.*, 46(2):433-440.
8. Nordgren, E. A. **1968**. Composition operator, *Can. J. Math.* 20:442-449.
9. Matache V. **2007**. Distance between compositions operators, *Extracta Math.*, 22(1):19-33.
10. Abood E. H. **1996**. The numerical range and normal operators, M. Sc. thesis, Dep. of Math., College of Science, University of Baghdad, Baghdad, Iraq.
11. Abdollahi, A. **2005**. The numerical ranges of a composition operator with conformal automorphism symbol, *Linear Algebra and its Applications*, 408:177-188.