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On Solving Singular Multi Point Boundary Value Problems with Nonlocal Condition

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Abstract
In this paper Hermite interpolation method is used for solving linear and non-linear second order singular multi point boundary value problems with nonlocal condition. The approximate solution is found in the form of a rapidly convergent polynomial. We discuss behavior of the solution in the neighborhood of the singularity point which appears to perform satisfactorily for singular problems. The examples to demonstrate the applicability and efficiency of the method have been given.

Keywords: Singular differential equation, Nonlocal condition, Hermite interpolation

حول حل مسائل القيم الحدودية متعددة النقاط الشاذة مع شروط غير محلية

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الخلاصة
في هذا البحث استخدمت طريقة اندراج هير مت لحل الرتبة الثانية الخطية وغير الخطية لمسائل القيم الحدودية الشاذة مع شروط غير محلية. ثم ايجاد حل تقريبي في شكل متعددة حدود متقاربة بسرعة .وناقشنا سلوك الحل في جوار نقطة منفردة حيث يظهر مرضيا للمسائل الشاذة .ثم اعطاء امثلة لاظهار تطبيق وكفائة الطريقة.

1. Introduction

In the study of non-linear phenomena in physics, engineering and other sciences, many mathematical models lead to singular multi point boundary value problems (SMPBVP) associated with non-linear ordinary differential equations (ODE) [1].

We will deal with no classical boundary value problems, the solution of singular multi point boundary value problems with nonlocal conditions specifications. The meaning of the problems is that the definite condition not only depends on the value of solution in the end of interval, but also depends on the value of solution in some points of the interior of interval .Started to fairly late study this kind of problem, initialed by II'in and Moiseev [2] studied the existence of solutions for a linear multipoint BVP.Gupta studied certain three point boundary value problems for non-linear ordinary differential equation [3]. Since then, more general multi point boundary value problem have been studied [4-7]. Within the following ten years,the study on nonlocal boundary value problems for ordinary differential equations has been made great progress. However, it is not good enough. In this paper we introduce Hermite interpolation for construction polynomial solutions of singular boundary value problems with nonlocal condition.

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2. Singular Differential Equation with BC [8]

The general form of the 2nd order singular ordinary differential equations

A(x)y''+B(x)y'+C(x)y=0 (1)

How to solve a linear ODE. The first thing we do is, rewrite the ODE as:

y''+P(x)y'+Q(x)y=0 (2)

where, of course, P(x) = B(x) / A(x) , and Q(x) = C(x) / A(x) .

There are two types of a point $x_o \in [0,1]$: Ordinary point and singular point .Also, there are two types of singular point : Regular and Irregular Points, A function y(x) is analytic at x_o if it has a power series expansion at x_0 that converges to y(x) on an open interval containing x_o . A point x_o is an ordinary point of the ODE (2), if the functions P(x) and Q(x) are analytic at x_o. Otherwise x_o is a singular point of the ODE, if the functions P(x) and Q(x) are analytic at x_o.

i.e. P(x) = $P_0 + P_1$ (x-x_o) + P_2 (x-$x_o)^2$+…….. = $\sum_{i=0}^{\infty} P_i\,(x-x_o)^i$ (3)

Q(x) = $Q_0 + Q_1$ (x-x_o) + Q_2 (x-$x_o)^2$+…….. = $\sum_{i=0}^{\infty} Q_i\,(x-x_o)^i$ (4)

If A , B and C are polynomials then a point x0 such that A(x_o)≠0 is an ordinary point. On the other hand if P(x) or Q(x) are not analytic at x_o then x_o is said to be a singular.A singular point x_o of the ODE (3) is a regular singular point of the ODE if the functions xP(x) and x^2Q(x) are analytic at x_o. Otherwise x_o is an irregular singular point of the ODE.

Now, we state the following theorem without proof which gives us a useful way of testing if a singular point is regular[9].

Theorem 1: If the $\lim_{x\to 0}$ P(x) and $\lim_{x\to 0}$ Q(x) exist, are finite, and are not 0 then x = 0 is a regular singular point. If both limits are 0, then x = 0 may be a regular singular point or an ordinary point. If either limit fails to exists or is ±∞ then x = 0 is an irregular singular point. There are four kinds of singularities:

- The first kind is the singularity at one of the ends of the interval [0, 1].
- The second kind is the singularity at both ends of the interval [0, 1].
- The third kind is the case of a singularity in the interior of the interval.
- The forth and final kind is simply treating the case of a regular differential equation on an infinite interval.

In this paper, we focus of the first three kinds.

3. Hermite interpolation

Hermite interpolation method used for construction polynomial solutions of singular multi point boundary value problems with nonlocal condition. Where the interpolation polynomial also matches the first derivatives $f^{(1)}$(x) $at\ x = x_k$,k=0,1,2,…n [10].This interpolation technique is important since it has the property that gives high order of accuracy.

Theorem 2 [10]: Suppose that f(x)∈ C^1[a,b] , and that $x_{o,}x_1,\dots,x_n \in [a,b]$ are distinct, then the unique polynomial of degree (at most) 2n + 1 denoted by Hermite interpolation H_{2n+1} , and such that $H_{2n+1}(x_j) = f(x_j), H'_{2n+1}(x_j) = f'(x_j) \quad j \in Z_{n+1}$ is given by :

$$H_{2n+1}(x) = \sum_{k=0}^{n}\left[1 - 2L'_k(x_k)(x - x_k)\right][L_k(x)]^2 f(x_k) + \sum_{k=0}^{n}(x - x_k)\,[L_k(x)]^2\, f'(x_k) \quad (5)$$

$$L_k(x) = \prod_{\substack{i=0\\ i\neq j}}^{n} \frac{x - x_i}{x_k - x_i}$$

An error bound for Hermite interpolation is provided by the expression:

E=$(x-x_o)^2(x-x_1)^2 \dots (x-x_n)^2 f^{(2n+1)}(x)/(2n+1)!$,where f(x)∈ $C^{2n+2}[a,b]$.

4. Solution for non-linear SMPBVP with Nonlocal Condition

In this section, we apply the Hermite interpolation method H_{2n+1} and Taylor series to solve regular singular point . A general form 2^{nd} order (SMPBVP) is:

$$x^m y'' = f(x, y, y'), \qquad 0 \le x \le 1 \quad (6a)$$

A associated with nonlocal condition of the form:

$$y(0) = \sum_{j=1}^{m-2} A_j y(\delta_j)\ ,\ y(1) = \sum_{j=1}^{m-2} B_j y(\eta_j) \quad (6b)$$

Where A_j,$B_j \epsilon$R and $\delta_j, \eta_j \epsilon (0,1)$, j=1,2……,m-2

Now, to solve the problem by suggested method doing the following steps:

Step one: Evaluate Taylor series of y(x) about x = 0:

$y(x) = \sum_{i=0}^{\infty} a_i x^i = a_o + a_1 x + \sum_{i=2}^{\infty} a_i x^i$ (7)

Where y(0)= a_o,y'(0)= a_1 ,$\frac{y''(0)}{2!} = a_2$,…,$\frac{y^{(i)}(0)}{i!} = a_i$,i=3,4,…

Evaluate Taylor series of y_j (x) about x =δ_j, $\forall j = 1,2, \dots, m-2$:

$y_j(x) = \sum_{i=0}^{\infty} b_{ij}(x-\delta_j)^i = b_{oj} + b_{1j}(x-\delta_j) + \sum_{i=2}^{\infty} b_{ij}(x-\delta_j)^i$ (8)

Where $y_j(\delta_j)$= b_{oj},$y'_j(\delta_j)$= b_{1j} ,$\frac{y''_j(\delta_j)}{2!} = b_{2j}$,…,$\frac{y^{(i)}{}_j(\delta_j)}{i!} = b_{ij}$,i=3,4,…and j=1,2,…,m-2

Evaluate Taylor series of y_j (x) about x =η_j, $\forall j = 1,2, \dots, m-2$:

$y_j(x) = \sum_{i=0}^{\infty} c_{ij}(x-\eta_j)^i = c_{oj} + c_{1j}(x-\eta_j) + \sum_{i=2}^{\infty} c_{ij}(x-\eta_j)^i$ (9)

Where $y_j(\eta_j)$= c_{oj},$y'_j(\eta_j)$= c_{1j} ,$\frac{y''_j(\eta_j)}{2!} = c_{2j}$,…,$\frac{y^{(i)}{}_j(\eta_j)}{i!} = c_{ij}$,i=3,4,…and j=1,2,…,m-2

And evaluate Taylor series of y(x) about x = 1:

$y(x) = \sum_{i=0}^{\infty} d_i(x-1)^i = d_o + d_1(x-1) + \sum_{i=2}^{\infty} d_i(x-1)^i$ (10)

Where y(1)= d_o,y'(1)= d_1 ,$\frac{y''(1)}{2!} = d_2$,…,$\frac{y^{(i)}(1)}{i!} = d_i$,i=3,4,…

Step two: Insert the series form (7) into equation (6) and put x = 0, then equate the coefficients of powers of x to obtain a_2 .Then derive equation(6) with respect to x,Then insert the series form (7)in to equation derived and put x=0 equate the coefficients of power of x to obtain a_3 .Iterate the process many times to obtain a_4 then a_5 and so on.

Step three: Make up x=δ_1 in the case j=1 of the series form (7) to obtain $y_1(\delta_1)$= b_{o1} ,to find $y'_1(\delta_1)$= b_{11} derive the series (7) and requite x=δ_1,then insert the series (8) into equation (6) and put x=δ_1,then equate the coefficients of power of (x-δ_1) to obtain b_{21} .to find b_{31} derive equation(6) with respect to x, Then insert the series form (8)in to equation derived and put x=δ_1 and equate the coefficient of power of (x-δ_1). Iterate the process many times to obtain b_{41} then b_{51} and so on. In the same way, we get b_{oj} , b_{1j} and b_{ij} ,i=2,3,…. And j=2,3,…,m-2.

Step four: Make up x=η_1 in the case j=1 of the series form (8) to obtain $y_1(\eta_1)$= c_{o1} ,to find $y'_1(\eta_1)$= c_{11} derive the series (8) and requite x=η_1,then insert the series (9) into equation (6) and put x=η_1,then equate the coefficients of power of (x-η_1) to obtain c_{21} .to find c_{31} derive equation(6) with respect to x, Then insert the series form (9)in to equation derived and put x=η_1 and equate the coefficient of power of (x-η_1). Iterate the process many times to obtain c_{41} then c_{51} and so on. In the same way, we get c_{oj} , c_{1j} and c_{ij} , i=2,3,…. And j=2,3,…,m-2.

Step five: Insert the series form (10) into equation (6) and put x = 1, then equate the coefficients of powers of(x-1) to obtain d_2 ,to find d_3 derive equation(6) with respect to x, Then insert the series form (10)in to equation derived and put x=1 and equate the coefficient of power of (x-1) . . Iterate the process many times to obtain d_4 then d_5 and so on.

Step six: The notation implies that the coefficients depend only on the indicated unknowns $y_j(\delta_j), y_j(\eta_j), j = 1,2, \dots, m-2$, then substitute these date (a_i, b_i, c_i, d_i) in Hermite interpolation polynomial H_{2n+1} equation(5). To find the unknowns coefficients by reduction order equation and use H_{2n+1} as a replacement of y(x) and substitute the nonlocal conditions, we have (m-2) unknown coefficients, we can find this for any |(m-2) by solving this system of algebraic equations using MATLAB,so insert the value of the unknown coefficients into equation(5),thus equation(5) represent the solution of the equation(6).

To illustrate the effectiveness of the presented method,we will consider the following example of non-linear(SMPBVP) with nonlocal conditions.

Example1: Consider the second-order non-linear (SMPBVP)[11]:

$x(x-1)y'' + 6y' + 2y + y^2 = 6\cosh(x) + (2 + x - x^2 + \sinh(x))\sinh(x), \quad 0 \le x \le 1$

With nonlocal conditions

$y(0) + y\left(\frac{2}{3}\right) = \sinh\left(\frac{2}{3}\right) \quad , y(1) + \frac{1}{2} y\left(\frac{4}{5}\right) = \frac{1}{2}\sinh\left(\frac{4}{5}\right) + \sinh(1)$

The exact solution y (x) =sinh(x).

Now, It is clear that x = 0 and 1 , are regular singular point and it is singularity of second kind . We solve this example using Taylor polynomials, with order 12. We get the values nonlocal conditions y(0)= -0.000000000062162 ,y(2/3)= 0.717158461073203 ,y(4/5)= 0.8881059822743, y(1)= 1.175201193600460, y'(0)= 1.000000000020721 ,y'(2/3)= 1.230575576985797, y'(4/5)= 1.337434941873226, y'(1)= 1.543080634846431

Then using these data to find $[1-2L_k'(x_k)(x-x_k)][L_k(x)]^2$ and $(x-x_k)\,[L_k(x)]^2$ when k=0,1,2,3. Every time we get polynomial of degree seven then compensate in equation (5) , we get:
$H_7(\mathrm{x}) =$ 0.0002259993923276 x^7 - 0.0000567571664938 x^6 + 0.008378912700215 x^5 - 0.00001360399735994 x^4+ 0.1666657491403187 x^3 + 0.000000893572893129 6842 x^2 + 1.000000000020721 x- 0.000000000062162

Table-1, gives the result $\mathrm{y}(x_i)$ exact solution and results $\mathrm{H}_7(x_i)$ also the error between them. Table-2 gives the maximum absolute error for Hermite interpolation method with third-kind Chebyshev wavelets Collocation method (3CWCM) and fourth-kind Chebyshev wavelets collocation method (4CWCM)[11]. Figure-1 we give a comparison between the exact solutions with suggested solution.

Table 1- The result of the method for $\mathrm{H}_7(\mathrm{x})$ of example 1

x_i	Exact Solution $\mathrm{y}(x_i)$	Hermite interpolation $\mathrm{H}_7(x_i)$	Error=| $\mathrm{y}(x_i)$- $\mathrm{H}_7(x_i)$|
0	0	-0.000000000062162	6.21620000000000e-11
0.1	0.100166750019844	0.100166757019349	6.99950536420246e-09
0.2	0.201336002541094	0.201336020424022	1.78829283048465e-08
0.3	0.304520293447143	0.304520314207979	2.07608362723377e-08
0.4	0.410752325802816	0.410752340466538	1.46637224029966e-08
0.5	0.521095305493747	0.521095311545534	6.05178696044817e-09
0.6	0.636653582148241	0.636653583098561	9.50319711812142e-10
0.7	0.758583701839533	0.758583701893823	5.42900169264726e-11
0.8	0.888105982187623	0.888105982274306	8.66829941159608e-11
0.9	1.026516725708175	1.026516725174939	5.33236788058389e-10
1	1.175201193643801	1.175201193600460	4.33413305245267e-11

Table 2- The result maximum absolute error of example 1

Method	Hermite interpolation	3CWCM	4CWCM
M.E	$2.07608362723.\ 10^{-8}$	$4.444 \cdot 10^{-10}$	$4.478 \cdot 10^{-9}$

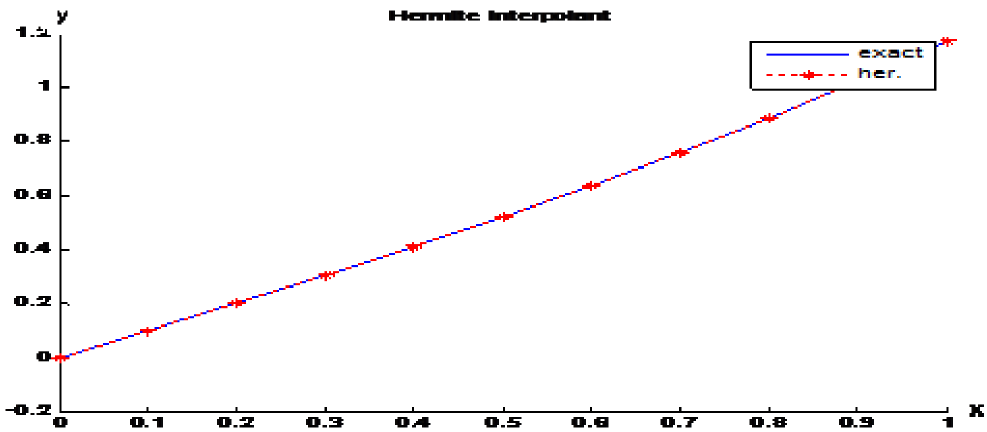

Figure 1- Comparison between the exact and suggested solution

5. Solution for linear SMPBVP

In this section, we apply the Hermite interpolation method H_{2n+1} and Taylor series to solve regular singular point. A general form 2^{nd} order (SMPBVP) is:

$$y'' + a(x)f(x,y,y') = b(x) \tag{11a}$$

A associated with nonlocal condition of the form:

$$y(0) = \sum_{j=1}^{m-2} A_j y(\delta_j)\ ,\ y(1) = \sum_{j=1}^{m-2} B_j y(\eta_j) \tag{11b}$$

WhereA_j,$B_j \epsilon$R and $\delta_j, \eta_j \epsilon(0,1)$, j=1,2……,m-2

$$a(x)=\begin{cases}a_1(x) & 0\le x\le\alpha\\ a_2(x) & \alpha<x\le 1\end{cases}, \qquad b(x)=\begin{cases}b_1(x) & 0\le x\le\alpha\\ b_2(x) & \alpha<x\le 1\end{cases} \tag{11c}$$

And a(x) have singularity α(0,1) because $\lim_{x\to\alpha+} a_1(x) \neq \lim_{x\to\alpha-} a_2(x)$.

Now, to solve the problem by suggested method defines on 1st sub domain then define on 2nd sub domain. Therefore, the construct $H_{2n+1}(x)$,to do this we repeat the same steps the previous in solution second order non-linear (SMPBVP) on 1st sub domain $[0,\alpha]$ and 2nd sub domain $(\alpha,1]$ .To obtain two polynomials represent the solution of equation (11)on sub domain $[0,\alpha]$,and $(\alpha,1]$.Finally, we sum the result of the two solutions, we get a solution to the equation (11) on domain [0,1] . To illustrate the effectiveness of the presented method, we will consider the following example of linear (SMPBVP).

Example2: Consider the second-order singular linear MBVP[11]:

$$y''+f(x)y=g(x)\,, \qquad 0\le x\le 1$$

With nonlocal conditions $y(0)+16y\left(\frac{1}{4}\right)=3\sqrt[4]{e}$, $y(1)+16y\left(\frac{3}{4}\right)=3\sqrt[4]{e^3}$

Where $f(x)=\begin{cases}3x, & 0\le x\le 1/2\\ 2x, & 1/2<x\le 1\end{cases}$, $g(x)=\begin{cases}x(2x-3x^2-3)e^x, & 0\le x\le 1/2\\ x(-3+x-2x^2)e^x, & 1/2<x\le 1\end{cases}$

The exact solution is: y(x)=x(1-x) e^x

It is clear that x = 1/2 is regular singular point and it is singularity of third kind. Method used when the 1st sub domain [0, 1/2] and got the following results using Taylor polynomials, with order 12

y(0)= -0.000000944194 , $y\left(\frac{1}{4}\right)$=0.2407538275585445 , y($\frac{1}{2}$)= 0.412179429344,y'(0)= 0.99999999501,$y'\left(\frac{1}{4}\right)$= 0.8827675573081319,y'($\frac{1}{2}$)= 0.4121806585940366

Then using these data to find $[1-2L'_k(x_k)(x-x_k)][L_k(x)]^2$ and $(x-x_k)\,[L_k(x)]^2$ when k=0,1,2

Every time we get polynomial of degree five then compensate in equation (5) , we get::

$H_5(x)$=- 0.1864703755818482 x^5 - 0.297887657485262 x^4 - 0.5084229230124038 x^3 + 0.0007136769436958884 x^2 + 0.9999999950157075 x - 0.0000009441941949273572

Method used again when the 2nd sub domain (1/2,1] and got the following results usingTaylor polynomials , with order 12

$y\left(\frac{3}{4}\right)$=0.396936731088709, y(1)= -0.000007037767897$y'\left(\frac{3}{4}\right)$=-0.6615619043526, y'(1)= -2.71828093006537

Then using these data to find $[1-2L'_k(x_k)(x-x_k)][L_k(x)]^2$ and $(x-x_k)\,[L_k(x)]^2$ when k=0,1,2.

Every time we get polynomial of degree five then compensate in equation (5) , we get:

$H_5(x)$=- 0.3910581463496783 x^5 + 0.2368196672084446 x^4- 1.091292272205465 x^3 + 0.3291622495789883 x^2 + 0.9052834502947542 x + 0.01107801369989036

Details about the solution to the equation on the domain [0,1] as following, Table-3, gives the result $y(x_i)$ exact solution and results $H_5(x_i)$ also the error between them. Table-4, gives the maximum absolute error for Hermite interpolation with third-kind Chebyshev wavelets Collocation method (3CWCM) and fourth-kind Chebyshev wavelets collocation method (4CWCM) [11]. Figure 2 we give a comparison between the exact solutions with suggested solution.

Table 3- The result of the Hermitei interpolation and exact solution for example 2

x_i	Exact Solution $y(x_i)$	Hermite interpolation $H_5(x_i)$	Error=| $y(x_i)$- $H_5(x_i)$|
0	0	-9.44194194927357e-07	9.44194194927357e-07
0.1	0.0994653826268083	0.0994661156842960	7.33057487725675e-07
0.2	0.195424441305627	0.195423927730433	5.13575194649452e-07
0.3	0.283470349590961	0.283469853275820	4.96315140152692e-07
0.4	0.358037927433905	0.358038794372705	8.66938799803929e-07
0.5	0.412180317675032	0.412179429343271	8.88331761372818e-07
0.6	0.437308512093722	0.437310510338862	1.99824513957836e-06
0.7	0.422888068568800	0.422887941273205	1.27295594953836e-07
0.8	0.356086548558795	0.356086372589765	1.75969030291601e-07
0.9	0.221364280004125	0.221363933503805	3.46500320796972e-07
1	0	-7.03777306577079e-06	7.03777306577079e-06

Table 4- The result maximum absolute error of example 2

Method	Hermite interpolation	3CWCM	4CWCM
M.E	$7.03777306577079.\ 10^{-6}$	$7.1 \cdot 10^{-6}$	$2.8 \cdot 10^{-5}$

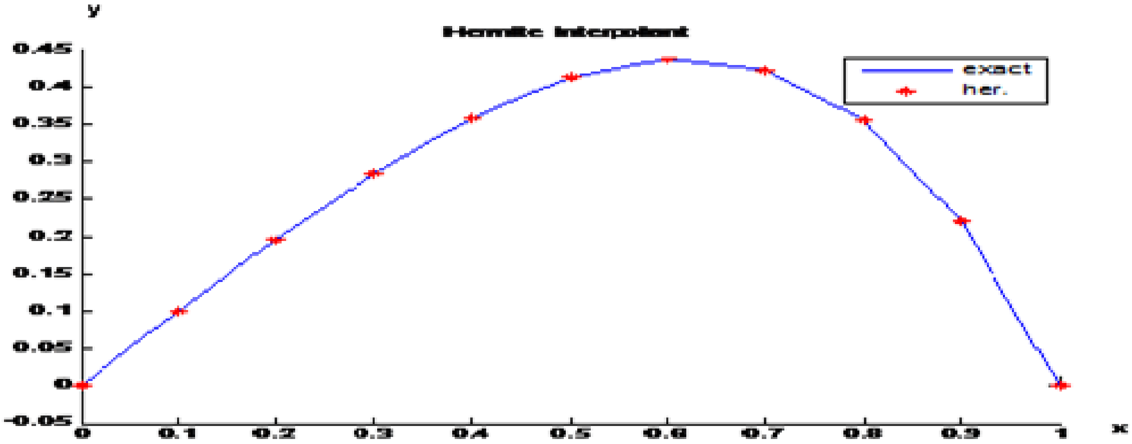

Figure 2- Comparison between the exact and suggested solution

6. Behavior of the solution in the neighborhood of the singularity x= 0 [12]

Our main concern in this section will be the study of the behavior of the solution in the neighborhood of singular point. Consider the following singular initial value problem:

$y''(x) + ((N-1)/x)\,y'(x) = f(y)$, $N \geq 1$, $0 < x < 1$, (12)

$y(0) = y_o$, $\lim_{x\to 0+} x\,y'(x) = 0$, , (13)

Where f(y) is continuous function .As the same manner in [12], let us look for a solution of this problem in the form:

$y(x)=y_o - C\,x^k(1+o(1))$

$y'(x)=-Ck\,x^{k-1}(1+o(1))$, $y''(x)=-C\,k(k-1)x^{k-2}\big(1+o(1)\big), x \to 0+$ (14)

where C is a positive constant and k > 1. If we substitute (14) in (12) we obtain : $C=(1/k)\,(f(y_o)/N)^{k-1}$, (15)

In order to improve representation (14) we perform the variable substitution:

$y(x) = y_0 - C\,x^k\,(1+g(x))$, (16)

we easily obtain the following result which is similar to the results in [12].

Theorem 3[12]: For each $y_o > 0$, problem (12), (13) has, in the neighborhood of x = 0, a unique solution that can be represented by:

$y(x, y_o) = y_o - C\,x^k(1 + g\,x^k + o(x^k))$,

where k, C and g are given by (15) and (16), respectively .We see that these results are in good agreement with the ones obtained by the method in [12], they are also consistent with the results presented in [13]. In order to estimate the convergence order of the suggested method at x = 0, we have carried out several experiments with different values of n and used the formula:

$c_{y_o} = -\log_2\,(|y_0^{n3} - y_0^{n2}| \,/\, |y_0^{n2} - y_0^{n1}|)$, (17)

Where y_0^{ni} is the approximate value of y_o obtained with ni, ni= 1,2, 3, 4,…

7. Conclusion

In this paper, we presented Hermite interpolation method to solve the singular multi point boundary value problem with nonlocal condition. It is clearly seen that present method is a powerful and accurate seen method for finding singular differential equation in the form of polynomial and presents a rapid convergence for the solutions. The numerical results showed that the Hermite interpolation method can solve the problem effectively and the comparison shows that the present method is in good agreement with the existing results in the literature.

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