ISSN: 0067-2904
GIF: 0.851

More on the Minimal Set of Periods

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Abstract

Let M be a n-dimensional manifold. A C^1- map $f : M \rightarrow M$ is called transversal if for all $m \in N$ the graph of f^m intersect transversally the diagonal of $M \times M$ at each point (x,x) such that x is fixed point of f^m. We study the minimal set of periods of f(M per (f)), where M has the same homology of the complex projective space and the real projective space. For maps of degree one we study the more general case of (M per (f)) for the class of continuous self-maps, where M has the same homology of the n-dimensional sphere.

Keywords: mainfolds , transveral map , Lefschetz number , Lefschetz number for periodic points .

إضافات لأصغر مجموعة دورية

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الخلاصة

ليكن M منطوي ذو بعد n . الدالة المستمرة والقابلة للاشتقاق $f: M \rightarrow M$ تسمى مستعرضة اذا كان لكل $m \in N$ ، بيان الدالة f^m يقطع قطر $M \times M$ بصورة مستعرضة لكل نقطة (x,x)، حيث ان x نقطة صامدة الى f^m. ستقوم بدراسة اصغر مجموعة دورية الى f ((M per (f))، حيث ان M يمتلك نفس هومولوجي المستوى الاسقاطي العقدي والمستوى الاسقاطي الحقيقي، بالنسبة للدوال ذات الدرجة واحد سندرس حالة اعم لحساب ((M per (f)) لمجموعة الدوال الذاتية المستمرة، حيث ان M يمتلك نفس هومولوجي الكراة ذات البعد n .

1. Introduction:-

Let M be a compact manifold of dimension n. Let $f : M \rightarrow M$ be a continuous map, a fixed point of f is a point x of M such that f(x) = x, denoted the totality of fixed points by Fix(f).

The point $x \in M$ is periodic of period m if $x \in \text{Fix}(f^m)$ but $x \notin \text{Fix}(f^k)$ for all k = 1 , … , m-1. Let per(f) denote the set of periods for all the periodic points of f . Llibre, J. in [1] gave the concept of the minimal set of periods of f in the class of continuous (resp. transversal , transversal holmorphic [2]) self – maps of M as the set

$\text{MPer}_C(f)= \bigcap_g per\ g$ (resp. $\text{MPer}_t(f) = \bigcap_g pr\ g$, $\text{MPer}_h(f) = \bigcap_g per\ g$).

Where g runs over all continuous self - maps of M of the same degree of f (resp. g runs over all transversal self – maps of M of the same degree of f , g runs over all transversal holomorphic self – maps of M of the same degree of f). In [1] Llibre completely described the minimal set of periods of transversal holomorphic self – maps , so he was deal with the complex manifolds such as a n-dimensional sphere S^n (n even) and a complex projective space . In this paper we investigated the minimal set of periods of self – maps which is only transversal on some spaces such as a real

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projective space and a complex projective space. For maps of degree one, we study the minimal set of periods of continuous maps of S^n for each n.

2. Transversal self – maps.

Let M be a C^1 – compact manifold. It is known a continuous map f : M → M induces endomorphisms $f_{*j} : H_j (M ; Q) \to H_j(M ; Q)$, (for j = 0,1, …… , n) on the rational homology groups of M (see , for instance [3]). The Lefschetz number of f is defined by:

$$L(f) = \sum_{k=0}^{n} (-1)^K trace\ (f_{*k})$$

By the Lefschetz fixed point theorem , if L(f) ≠ 0 then f has fixed points (see , for instance [4]). In [1] LLibre investigated the minimal set of periods of f in the class of transversal holomorphic self maps of M , where M has the same homology of the n – dimensional Sphere (n even) . For maps of degree one on S^n , we have the following theorem :

Theorem 2.1. Let f : $S^n \to S^n$ be a continuous map of degree one, then :

$$\text{MPer (f)} = \begin{Bmatrix} 1 & n\ even \\ \emptyset & n\ odd \end{Bmatrix}$$

Proof: Since l (f) = $L(f) = \sum_{K=0}^{n} (-1)^K\ trace\ (f_{*k})$
= 1 + $(-1)^n.1 = 2 \neq 0$

Hence, by the Lefschetz fixed point theorem 1 ∈ per f and f is arbitray implies 1 ∈ $MPer_c(f)$. Now since

$MPer_c(f) = \cap_g per(g)$

Where g run over all continuous self – map of S^n of degree 1 . Thus one of these maps is the identity map which is continuous of degree one and it is not difficult to show that it has no periodic point of period greater than one , hence

$\cap_g perg = MPer_c(f) = 1$

For n odd , in [5] was proved that S^n admits infinite number of fixed point free homeomorphisms. Thus infinite number of these maps containd in the collection maps of degree one (of Course homeomorphisms degree ±1) , which leads to 1 ∉ $MPer_c(f)$. Similarly as case (a) the identity map has no periodic point greater than one , implies $MPer_c(f) = \emptyset$ which complete the proof ■

We can consider the Lefschetz number of (f^m is an iterate of f) but in general it is not true that $L(f^m) \neq 0$, then f has periodic points of period m. It could have periodic points with period some proper division of m. Therefore, we will use the Lefschetz numbers for periodic points [6] for analyzing if a given period belongs to the set of periods of a self – map. More precisely, for every m ∈ N the Lefschetz number of period m, $l(f^m)$ is defined as follows :

$$l\ (f^m) = \sum_{r/m} \mu(r)\ L\left(f^{\frac{m}{r}}\right) \quad (1)$$

Where $\sum_{r/m}$ denoted the sum over all positive divisors r of m , and μ is the Moebius function defined by :

$$\mu(m) = \begin{cases} 1 & if\ m = 1 \\ 0 & if\ k^2/m \quad \text{for Some } k \in \mathbb{N} \\ (-1)^r & if\ m = p_1 \dots p_r \quad \text{distinct prime factors} \end{cases}$$

According to the inversion formula (see for instance[7]).

$L\ (f^m) = \sum_{r/m} l(f^r)$

The Leschetz number of period m, will become interesting after showing that for some classes of self – maps we have , if $l(f^m) \neq 0$ then m ∈ Per(f). This is almost the case when f is a transversal self map since for a transversal self – map f , the fixed points of f^m are isolated and M compact , the cardinal of Fix (f^m) is finite for every m ∈ N . Dold [8] showed that m divides $l(f^m)$ for every m ∈ N. The following result will play an important role in this paper and it was proved by using the Leschetz zeta function, see [6].

Theorem 2.2 [6] : Let f be a transversal self map. Suppose that $l(f^m) \neq 0$ for some $m \in N$.

(a) If m is odd then $m \in$ per f

(b) If m is even then $\left\{\frac{m}{2}, m\right\} \cap per\ f \neq \emptyset$

This results on the transversal self – maps on arbitrary compact manifolds given in theorem (2-2) are in general difficult to apply because of the computation of $l(f^m)$. Of course, if the homological rational groups are simple then these computations become easier. We will investigate in this section the complex projective space while the real projective space in the next section.
The complex projective Space of dimension n is the quotient of $\mathbb{C}^{n+1}$ by the equivalence relaction :

$(z_0, z_1, \ldots, z_n) \sim (\lambda z_0, \lambda z_1, \ldots, \lambda z_n)$; $\lambda \in \mathbb{C}, \lambda \neq 0$

Point in ℂP(n) are represented in homogeneous coordinates by
$[z_0 : z_1 : \ldots : z_n]$, $z_i \neq 0$ for some $0 \leq i \leq n$; which denotes an equivalence class . The Space ℂP(n) is then the natural compactification of $\mathbb{C}^n$ and can be thought as adding at infinity a subspace of dimension n-1. For example, ₵p(1) , the Riemann sphere , is the compactification of ℂ , obtained by adding the point at infinity.

Now, it is well known facts from topology (see for instance[9]) the homology groups of ℂP(n) with rational coefficients are :

$$H_i(₵P(n); Q) = \begin{cases} Q & i = 0,2,\ldots .2n \\ O & otherwise \end{cases}$$

Moreover, any Continuous mapping f of ℂP(n) with degree d induces homomorphisms on the homology groups.

$$f_{*2k} = (d^k) \quad for\ k = 0,1,\ldots,n$$

Hence

$$L(f) = \sum_{i=1}^{2n} (-1)^i\ trace\ (f * i)$$
$$= 1 + d + d^2 + \ldots + d^n .$$

The following theorem was proved in [1].

Theorem 2.3 Let M be a compact Complex manifold and f: M → M be a non constant transversal holomorphic map, then $MPer_h(f) = Per(f)$.
As an immediate consequence of this theorem is $Mper_h(f) = per(f)$, where f :ℂ P(n) → ℂ P(n) .
We shall compute the minimal set of periods for self – maps on ℂ P(n) which is transversal not necessary holomorphic for some cases and we have the following theorem.

Theorem 2.4 Let f:ℂ(n) → ℂP(n) be a transversal map of degree d ,

(a) If m is prime then $m \in MPer_t(f)$

(b) If m is even of the form 2^k r (r prime). Then

$\{2^k r, 2^{k-1} r\} \cap MPer_t(f) \neq \emptyset$

Proof: (a) Suppose first $d \geq 2$ and $m \geq 2$. Now Since the non zero homology groups of ℂP(n) are one – dimensional and degree(f^m) = (degree f$)^m$, so we have for any m

$L(f^m) = 1 + d^m + d^{2m} + \ldots\ldots + d^{nm}$, but m is prime implies

$$l(f^m) = \sum_{r/m} \mu(r)\ L(f^{\frac{m}{r}})$$
$$= \mu(1) L(f^m) + \mu(m) L(f)$$
$$= 1 + d^m + d^{2m} + \ldots\ldots + d^{nm} - 1 - d + d^2 - \cdots - d^n$$
$$=(d^{nm} - d^n) + \left(d^{(n-1)m} - d^{n-1}\right) + \cdots + (d^m - d) > 0$$

Hence $m \in MPer_t(f)$ (theorem 2.2) which Complete the proof
(b) Since $m = 2^k r$ and r (prime) So we have two steps:

Step (1) If r =1 , then $\mu(m) = \mu(2^k) = 0$ implies

$$l(f^m) = \mu(1)\, L(f^m) + \mu(2) L\left(f^{\frac{m}{2}}\right)$$
$$= L(f^m) + (-1) L\left(f^{\frac{m}{2}}\right)$$

Clear now $l(f^m)$ greater than zero and we have done by theorem 2.2

Step(2) for r ≥ 3 we have

$$l(f^m) = \mu(1)\, L(f^m) + \mu(2) L\left(f^{\frac{m}{2}}\right) + \mu(r)\, L\left(f^{\frac{m}{r}}\right) + \mu(2r)\, L\left(f^{\frac{m}{2r}}\right)$$

If μ(r) = +1 , then by Moebius function μ(2r) = -1 and conversely .

Case (1) Suppose μ(r) = 1 , μ(2r) = -1 leads

$$\mu(1)\, L(f^m) > \mu(2) L\left(f^{\frac{m}{2}}\right) \; and$$
$$\mu(r)\, L\left(f^{\frac{m}{r}}\right) > \mu(2r)\, L\left(f^{\frac{m}{2r}}\right)$$

Hence $L(f^m) > 0$ i.e. $L(f^m) \neq 0$

Case (2) If μ(r) = -1 and μ (2r) = +1 , then $l(f^m) = 1 + d^m + d^{2m} + \dots\dots\dots + d^{nm} - 1 - d^{\frac{m}{2}} - d^{\frac{2m}{2}} - \dots\, d^{\frac{nm}{2}} - 1 - d^{\frac{m}{r}} - d^{\frac{2m}{r}} - \dots\, d^{\frac{nm}{r}} + 1 + d^{\frac{m}{2r}} + \cdots + d^{\frac{nm}{2r}}$

Now by Mathematical induction we have

$$d^{nm} \geq 2\, d^{n\frac{m}{2}} \;, n \geq 1$$

So

$$d^{nm} + d^{(n-1)m} + \cdots + d^{2m} + d^m \geq 2\left(d^{\frac{nm}{2}} + d^{\frac{(n-1)m}{2}} + \dots + d^{\frac{m}{2}}\right),$$

and Since r ≥ 3 , implies

$$d^{nm} + d^{(n-1)m} + \cdots + d^{2m} + d^m > d^{\frac{nm}{2}} + d^{\frac{(n-1)m}{2}} + \cdots + d^{\frac{m}{2}} + d^{\frac{nm}{r}} + d^{\frac{(n-1)m}{r}} + \dots + d^{\frac{m}{r}}$$

Which certainly implies

$$d^{nm} + d^{(n-1)m} + \cdots + d^{2m} + d^m + d^{\frac{m}{2r}} + d^{\frac{2m}{2r}} + \cdots + d^{\frac{nm}{2r}} > d^{\frac{nm}{2}} + d^{\frac{(n-1)m}{2}} + \dots + d^{\frac{m}{2}} + d^{\frac{nm}{r}} + d^{\frac{(n-1)m}{r}} + d^{\frac{m}{r}}$$

That's mean $l(f^m) > 0$. Thus case (1) and (2) gives $l(f^m) \neq 0$ and by theorem (2.2) we have $\{2^k r, 2^{k-1} r\} \cap MPer_t(f) \neq 0$ ■

Note: By use Lefschetz number of period m case m=1 and d=0,1, theorem (2.4) follows immediately from theorem 2.2.

3. Real projective space

Real projective space , or RP(n) , is the topological space of lines passing through the origin 0 in R^{n+1} . It is a compact, smooth manifold of dimension n (for more details , see [3],[4]) . The rational homology group of RP(n) is given by :

$$H_k(\mathrm{R}P(n);\, Q) = \begin{cases} Q & k = 0, \mathrm{n} \\ 0 & otherwise \end{cases}$$

We will investigate in this section the minimal set of period of RP(n) (n odd). Now since any continuous mapping f of RP(n) induces homomorphisms on the homology groups , hence

L(f) $= \sum_{i=0}^{n} (-1)^i$ trace f_{*i} (n odd)
= 1 – d (d = degree f)
= 0 iff d=1

That's mean 1 ∈ $Per_c(f)$ for any map with d ≠1 . So for m > 1 we have the following Theorem.

Theorem 3.1 For m even of the from $2^k P$ (p prime) , let f:RP(n) → RP(n) be atransversal map of degree d , s.t. d ∉ {-1,0,1} , then either $2^k P$ or $2^{k-1} P$ belongs to $MPer_t$ (f) .

Proof : By use formula (1) we have

$$l(f^m) = \mu(1) L(f^m) + \mu(2) L\left(f^{\frac{m}{2}}\right) + \mu(P) L\left(f^{\frac{m}{P}}\right) + \mu(2P) L\left(f^{\frac{m}{2P}}\right)$$
$$= (1 - d^m) + (-1)\left(1 - d^{2^{k-1}p}\right) + (-1)\left(1 - d^{2^k}\right) + (1 - d^{2^{k-1}})$$
$$= 1 - d^{2^{k-1}p} - 1 + d^{2^{k-1}p} - 1 + d^{2^k} + 1 - d^{2^{k-1}}$$
$$= d^{2^k} + d^{2^{k-1}p} - d^{2^{kp}} - d^{2^{k-1}}$$
$$= d^{2^{k-1}}\left(d^2 \;\; + d^p\right) - d^{2^{k-1}}\left(1 + d^{2\;\;p}\right)$$

$= d^{2^{k-1}}(d^2 \ + d^p - 1 - d^{2p} \ )$
$= d^{2^{k-1}}\left(d^p\left(-d^2 \ +1\right)+(d^2 \ -1\right)$
$= d^{2^{k-1}}\left(d^p\left(1 - d^2 \ \right)-\left(1-d^2 \ \right)\right)$
$= d^{2^{k-1}}\left(\left(1 - d^2 \ \right)+\left(d^P \ -1\right)\right)$
= 0 if and only if d ∈ {-1 , 0 ,1}

Implies L(f^m) ≠ 0 and by theorem 2.2 we have done ∎

Theorem 3.2 For (m odd) , let f: RP(n) → RP(n) be a transversal map of degree ≠ 1 , then m ∈ MPer$_t$(f) .

Proof : first Similarly as theorem (3.1) we have

$$l(f^m) = \sum_{t/m}\mu(t)\ L\left(f^{\frac{m}{t}}\right)$$
= 1- deg f …… (2)

By use the fundamental theorem of Arithmetic [7], we have

$m = p_1^{r_1}\, p_2^{r_2} \dots\dots\dots\dots . p_S^{r_s}$, where $p_1 , p_2 , \dots\dots , p_s$ are Prime numbers and r$_i$ ≥ 1 (i = 1 , ….. , s) . Thus

$$l(f^m) = \mu(1)\,L(f^m) + \sum_{i=1}^{s}\mu(p_i)\ L\left(f^{\frac{m}{P_i}}\right) + \sum_{i_1,i_2=1}^{s}\mu\left(p_{i_1}p_{i_2}\right)\ L\left(f^{\frac{m}{P_{i_1}P_{i_2}}}\right) + \sum_{i_1,i_2,i_3=1}^{s}\mu\left(p_{i_1}p_{i_2}p_{i_3}\right)\ L\left(f^{\frac{m}{P_{i_1}P_{i_2}P_{i_3}}}\right)+\dots+\sum_{i_1,i_2,\dots i_{s-1}=1}^{s}\mu\left(p_{i_1}p_{i_2}\dots p_{i_{s-1}}\right)\ L\left(f^{\frac{m}{P_{i_1}P_{i_2}\cdots P_{i_{s-1}}}}\right)+ \mu(p_1\dots p_s)L\left(f^{\frac{m}{P_1\cdots P_s}}\right)$$

Hence

$$l(f^m) = L(f^m)-\sum_{i=1}^{s}L\left(f^{\frac{m}{P_i}}\right) + \sum_{i_1,i_2=1}^{s}L\left(f^{\frac{m}{P_{i_1}P_{i_2}}}\right)-\sum_{i_1,i_2,i_3=1}^{s}L\left(f^{\frac{m}{P_{i_1}P_{i_2}P_{i_3}}}\right)+\cdots+ (-1)^s\,L\left(f^{\frac{m}{P_1\dots\dots\dots P_s}}\right)$$

By (2) we have

$$l(f^m) = [1-degf^m\,]-\left(\sum_{i=1}^{s}\left(1-\deg f^{\frac{m}{P_i}}\right)\right)+\left[\sum_{i_1,i_2=1}^{s}\left(1-\deg f^{\frac{m}{P_{i_1}P_{i_2}}}\right)\right]+\cdots + (-1)^{s-1}\left[\sum_{i_1,i_2,\dots,i_{s-1}=1}^{s}\left(1-\deg f^{\frac{m}{P_{i_1}P_{i_2}\cdots P_{i_{s-1}}}}\right)\right]+\cdots + (-1)^{s+1}\left[1-\deg\left(f^{\frac{m}{P_1\dots P_s}}\right)\right]$$

Implies

$$l(f^m) = [1-degf^m\,]-\left(s-\sum_{i=1}^{s}\deg f^{\frac{m}{P_i}}\right)+\left[C_2^s-\sum_{i_1,i_2=1}^{s}\deg f^{\frac{m}{P_{i_1}P_{i_2}}}\right] - \left[C_3^s-\sum_{i_1,i_2,i_3=1}^{s}\deg f^{\frac{m}{P_{i_1}P_{i_2}p_{i_3}}}\right]+\left[C_4^s-\sum_{i_1,i_2,i_3,i_4=1}^{s}\deg f^{\frac{m}{P_{i_1}P_{i_2}p_{i_3}p_{i_4}}}\right]+\cdots + (-1)^{s+1}\ \deg\left(f^{\frac{m}{P_1\cdots P_s}}\right)$$

Now by the properties of combinations [10] we have

1-S + $C_2^s - C_3^s + \cdots + (-1)^s C_s^s = 0$

And by induction we have

$$\deg f^m \ = \ (\deg f)^m > \sum_{r/m}(\deg f)^r \ \ (\deg f > 1\,, m > 1)$$

Which gives $(f^m) \neq 0$, hence by theorem (2.2) we have done ■

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