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P-Polyform Modules

Maria Mohammed Baher*, Muna Abbas Ahmed

Department of Mathematics, College of Science for Women, University of Baghdad, Baghdad, Iraq

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Abstract

An R-module M is called polyform if every essential submodule of M is rational. The main goal of this paper is to give a new class of modules named P-polyform modules. This class of modules are contained properly in the class of polyform modules. Several properties of this concept are introduced and compared with those which is known in the concept of polyform modules. Another characterization of the definition of P-polyform modules is given as an analogue to that in the concept of polyform modules. So we proved that a module M is P-polyform if and only if $\mathrm{Hom}_R(\frac{K}{N}, M)= 0$, for each essential submodule N of M, which is pure in M, and a submodule K of M, with $N\subseteq K\subseteq M$. The relationships between this class of modules and some other related concepts are discussed such as monoform, QI-monoform, monoform, essentially quasi-Dedekind, essentially prime, purely quasi-Dedekind, ESQD and St-polyform modules. Furthermore, purely St-polyform is defined and its relationship with the P-polyform module is studied.

Keywords: Polyform modules, P-polyform modules, Rational submodules, P-rational submodules, Pure submodules, Essential submodules.

المقاسات متعددة الصيغ من النمط -P

مارية محمد بحر * , منى عباس أحمد

قسم الرياضيات، كلية العلوم للبنات، جامعة بغداد، بغداد، العراق

الخلاصة

يُقال للمقاس M بأنه متعدد الصيغ اذا كان كل مقاس جوهري فيه نسبياً. الهدف الرئيس من هذا البحث هو إعطاء صنف جديد من المقاسات يدعى بالمقاسات متعددة الصيغ من النمط –P. ان هذا النوع من المقاسات محتوى بشكل فعلي في المقاسات متعددة الصيغ. العديد من الخصائص المهمة قُدمت حول المقاسات متعددة الصيغ من النمط –P. كما قدم تشخيص أخر للمقاسات متعددة الصيغ من النمط –P. مناظرا لما هو معروف في المقاسات متعددة الصيغ. على سبيل المثال برهنا ان المقاس M يكون متعدد الصيغ من النمط –P. اذا وفقط اذا تحقق $\mathrm{Hom}_R\left(\frac{K}{N}, M\right) = 0$ لكل مقاس جزئي جوهري N بحيث يكون نقياً في M ولكل $K\leq M$ حيث ان $N\subseteq K\subseteq M$.

*Email: mariya.abd1703@csw.uobaghdad.edu.iq

نوقشت أيضاً علاقة هذا الصنف من المقاسات بعدد من المقاسات الاخرى مثل المقاسات احادية الصيغة، مقاسات احادية الصيغة من النمط–QI، المقاسات شبه الديديكاندية الجوهرية، المقاسات الأولية الجوهرية، المقاسات شبه الديديكاندية الأولية، ESQD. فضلاً عن ذلك تم تعريف المقاسات متعددة الصيغ من النمط – St النقية ودُرست علاقاتها بالمقاسات متعددة الصيغ من النمط –P.

1. Introduction

Polyform modules play a large and important role in module theory. A non-zero submodule N of M is said to be large or essential (briefly $N\leq_e M$), if $N\cap L\neq 0$ for every non-zero submodule L of M, [1, Definition 5.1.1, P.106]. An R-module M is called injective if for every monomorphism $f:M\longrightarrow B$ and for every homomorphism $g: M\longrightarrow C$ there is a homomorphism $h: B\longrightarrow C$ with $g=h\circ f$, [2, P.9]. An essential monomorphism is a monomorphism $f: S\longrightarrow T$ such that $f(S)\leq_e T$, [1, Definition 5.6.5(1)]. An injective hull of M is denoted by E(M), and it is defined as a monomorphism $f: M\longrightarrow E(M)$ with E(M) is an injective module and f is an essential monomorphism, [1, P.124]. A submodule N of an R-module M is called rational (simply $N\leq_r M$) if $\mathrm{Hom}_R(\frac{M}{N}, E(M))=0$, where E(M) is the injective hull of M, [3, P.274]. An R-module M is called polyform if every essential submodule of M is rational, [4]. A submodule N of M is called pure if $N\cap IM=IN$ for every ideal I of R, [5, P.18]. A submodule N of M is called a P-rational submodule if N is a pure submodule and $\mathrm{Hom}_R(\frac{M}{N}, E(M))=0$, [6]. Note that every P-rational submodule is rational.

By using the concept of P-rational submodules, a new type of module is introduced and studied in this paper, it is called P-polyform modules, and it is contained properly in the class of polyform modules. We try to investigate several results of P-polyform modules as analogues to results which is known in the polyform modules.

This paper consists of four sections. Section two discussed the main properties of P-polyform modules. Among these results are the following:

- Let M be a prime R-module, and N be a non-zero pure and essential submodule of M such that $\mathrm{ann}_R\left(\frac{M}{N}\right)\not\subseteq \mathrm{ann}_R(M)$, then M is P-polyform, see Proposition 2.5.
- If M is a P-polyform and $N\leq_e M$, then N is a P-polyform module, see Proposition 2.6.

Moreover, another characterization of the definition of a P-polyform module is given as follows:

- Let M be an R-module. Then M is a P-polyform module if and only if $\mathrm{Hom}_R(\frac{K}{N}, M)= 0$, for every essential submodule N of M, which is pure in M, and a submodule K of M, with $N\subseteq K\subseteq M$. See Theorem 2.8.
- For any R-module M, consider the following statements:

1. M is a P-polyform module.
2. M is a polyform module.
3. For any submodule N of M, and each non-zero homomorphism $f: N\rightarrow M$, implies kerf is closed in N.
4. For any non-zero pure submodule N of M, and any non-zero homomorphism $f: N\rightarrow M$, implies $\mathrm{kerf}\not\leq_e N$.

Then (1) ⇒ (2) ⇒ (3) ⇒ (4).
See Theorem 2.10.

Section three is focused on the relationships of the P-polyform modules with quasi-invertibility monoform modules, as the following shows:

- Let M be a multiplication and prime module. If M is P-polyform then M is quasi-invertibility monoform, see Proposition 3.2.
- Let M be a multiplication and prime R-module. Consider the following:

1. M is a P-polyform module.
2. M is a polyform module.
3. M is a quasi-invertibility monoform module.
Then (1) ⇒ (2) ⇔ (3), and if R is a regular ring then (3) ⇒ (1).
See Theorem 3.5.

- Let M be a quasi-injective module with $J(End_R(M))=(0)$, consider the following:

1. M is a P-polyform module.
2. M is a polyform module.
3. M is a quasi-invertibility monoform module.
Then (1) ⇒ (2) ⇔ (3).
See Proposition 3.6.

- Let R be a quasi-Dedekind ring. Consider the following:

1. R is a P-polyform ring.
2. R is a polyform ring.
3. R is a quasi-invertibility monoform ring.
4. R is a monoform ring.
Then (1) ⇒ (2) ⇒ (3) ⇒ (4).
See Proposition 3.7.

- Let R be an essentially quasi-Dedekind ring. Consider the following:

1. M is a P-polyform ring.
2. M is a quasi-invertibility monoform ring.
3. M is a polyform ring.
Then (1) ⇒ (2) ⇒ (3).
See Proposition 3.9.

Section four is studied the relationships between the P-polyform modules and other related modules such as in the following results:

- Let M be a uniform module over a regular ring. Then M is P-polyform if and only if M is a monoform module, see Proposition 4.3.
- Every P-polyform module is essentially quasi-Dedekind, see Proposition 4.4.
- Every P-polyform module is ESQD, see Proposition 4.9.
- Let M be a uniform and essentially quasi-Dedekind module. Consider the following statements:

1. M is a Purely St-polyform module.
2. M is a P-polyform module.
3. M is a monoform module.
4. M is a quasi-invertibility monoform module.
5. M is a polyform module.
Then (1) ⇒ (2) ⇒ (3) ⇒ (4) ⇒ (5).
See Theorem 4.13.

It is noteworthy that all rings R in this paper are commutative with identity, and all modules are unitary left R-modules.

2. P-Polyform Modules

This section is devoted to investigating the main properties of P-polyform modules.

Definition 2.1: An R-module M is called P-polyform if every essential submodule of M is P-rational in M. Equivalently, $\mathrm{Hom}_R(\frac{M}{N}, E(M))=0$ for every essential submodule N of M, which must be pure in M, where E(M) is the injective hull of M. A ring R is called P-polyform if R is P-polyform R-module.

Examples and Remarks 2.2

1. Every P-polyform module is a polyform module. This is followed by the direct implication from P-rational to rational submodules, [6].
2. The converse of (1) is not true in general, for example, the module of rational numbers $\mathbb{Q}$ is a polyform module, but not P-polyform, as it will be verified in (4) below.
3. Each simple module is a P-polyform, since the only essential submodule in a simple module is itself which is a P-rational submodule, that is $\mathrm{Hom}_R(\frac{M}{M}, E(M)) \cong \mathrm{Hom}_R(0, E(M))=0$. Thus, M is P-polyform.

Remember that an R-module M is said to be nonsingular if $\mathbb{Z}(M)=0$, where $\mathbb{Z}(M)=\{x\in M: \mathrm{ann}_R(x)\leq_e R\}$, [2, P.31].

4. It is known that every nonsingular module is polyform. This property is not held in the class of P-polyform modules, for example, the $\mathbb{Z}$-module of rational numbers $\mathbb{Q}$ is nonsingular, but not a P-polyform module, since $\mathbb{Z} \leq_e \mathbb{Q}$, but $\mathbb{Z}$ is not P-rational in $\mathbb{Q}$, [6, Remark 2.2 (2)].
5. Let M be a P-polyform module, then $N\leq_e M$ if and only if $N\leq_{pr} M$.
   Proof: It follows directly from the definition of a P-polyform module.

Recall that an R-module M is called semisimple if every submodule of M is a direct summand of M, [1, P.107].

6. Every semisimple module is P-polyform because the only essential submodule of a semisimple module M is itself which is P-rational, see [6, Remark 2.3 (8)].
7. $\mathbb{Z}_4$ is not P-polyform $\mathbb{Z}$-module, since $<\bar{2}>\leq_e \mathbb{Z}_4$ but not P-rational in $\mathbb{Z}_4$, [6, Example 2.3 (4)].

Remember that a non-zero module M is called pure simple if the only pure submodules of M are (0) and M itself, [7].

8. The $\mathbb{Z}$-module $\mathbb{Z}_{p^\infty}$ is not P-polyform, in fact, $\mathbb{Z}_{p^\infty}$ is a pure simple module, and by, [6, Remark 2.3 (9)], the only P-rational submodule of $\mathbb{Z}_{p^\infty}$ is $\mathbb{Z}_{p^\infty}$. On the other hand, $<\frac{1}{p}+\mathbb{Z}>\leq_e \mathbb{Z}_{p^\infty}$, but $<\frac{1}{p}+\mathbb{Z}>\not\leq_{pr} \mathbb{Z}_{p^\infty}$. Hence $\mathbb{Z}_{p^\infty}$ is not P-polyform.
9. If $\frac{M}{N}$ is a P-polyform R-module, then M is not necessarily P-polyform for example, $\mathbb{Z}_6 \cong \frac{\mathbb{Z}}{6\mathbb{Z}}$ is a P-polyform $\mathbb{Z}$-module, since it is semisimple, while $\mathbb{Z}$ is not P-polyform $\mathbb{Z}$-module since $2\mathbb{Z} \leq_e \mathbb{Z}$ but $2\mathbb{Z} \not\leq_{pr} \mathbb{Z}$, [6, Remark 2.3 (3)].

A ring R is said to be regular (in the sense of Von Neumann) if for every $a\in R$ there is an $x\in R$ such that $axa=a$, [2, P.10].

10. If M is a module over a regular ring, then the two classes polyform and P-polyform modules coincide.

Proof: It follows directly by [6, Remark 2.3(11)].

Recall that an R-module M is called a scalar if for each $f \in End_R(M)$, there exists $r \in R$ such that $f(x)=rx$ for all $x \in M$, [8], where $End_R(M)$ is the endomorphism ring of M.

The following proposition deals with the connection between P-polyform rings and the endomorphism ring of R-modules.

Proposition 2.3: Let M be a faithful scalar R-module. Then R is P-polyform if and only if $End_R(M)$ is a P-polyform ring.
Proof: Since M is a faithful scalar R-module, then $End_R(M) \cong R$, [9], so if R is a P-polyform ring, then $End_R(M)$ is a P-polyform ring, and vice versa.

An R-module M is called multiplication, if every submodule of M is of the form IM, for some ideal I of R, [10]. Since every finitely generated multiplication module is scalar, [8, Corollary 1.1.11, P.12], then we obtain the following.

Corollary 2.4: Let M be a finitely generated faithful multiplication module then R is P-polyform if and only if $End_R(M)$ is a P-polyform ring.

Remember that an R-module M is called prime if $ann_R(M)=ann_R(N)$, for every non-zero submodule N of M, [11].

Proposition 2.5: Let M be a prime R-module, and N be a non-zero pure and essential submodule of M such that $ann_R\left(\frac{M}{N}\right) \not\subseteq ann_R(M)$, then M is P-polyform.
Proof: Let N be an essential submodule of M, and $0 \neq x \in M$. Since M is prime then $ann_R(M)=ann_R(x)$ $\forall x \in M$. But $ann_R\left(\frac{M}{N}\right) \not\subseteq ann_R(M)$, therefore $ann_R\left(\frac{M}{N}\right) \not\subseteq ann_R(x)$. Now, $N \leq_e M$, and since N is pure in M, then by, [6, Proposition 2.21], $N \leq_{pr} M$. Thus, M is a P-polyform module.

Proposition 2.6: If M is a P-polyform and $N \leq_e M$, then N is a P-polyform module.

Proof: Assume that $K \leq_e N$ and f: $\frac{N}{K} \rightarrow E(N)$ is a homomorphism where $K \leq N$. To prove f=0.

Consider the following diagram:

$$\begin{tikzcd}
\frac{N}{K} \arrow[r, "i_1"] \arrow[d, "f"'] & \frac{M}{K} \arrow[ddl, "h"] \\
E(N) \arrow[d, "i_2"'] & \\
E(M) &
\end{tikzcd}$$

Since E(M) is injective, then there exists h: $\frac{M}{K}\rightarrow$E(M) such that:

$$i_2\circ f = h\circ i_1, \ldots\ldots\ldots.. \text{(i)}$$

Now, $K\leq_e N$ and $N\leq_e M$, then $K\leq_e M$ [2, Proposition 1.1, P.16]. Since M is P-polyform then $K\leq_{pr}M$, this means $\mathrm{Hom}_R\left(\frac{M}{K}, E(M)\right)$=0. Beside that K is pure in M, so that K is pure in N, [5, Remark 2.8 (6), P.16]. Now, from (i), we get f=0, that is $K\leq_{pr}N$. Thus, N is P-polyform.

Another proof:

Suppose that $K\leq_e N$. Since $N\leq_e M$, then $K\leq_e M$, [2, Proposition 1.1, P.16]. But M is P-polyform, then $K\leq_{pr}M$, therefore N is a P-polyform module.

Corollary 2.7: If the injective hull of any R-module M is P-polyform, then M is P-polyform.
Proof: Since $M\leq_e E(M)$, and E(M) is P-polyform then the result followed by Proposition 2.6.

The following theorem gives another characterization of the definition of the P-polyform module.

Theorem 2.8: Let M be an R-module. Then M is a P-polyform module if and only if $\mathrm{Hom}_R(\frac{K}{N}, M)$= 0, for each essential submodule N of M, which is pure in M, and a submodule K of M, with N⊆K⊆M.
Proof:
⟹) Assume that N is an essential and pure submodule of M, and let K≤M with N⊆K⊆M. Suppose there exists a non-zero homomorphism f: $\frac{K}{N}\rightarrow$M. Consider the following diagram:

$$\begin{tikzcd}
0 \arrow[r] & \frac{K}{N} \arrow[r, "i"] \arrow[d, "f"'] & \frac{M}{N} \arrow[d, "g"] \\
& M \arrow[r, "j"'] & E(M)
\end{tikzcd}$$

where i and j are the inclusion homomorphism. By injectivity of E(M), we obtain g∘i=j∘f. If f≠0, then g∘i≠0, so that g≠0. This causes a contradiction since M is P-polyform, thus f=0, hence $\mathrm{Hom}_R(\frac{K}{N}, M)$=0.

⟸) Let N be an essential submodule of M, and f: $\frac{M}{N}\rightarrow$ E(M) be a non-zero homomorphism. Now, $f(\frac{M}{N})=\frac{K}{N}$ where K is a submodule of M such that N⊆K⊆M. Define φ: $\frac{K}{N}\rightarrow$ E(M) by φ(t+N)=f(t+N) for each (t+N)∈ $\frac{K}{N}$. It can be easily shown that φ is well-defined and homomorphism. In addition, since f≠0, then φ≠0, which is a contradiction, therefore M is P-polyform.

Remember that a submodule N of an R-module M is called closed (simply $N\leq_c M$) if N has no proper essential extension in M, [2, P.18]. A module M is called F-regular if every submodule of M is pure, [12].

Lemma 2.9: Let M be an R-module. Consider the following statements:

1. For any submodule N of M and each non-zero homomorphism f: N→M, implies that kerf is closed in N.
2. For any non-zero pure submodule N of M and each non-zero homomorphism f: N→M, implies that kerf is not essential in N.

Then (1) ⇒ (2), and if M is an F-regular module, then (2) ⇒ (1).

Proof:

(1) ⇒ (2): Suppose there exists a non-zero pure submodule N of M and a non-zero homomorphism f: N→M such that kerf$\leq_e$N. By assumption kerf$\leq_c$N, hence kerf=N, that is f=0. But this is a contradiction, thus kerf$\not\leq_e$N.

(2) ⇒ (1): Assume that M is an F-regular module, and let N be a submodule of M, f: N→M be a non-zero homomorphism. Suppose that kerf$\not\leq_c$N, so there exists a submodule K of N containing kerf such that kerf is essential in K. Now, consider the following sequences of homomorphism:

$$K \xrightarrow{i} N \xrightarrow{f} M$$

where i is the inclusion homomorphism. Since f≠0 then f∘i≠0. In addition, kerf ⊆ K, then kerf = ker(f∘i) $\leq_e$ K. On the other hand, M is F-regular, therefore K is pure in M. When K=0, and because kerf is contained in K, then kerf=(0)$\leq_c$N, which is a contradiction with our assumption, thus K≠0. So, we obtain a non-zero pure submodule K of M and a non-zero homomorphism (f∘i): K→M such that ker(f∘i)$\leq_e$K. But this contradicts with (2), therefore kerf must be closed in N.

Theorem 2.10: For any R-module M, consider the following statements:

1. M is a P-polyform module.
2. M is a polyform module.
3. For any submodule N of M, and each non-zero homomorphism f: N→M, implies kerf is closed in N.
4. For any non-zero pure submodule N of M, and any non-zero homomorphism f: N→M, implies kerf$\not\leq_e$N.

Then (1) ⇒ (2) ⇒ (3) ⇒ (4).

Proof:

(1) ⇒ (2): It is obvious.

(2) ⇒ (3): [13, Proposition 4.9, P.34]. .

(3) ⇒ (2): Let N be an essential submodule of M, and 0 ≠ f∈ $\mathrm{Hom}_R(\frac{M}{N}$, E(M)). Suppose that f≠0, so there is m+N∈ $\frac{M}{N}$ such that f(m+N)=$m_1$where m_1∈ E(M). But M $\leq_e$ E(M), so ∃ r∈R with 0≠ rm_1∈ M. Put rm_1=t. Define g: N+Rm→Rt by g(n+rm)=rt for each n∈N, r∈R. To prove g is well-defined, suppose that $n_1 + r_1$m=$n_2 + r_2$m, where n_1, n_2∈N, r_1,r_2∈R, that is n_1- n_2= (r_1-r_2) m∈N. But:

$$f\,[(r_1\text{-}r_2)(m+N)] = f\,[(r_1\text{-}r_2)m+N] = 0, \qquad \ldots\ldots\ldots\ldots\ldots\ldots(1)$$

Also,

$$f\,(r_1\text{-}r_2)(m+N) = (r_1\text{-}r_2)\,f\,(m+N) = (\,r_1\text{-}r_2)m_1, \qquad \ldots\ldots\ldots\ldots\ldots(2)$$

from (1) and (2) we get (r_1-r_2)m_1=0. This implies r_1m_1-r_2m_2, therefore r_1rm_1- r_2rm_2, hence r_1t=r_2t. thus g($n_1 + r_1$m)=r_1t = g($n_2 + r_2$m)= r_2t. That is g is well-defined. Besides that, g is a non-zero homomorphism. Also, N ⊆ kerg since ∀n∈N, n=n+0m, so that g(n)=0t=0. Now, N⊆ kerg ⊆ M, and N$\leq_e$M, then kerg $\leq_e$ M, [2, Proposition 1.1, P.16]. Moreover, we have kerg

$\subseteq$ N+Rm $\subseteq$M, therefore N+Rm $\leq_e$ M, [2, Proposition 1.1, P.16]. By the assumption kerg is closed in N+Rm, thus kerg=N+Rm. This implies that g=0 which is a contradiction, thus f=0. That is M is a polyform module.
(3) ⇒ (4): It is just Lemma 2.9.

3. P-Polyform Modules and Quasi-invertibility Monoform Modules

We explore an interesting relationship between P-polyform and quasi-invertibility monoform modules which is introduced by M.A. Ahmed, [14]. For that reason, this section is dedicated to investigating several important results about this connection.

Remember that a submodule N of an R-module M is called a quasi-invertible submodule of M (simply we used the symbol N$\leq_{qu}$M) if $\mathrm{Hom}_{\mathrm{R}}(\frac{\mathrm{M}}{\mathrm{N}}$, M)=0, [15, P.6].

An R-module M is called Quasi-invertibility monoform (simply, QI-monoform), if every non-zero quasi-invertible submodule of an R-module M is rational in M, [14]. In the category of rings, we have the following.

Proposition 3.1: Every P-polyform ring is a QI-monoform ring.
Proof: Assume that R is a P-polyform ring, and N is a non-zero quasi-invertible ideal of R. Since every quasi-invertible ideal is essential, [15, Corollary 2.3, P.12], and since R is P-polyform, then every essential is P-rational ideal of R, hence N$\leq_r$R, [6, Remark 2.2]. So that R is QI-monoform.

The converse of Proposition 3.1 is not true in general, for example: $\mathbb{Z}_4$ is QI-monoform but not P-polyform see Example 2.2 (7).

In the category of modules, there is no direct implication between P-polyform and QI-monoform, but under certain conditions, we get some results as the following Proposition.

Proposition 3.2: Let M be a multiplication and prime module. If M is P-polyform then M is QI-monoform.
Proof: Suppose that M is a P-polyform module, and N is a non-zero quasi-invertible submodule of M. Since M is multiplication and prime, then N$\leq_e$M, [15, Corollary 3.12, P.19]. By assumption N$\leq_{pr}$M, and consequently, N$\leq_r$M, [6, Remark 2.2]. Thus, M is QI-monoform.

Also, by the same argument of the poof of Proposition 3.2, and with replacing [15, Corollary 3.12, P.19], instead of [15, Theorem 3.11, P.18], we can prove the following.

Proposition 3.3: Let M be a multiplication module with a prime annihilator. If M is P-polyform then M is QI- monoform.

Furthermore, we can use [15, Theorem 3.8, P.17] instead of [15, Theorem 3.11, P.18], to prove the following.

Proposition 3.4: Let M be a quasi-injective R-module with J(End$_{\mathrm{R}}$(M))=(0), if M is P-polyform, then M is QI-monoform.
Proof: Assume that M is a P-polyform module and N$\leq_{qu}$M. Since M is quasi-injective and J(End$_{\mathrm{R}}$(M))=(0), then N$\leq_e$M, [15, Theorem 3.8, P.17]. On the other hand, M is P-polyform, so N$\leq_{pr}$M, hence N$\leq_r$M, [6, Remark 2.2]. Thus, the proof is complete.

Theorem 3.5: Let M be a multiplication and prime R-module. Consider the following:

1. M is a P-polyform module.
2. M is a polyform module.
3. M is a QI-monoform module.

Then (1) ⇒ (2) ⇔ (3), and if R is a regular ring, then (3) ⇒ (1).

Proof:

(1) ⇒ (2): It is obvious.

(2) ⇔ (3): Since M is multiplication and prime, then the result follows by [14, Proposition 4.4].

(3) ⇒ (1): Assume that M is QI-monoform, and let $N \leq_e M$. Since M is multiplication and prime, then $N \leq_{qu} M$, [15, Corollary 3.12, P.19]. But M is QI-monoform then $N \leq_r M$, [14]. In contrast, R is a regular ring then M is F-regular, [5, P.29], this implies that $N \leq_{pr} M$, [6, Remark 2.3 (10)]. Thus, M is P-polyform.

Proposition 3.6: Let M be a quasi-injective module with $J(End_R(M))=(0)$, consider the following:

1. M is a P-polyform module.
2. M is a polyform module.
3. M is a QI-monoform module.

Then (1) ⇒ (2) ⇔ (3).

Proof:

(1) ⇒ (2): It is clear.

(2)⇔(3): Since M is a quasi-injective module with $J(End_R(M))=(0)$, then according to [14, Proposition 4.5], the equivalence between polyform and QI-monoform is achieved.

Proposition 3.7: Let R be a quasi-Dedekind ring. Consider the following:

1. R is a P-polyform ring.
2. R is a polyform ring.
3. R is a QI-monoform ring.
4. R is a monoform ring.

Then (1) ⇒ (2) ⇒ (3) ⇒ (4).

Proof:

(1) ⇒ (2): It is straightforward.

(2) ⇒ (3) ⇒ (4): Since R is a quasi-Dedekind ring, then by [14, Theorem 4.13], the required result will be achieved.

As a consequence of Proposition 3.7, we have the following.

Corollary 3.8: Let R be an integral domain. Consider the following:

1. R is a P-polyform ring.
2. R is a polyform ring.
3. R is a QI-monoform ring.
4. R is a monoform ring.

Then (1) ⇒ (2) ⇒ (3) ⇒ (4).

Proof: Since every integral domain is quasi-Dedekind, [15, Example 1.4, P.24], then the result follows directly by Proposition 3.7.

Recall that an R-module M is an essentially quasi-Dedekind module, if $Hom_R\left(\frac{M}{N}, M\right)=0$, for all $N \leq_e M$, [16].

Proposition 3.9: Let R be an essentially quasi-Dedekind ring. Consider the following:

1. R is a P-polyform ring.
2. R is a QI-monoform ring.
3. R is a polyform ring.

Then (1) ⇒ (2) ⇒ (3).

Proof:

(1) ⇒ (2): It is just Proposition 3.1.

(2) ⇒ (3): Since R is an essentially quasi-Dedekind ring, then from [14, Theorem 4.9], R is a polyform ring.

4. P-Polyform Modules and Other Related Concepts

In this section, the relationships between a P-polyform module and some related concepts are discussed such as monoform, essentially quasi-Dedekind, essentially prime and ESQD. Moreover, a purely St-polyform module is introduced and its relationship with a P-polyform module is considered.

It is known that every monoform module is polyform. This fact is not satisfied for P-polyform modules. For example, the $\mathbb{Z}$-module of rational numbers $\mathbb{Q}$ is monoform, [17], while $\mathbb{Q}$ is not P-polyform as shown in Example 2.2 (4).

Proposition 4.1: Any monoform module over a regular ring is P-polyform.

Proof: Let M be a monoform module over a regular ring and $N\leq_e M$. Since R is a regular ring then M is F-regular, [5, P.29]. On the other hand, the module M is monoform, then every non-zero submodule of M is rational. But M is F-regular then $N\leq_{pr} M$. Therefore, M is P-polyform.

Proposition 4.2: Let M be a uniform module. If M is P-polyform then M is monoform.

Proof: Suppose that M is a P-polyform module, and let N be a non-zero submodule of M. Since M is uniform, then $N\leq_e M$, but M is P-polyform. So that $N\leq_{pr} M$, hence $N\leq_r M$, [6, Remark 2.2]. Thus, M is a monoform module.

However, if R is a regular ring, then the converse of Proposition 4.2 is true, as the following shows.

Proposition 4.3: Let M be a uniform module over a regular ring. Then M is P-polyform if and only if M is a monoform module.

Proof:

⟹) It follows directly by Proposition 4.2.

⟸) Assume that M is a monoform module, and let N be a non-zero submodule of M, then $N\leq_r M$. Since R is a regular ring, then M is F-regular, [5, P.29]. This implies that $N\leq_{pr} M$, [6, Remark 2.3 (10)]. Thus, M is P-polyform.

Proposition 4.4: Every P-polyform module is essentially quasi-Dedekind.

Proof: Let M be a P-polyform module, and N be an essential submodule of M. Suppose that f: $\frac{M}{N} \longrightarrow M$ is a homomorphism. Consider the following sequence of homomorphism:

$$\frac{M}{N} \xrightarrow{f} M \xrightarrow{i} E(M)$$

where i is the inclusion homomorphism and E(M) is the injective hull of M. Since M is P-polyform, then $\mathrm{Hom_R}(\frac{M}{N}, E(M))=0$. So that i∘f=0, hence f=0. Thus $\mathrm{Hom_R}(\frac{M}{N}, M)=0$ for each $N\leq_e M$, that is M essentially quasi-Dedekind.

An R-module M is essentially prime if $\mathrm{ann_R}(M) = \mathrm{ann_R}(N)$ for each $N\leq_e M$, [18, P.47].

Corollary 4.5: Every P-polyform module is essentially prime.
Proof: Since every essentially quasi-Dedekind module is essentially prime, [18, Proposition 2.1.8, P.47], then the result follows from Proposition 4.4.

Recall that an R-module M is purely quasi-Dedekind if every proper non-zero pure submodule of M is quasi-invertible, [19].

Remark 4.6: The two concepts P-polyform and purely quasi-Dedekind are independent. For example $\mathbb{Z}$-module $\mathbb{Z}_4$ is purely quasi-Dedekind, [19, Remark 2.3 (7)], but it is not P-polyform see Example 2.2 (7). In contrast, the $\mathbb{Z}$-module $\mathbb{Z}_6$ is P-polyform, since it is semisimple, but it is not purely quasi-Dedekind, [19, Remark 2.3 (2)].

Remember that a submodule N of an R-module M is SQI if for each $f\in\mathrm{Hom_R}\left(\frac{M}{N}, M\right)$, implies that $f\left(\frac{M}{N}\right)$ is small in M, and an R-module M is an SQD-module if each non-zero submodule N of M is an SQI-submodule, [20].

Remark 4.7: The two concepts P-polyform and SQD module are independent. For example, $\mathbb{Z}_4$ is SQD module but not P-polyform.

This leads us to define the following.

Definition 4.8: An R-module M is called ESQD if every essential submodule of M is SQI.

Proposition 4.9: Every P-polyform module is ESQD.
Proof: Let M be P-polyform, and N is an essential submodule of M. Since M is P-polyform, then $N\leq_{pr}M$, so that $N\leq_r M$. Since every rational submodule is purely quasi-invertible, therefore N is an SQI submodule.

Following [21], a submodule N of an R-module M is St-closed (simply $N\leq_{stc}M$), if N has no proper semi-essential extensions in M, where a submodule N is said to be semi-essential if N∩P≠0 for every non-zero prime submodule P of M. A module M is called St-polyform, if for each submodule N of M and all homomorphism f: N→M, kerf is an St-closed submodule of M, [22].

Remark 4.10: The two concepts P-polyform and St-polyform are independent. For example, $\mathbb{Z}_2$ is P-polyform but not St-polyform, [22]. In contrast, St-polyform is not P-polyform because of the purity property.

This motivates us to define the following.

Definition 4.11: An R-module M is called purely St-polyform if each semi-essential submodule of M is P-rational in M.

Remark 4.12: Every purely St-polyform is P-polyform.
Proof: Since every essential submodule is semi-essential, [21], then the result is obtained.

Theorem 4.13: Let M be a uniform and essentially quasi-Dedekind module. Consider the following statements:

1. M is a purely St-polyform module.
2. M is a P-polyform module.
3. M is a monoform module.
4. M is a QI-monoform module.
5. M is a polyform module.

Then (1) ⇒ (2) ⇒ (3) ⇒ (4) ⇒ (5).
Proof:
(1) ⇒ (2): It is just Remark 4.12.
(2) ⇒ (3): Since M is uniform and a P-polyform module, so by Proposition 4.2, M is monoform.
(3) ⇒ (4) ⇒ (5): Since R is a uniform and essentially quasi-Dedekind, then the result follows by [14, Theorem 4.11].

Since every nonsingular module is essentially quasi-Dedekind [14, Remark 4.8 (3)], then we deduce the following.

Corollary 4.14: Let M be a uniform and nonsingular module. Consider the following:

1. M is a purely St-polyform module.
2. M is a P-polyform module.
3. M is a monoform module.
4. M is a QI-monoform module.
5. M is a polyform module.

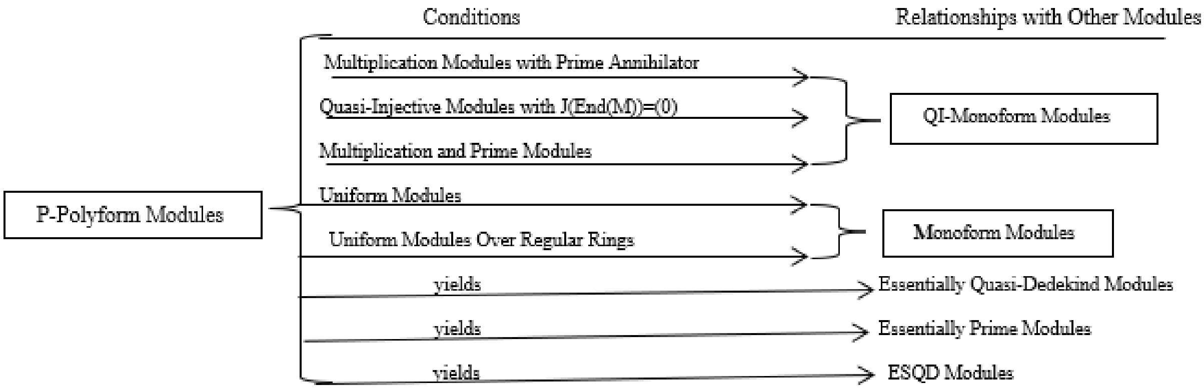
Then (1) ⇒ (2) ⇒ (3) ⇒ (4) ⇒ (5).

5. Conclusions

In this work, the class of polyform modules has been restricted to a new class. It is called P-polyform modules. The main results of this paper are summarized as follows:

1. Several results were introduced which described the main properties of P-polyform modules.
2. Sufficient conditions were investigated under which P-polyform and polyform modules are identical.
3. Another characterization and partial characterization of P-polyform modules are given.
4. Sufficient conditions under which P-polyform and QI-monoform modules are identical are discussed.
5. Many connections between P-polyform and other related concepts were studied and established.
6. Some classes of modules which contain a P-polyform module are examined, such as essentially quasi-Dedekind and essentially prime modules.

Finally, all these relationships can be represented in the following diagram:



Relationships Between P-polyform and Other Related Modules

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