DOI: 10.24996/ijs.2024.65.7.33

ISSN: 0067-2904

Coefficients Estimates for New Subclasses of Bi-univalent Functions Using Convolution Operator

Asmaa KH. Abdul-Rahman[1,*] , Thamer Khalil MS. Al-Khafaji[2], Lieth A. Majed[1]
[1]*Department of Mathematic, College of Sciences, Diyala of university, Diyala, Iraq*
[2]*Department of Renewable Energy, College of Energy & Environmental Sciences, Al-Karkh University of Science, Baghdad, Iraq*

Received: 10/2/2023 Accepted: 4/7/2023 Published: 30/7/2024

Abstract

We offered and suggested some classes $\mathcal{N}^{\mu,\delta}_{\Omega,b,\alpha,\gamma}(\psi,\tau,\varphi)$ and $C^{\mu,\delta}_{\Omega,b,\alpha,\gamma}(\tau,a,\psi,\varphi)$ of bi-univalent functions in the unit disk υ, to apply the effects to the classes through convolution of the operators $R^{\delta}_{q}f(z)$ and $I^{\lambda}_{s,a,\mu}f(z)$. Which satisfies the condition quasi-subordination. We got estimates the first two Taylor-Maclaurin coefficients $|a_2|$ and $|a_3|$ by convolution of the operators $R^{\delta}_{q}f(z)$ and $I^{\lambda}_{s,a,\mu}f(z)$.

Keywords: normalized analytic, bi-univalent function, estimate, integral operator.

مخمنات المعاملات للدوال ثنائية التكافؤ المعرفة باستخدام مؤثر الالتواء

اسماء خوام عبد الرحمن[1] ,. ثامر خليل محمد صالح[2] , ليث عبد اللطيف مجيد [1]
[1]قسم الرياضيات , كلية العلوم , جامعة ديالى , ديالى , العراق
[2]قسم الطاقة المتجددة ، كلية علوم الطاقة و البيئة ، جامعة الكرخ للعلوم ، بغداد ، العراق

الخلاصه

عرضنا واقترحنا بعض الاصناف $\mathcal{N}^{\mu,\delta}_{\Omega,b,\alpha,\gamma}(\psi,\tau,\varphi)$ و $C^{\mu,\delta}_{\Omega,b,\alpha,\gamma}(\tau,a,\psi,\varphi)$ لمخمنات الدوال في قرص الوحدة υ لتطبيق التاثيرات على الاصناف من خلال التواء الموثرات والتي تحقق شرط شبه التابعية $R^{\delta}_{q}f(z)$ و $I^{\lambda}_{s,a,\mu}f(z)$ هنا حصلنا على اول معاملين ماكلورين–تايلر $|a_2|,|a_3|$ من خلال التواء الموثرين $R^{\delta}_{q}f(z)$ و $I^{\lambda}_{s,a,\mu}f(z)$.

1. Introduction

In an open unit disk $\upsilon=\{z:|z|<1\}$ let $\mathcal{N}$ be the class of normalized analytic function f defined with Taylor series

$$f(z)=z+\sum_{k=2}^{\infty}a_k z^k. \quad (1)$$

*Email: asmaaalshaiby@gmail.com

In addition to, $\mathcal{N}$ which are univalent and normalized by the conditions in and suppose פ denote the class of all functions in $\mathcal{N}$. According to the Koebe One-Quarter Theorem [1] such that the inverse f^{-1} of every $f \in$ פ satisfies:

$$f^{-1}\big(f(z)\big) = z, \quad (z \in \upsilon).$$

and

$$f^{-1}\big(f(\theta)\big) = \theta\,, \qquad \left(|\theta| < r_0(f); r_0(f) \geq \frac{1}{4}\right),$$

where

$$f^{-1}(\theta) = g(\theta) = \theta - a_2\theta^2 + (2a_2^2 - a_3)\theta^3 - (5a_2^3 - 5a_2a_3 + a_4)\theta^4 + \cdots \,. \qquad (2)$$

Let Ω the class of all bi-univalent function $f \in \mathcal{N}$ in υ.
Some similar studies have been published but with different operators they are as follows see [2-9].

2. Preliminary

For functions $f, g \in \mathcal{N}$ depending on the equation (1). Now the function $g(z)$

$$g(z) = z + \sum_{k=2}^{\infty} b_k z^k \quad (z \in \upsilon).$$

The convolution of the function f and K is defined as:

$$(f * g)(z) = z + \sum_{k=2}^{\infty} a_k b_k z^k.$$

In [10] the Rusyeyweyh type q-analogue operator R_q^{δ} is defined by

$$R_q^{\delta} f(z) = z + \sum_{k=2}^{\infty} \frac{[k+\delta-1]_q!}{[\delta]_q!\ [k-1]_q!} a_k b_k z^k,$$

where $\delta \geq 0$. Also, as $q \to 1^-$, we have

$$\lim_{q\to 1^-} R_q^{\delta,q} f(z) = z + \lim_{q\to 1^-}\left(\sum_{k=2}^{\infty} \frac{[k+\delta-1]_q!}{[\delta]_q!\ [k-1]_q!} a_k b_k z^k\right).$$

$$\lim_{q\to 1^-} R_q^{\delta} f(z) = R^S f(z).$$

Komatu [11] introduced and investigated a family of integral operator $J_{\mu}^{\lambda}: \Omega \to \Omega$. That is obtained as fallows

$$J_{\mu}^{\lambda} f(z) = z + \sum_{k=1}^{\infty} \left[\frac{\mu}{\mu + k - 1}\right]^{\lambda} a_k b_{k\,,} z \in \upsilon, k > 1, \lambda \geq 0.$$

The Hurwitz – Lerch Zeta function [11]

$$\emptyset(z, s, a) = \sum_{k=0}^{\infty} \frac{z^k}{[t+a)^s} \quad a \in C - z_0, s \in C \ when\ 0 < |z| < 1.$$

Interims of convolutioin. Where $G_{s,a(z)}$ is given by
$G_{s,a(z)} = [1+a]^s(\emptyset(z,s,a) - a^{-s}), z \in \upsilon.$
The linear operator $I_{s,a,\mu} f(z): \sum \to \sum$ is defined

$$I_{s,a,\mu} f(z) = G_{s,a(z)} * J_{\mu}^{\lambda} f(z) = z + \sum_{k=1}^{\infty} \left[\frac{1+a}{t+a}\right]^{s} \left[\frac{\mu}{\mu+k-1}\right]^{\lambda} a_k b_k\ z^k.$$

The convolution of the operator $R^s f(z)$ and $I_{s,a,\mu}^{\lambda} f(z)$ are defined by
פ$_{\mathbf{s,a},\mu}^{\delta,\lambda} f(z) = R^S f(z) * I_{s,a,\mu}^{\lambda} f(z).$

$\mathfrak{D}_{\mathbf{s,a,\mu}}^{\delta,\lambda} f(z) = z + \sum_{k=2}^{\infty} \frac{[k+\delta-1]_q!}{[\delta]_q!\,[k-1]_q!} \left[\frac{1+a}{t+a}\right]^s \left[\frac{\mu}{\mu+k-1}\right]^{\lambda} a_k z^k.$
$\mathfrak{D}_{\mathbf{s,a,\mu}}^{\delta,\lambda} f(z) = z + \sum_{k=2}^{\infty} \theta_{k,\delta} a_k z^k.$
Where

$$\theta_{k,\delta} = \left[\frac{[k+\delta-1]_q!}{[\delta]_q!\ [k-1]_q!}\left[\frac{1+a}{t+a}\right]^s \left[\frac{\mu}{\mu+k-1}\right]^{\lambda}\right].$$

Now we can say there exists analytic functions ϕ and F, with $|\phi(z)| \le 1, F(0) = 0$ and $|F(z)| < 1$, then function f is a quasi-subordinate to g in υ, such that $f(z) = \phi(z) g(F(z))$, so

$$f(z) \prec_q g(z), \quad (z \in \upsilon).$$

We denote this a quasi-subordination by [12], as follows:

We will notes if $\phi(z) = 1$, then $f(z) = g(F(z))$, hence $f(z) \prec g(z)$ in [7] and [8]. Furthermore, if $F(z) = z$, then $f(z) = \phi(z) g(z)$, and in case, f is majorized by g, written $f(z) \ll g(z)$ in υ. By the method of subordination we defined and studied classes $\mathfrak{D}^*(\phi)$ and $G^*(\phi)$ of starlike functions by:

$$\mathfrak{D}^*(\phi) = \left\{ f \in \mathcal{N}: \frac{zf'(z)}{f(z)} \prec \phi(z), z \in \upsilon \right\},$$

and

$$G^*(\phi) = \left\{ f \in \mathcal{N}: 1 + \frac{zf''(z)}{f'^{(z)}} \prec \phi(z), z \in \upsilon \right\},$$

where

$$\phi(z) = k_0 + k_1 z + k_2 z^2 + \cdots, \quad (z \in \upsilon).$$

Consider

$$\varphi(z) = 1 + s_1 z + s_2 z^2 + s_3 z^3 + \cdots, \quad s \in \upsilon,$$

and univalent analytic function with a positive real part in υ, symmetric with respect to the real axis, and starlike with respect to $\varphi(zero) = 1$ and $\varphi'(zero) > zero$. By $\mathfrak{D}^*{}_{\Omega}(\phi)$ and $G^*{}_{\Omega}(\phi)$, we introduce and study here certain subclasses of the class Ω [13].

Definition 1: Let f belong to $\mathcal{N}$ is called in the class $\mathcal{N}_{\sum,b,\alpha,\gamma}^{\mu,\delta}(\psi,\tau,\varphi)$ $(0 \le \psi \le 1, 0 \le \tau \le 1)$, if satisfied the quasi-subordination:

$$\frac{z\left(\mathfrak{D}_{\mathrm{s,a,\mu}}^{\delta,\lambda} f(z)\right)' + \psi z^2 \left(\mathfrak{D}_{\mathrm{s,a,\mu}}^{\delta,\lambda} f(z)\right)'' + \tau z \left(\mathfrak{D}_{\mathrm{s,a,\mu}}^{\delta,\lambda} f(z)\right)'' \mathfrak{D}_{\mathrm{s,a,\mu}}^{\delta,\lambda} f(z) \left(\mathfrak{D}_{\mathrm{s,a,\mu}}^{\delta,\lambda} f(z)\right)' \mathfrak{D}_{\mathrm{s,a,\mu}}^{\delta,\lambda} f(z)}{\mathfrak{D}_{\mathrm{s,a,\mu}}^{\delta,\lambda} f(z)} - 1 \prec_q \varphi(z) - 1, \qquad (3)$$

$$\frac{\theta\left(\mathfrak{D}_{\mathrm{s,a,\mu}}^{\delta,\lambda} g(\theta)\right)' + \psi \theta^2 \left(\mathfrak{D}_{\mathrm{s,a,\mu}}^{\delta,\lambda} g(\theta)\right)'' + \tau \theta \left(\mathfrak{D}_{\mathrm{s,a,\mu}}^{\delta,\lambda} g(\theta)\right)'' \mathfrak{D}_{s,a,\mu}^{\delta,\lambda} g(\theta) + \left(\mathfrak{D}_{\mathrm{s,a,\mu}}^{\delta,\lambda} g(\theta)\right)' \ \mathfrak{D}_{s,a,\mu}^{\delta,\lambda}\, g(\theta)}{\mathfrak{D}_{s,a,\mu}^{\delta,\lambda} g(\theta)} - 1 \prec_q \varphi(\theta) - 1, \qquad (4)$$

when g is the inverse function of f and $z, \theta \in \upsilon$.

Properties

1- For all $\tau = zero$ and $((0 \le \psi \le 1)$ a function $f \in \Omega$ defined (1) is called in the class $\mathcal{N}_{\Omega,b,\alpha,\gamma}^{\mu,\delta}(\psi, o, \varphi)$ if satisfied the condition quasi-subordination :

$$\frac{z\left(\mathfrak{D}_{\mathbf{s,a},\mu}^{\delta,\lambda}f(z)\right)' + \psi z^2\left(\mathfrak{D}_{\mathbf{s,a},\mu}^{\delta,\lambda}f(z)\right)'' + \left(\mathfrak{D}_{\mathbf{s,a},\mu}^{\delta,\lambda}f(z)\right)'\mathfrak{D}_{s,a,\mu}^{\delta,\lambda}f(z)}{\mathfrak{D}_{s,a,\mu}^{\delta,\lambda}f(z)} - 1 \prec_q \varphi(z) - 1,$$

$$\frac{\theta\left(\mathfrak{D}_{\mathrm{s,a},\mu}^{\delta,\lambda}g(\theta)\right)' + \psi\theta^2\left(\mathfrak{D}_{\mathrm{s,a},\mu}^{\delta,\lambda}g(\theta)\right)'' + \mathfrak{D}_{\mathrm{s,a},\mu}^{\delta,\lambda}g(\theta)'\mathfrak{D}_{\mathrm{s,a},\mu}^{\delta,\lambda}g(\theta)}{\mathfrak{D}_{\mathrm{s,a},\mu}^{\delta,\lambda}g(\theta)} - 1 \prec_q \varphi(\theta) - 1.$$

2- For all $\delta = zero, (\lambda, s, a, \mu = 0)$ and $((0 \le \psi \le 1)$ a function $f \in \Omega$ defined (1) is called in the class $\mathcal{N}_\Omega(\psi, o, \varphi)$ if satisfied the quasi-subordination:

$$\frac{z(f(z))' + \psi z^2(f(z))'' + \tau z(f(z))''f(z) + (f(z))'f(z)}{f(z)} - 1 \prec_q \varphi(z) - 1,$$

$$\frac{\theta(g(\theta))' + \psi\theta^2(g(\theta))'' + \tau\theta(g(\theta))'' + (g(\theta))'}{g(\theta)} - 1 \prec_q \varphi(z) - 1.$$

Lemma 1: [1]If $p \in P$, then $|p_i| \le 2\ \forall i$, where P is the family of all analytic p, for which $Re\{p(z)\} > 0, z \in \upsilon$, where

$$p(z) = 1 + p_1 z + p_2 z^1 + p_3 z^3 + \cdots, \quad (z \in \upsilon).$$

3. Main results

Theorem 1: If $f \in \mathcal{N}_{\Omega,b,\alpha,\gamma}^{\mu,\delta}(\psi, \tau, \varphi)$, then

$$|a_2| \le \min\{\frac{|k_0|s_1}{(3 + 2\psi + 2\tau)\beta_{2,\mu}}, \left(\frac{|k_0|s_1 + |k_0|(s_2 - s_1)}{(10 + 12\psi + 12\tau)\beta_{3,\mu} - 2(1 + 2\psi)\beta_{3,\mu}}\right)^{\frac{1}{2}}\}, \tag{5}$$

$$|a_3| \le \min\{\frac{|k_0|s_1}{(5 + 6\psi + 6\sigma)\beta_{2,\mu}} + \frac{|k_0^2|s_1}{(6 + 4\psi + 4\tau)\beta_{2,\mu}}, \frac{|k_0|s_1}{(5 + 6\psi + 6\tau)\beta_{2,\mu}} + \frac{|k_0|s_1 + |k_0|(s_2 - s_1)}{(30 + 24\psi + 24\sigma)\beta_{3,\mu} - 2(1 + 2\psi)\beta_{3,\mu}}\}. \tag{6}$$

Proof**:** Since $f \in \mathcal{N}_{\Omega,b,\alpha,\gamma}^{\mu,\delta}(\psi, \tau, \varphi)$, then there exists analytic function, F, φ in υ and $\varphi, F: \upsilon \to \upsilon$, with $|\varphi(z)| \le 1$, such that $\varphi(o) = F(o) = o$ and $|F(z)| \le 1$, satisfied:

$$\frac{z\left(\mathfrak{D}_{\mathrm{s,a},\mu}^{\delta,\lambda}f(z)\right)' + \psi z^2\left(\mathfrak{D}_{\mathrm{s,a},\mu}^{\delta,\lambda}f(z)\right)'' + \tau z\left(\mathfrak{D}_{\mathrm{s,a},\mu}^{\delta,\lambda}f(z)\right)''\mathfrak{D}_{\mathrm{s,a},\mu}^{\delta,\lambda}f(z) + \left(\mathfrak{D}_{\mathrm{s,a},\mu}^{\delta,\lambda}f(z)\right)\left(\mathfrak{D}_{\mathrm{s,a},\mu}^{\delta,\lambda}f(z)\right)'}{\mathfrak{D}_{\mathrm{s,a},\mu}^{\delta,\lambda}f(z)} - 1 = \phi(z)\big(\varphi(F(z)) - 1\big), \tag{7}$$

$$\frac{\theta\left(\mathfrak{D}_{\mathrm{s,a},\mu}^{\delta,\lambda}g(\theta)\right)' + \psi\theta^2\left(\mathfrak{D}_{\mathrm{s,a},\mu}^{\delta,\lambda}g(\theta)\right)'' + \tau\theta\left(\mathfrak{D}_{\mathrm{s,a},\mu}^{\delta,\lambda}g(\theta)\right)''\mathfrak{D}_{s,a,\mu}^{\delta,\lambda}g(\theta) + \left(\mathfrak{D}_{\mathrm{s,a},\mu}^{\delta,\lambda}g(\theta)\right)'\mathfrak{D}_{s,a,\mu}^{\delta,\lambda}g(\theta)}{\mathfrak{D}_{s,a,\mu}^{\delta,\lambda}g(\theta)} - 1 = \phi(\theta)\big(\varphi(V(\theta)) - 1\big), \tag{8}$$

so g is the inverse of f and z $\in \upsilon$.

$$x(z) = \frac{1 + F(z)}{1 - F(z)} = 1 + x_1 z + x_2 z^2 + x_3 z^3 + \cdots, \tag{9}$$

$$y(\theta) = \frac{1 + V(\theta)}{1 - V(\theta)} = 1 + y_1\theta + y_2\theta^2 + y_3\theta^3 + \cdots, \tag{10}$$

or

$$F(z) = \frac{x(z) - 1}{x(z) + 1} = \frac{1}{2}[x_1 z + \left(x_2 - \frac{x_1^2}{2}\right)z^2 + \cdots, \tag{11}$$

$$V(\theta)=\frac{y(z)-1}{y(z)+1}=\frac{1}{2}[y_1\theta+\left(y_2-\frac{y_1^2}{2}\right)\theta^2+\cdots. \tag{12}$$

i.e., the analytic function $x(z)$ and $y(\theta)$ are in υ with $x(0)=y(0)=1$, since $F,V:\upsilon\to\upsilon,\ x(z)$ and $y(\theta)$ have a positive real in υ, and $|x_i|\le 2$ and $|y_i|\le 2$, (i = 1,2)

$$\frac{z\left(\mathfrak{D}_{s,a,\mu}^{\delta,\lambda}f(z)\right)'+\psi z^2\left(\mathfrak{D}_{s,a,\mu}^{\delta,\lambda}f(z)\right)''+\tau z\left(\mathfrak{D}_{s,a,\mu}^{\delta,\lambda}f(z)\right)''\mathfrak{D}_{s,a,\mu}^{\delta,\lambda}f(z)+\left(\mathfrak{D}_{s,a,\mu}^{\delta,\lambda}f(z)\right)'\mathfrak{D}_{s,a,\mu}^{\delta,\lambda}f(z)}{\mathfrak{D}_{s,a,\mu}^{\delta,\lambda}f(z)}-1=\phi(z)\left(\varphi\left(\frac{x(z)-1}{x(z)+1}\right)-1\right), \tag{13}$$

$$\frac{\theta\left(\mathfrak{D}_{s,a,\mu}^{\delta,\lambda}g(\theta)\right)'+\psi\theta^2\left(\mathfrak{D}_{s,a,\mu}^{\delta,\lambda}g(\theta)\right)''+\tau\theta\left(\mathfrak{D}_{s,a,\mu}^{\delta,\lambda}g(\theta)\right)''\mathfrak{D}_{s,a,\mu}^{\delta,\lambda}f(\theta)+\left(\mathfrak{D}_{s,a,\mu}^{\delta,\lambda}g(\theta)\right)'\mathfrak{D}_{s,a,\mu}^{\delta,\lambda}g(\theta)}{\mathfrak{D}_{s,a,\mu}^{\delta,\lambda}g(\theta)}-1=\phi(\theta)\left(\varphi\left(\frac{y(\theta)-1}{y(\theta)+1}\right)-1\right), \tag{14}$$

we can write (13) and (14) of the form (15) and (16)

$$k_0s_1x_1z+\frac{1}{2}z^2k_1s_1x_1+\frac{1}{2}z^2k_0s_1\left(x_2-\frac{x_1^2}{2}\right)+\frac{1}{4}k_0s_2x_1^2z^2+\cdots, \tag{15}$$

$$\frac{1}{2}k_0s_1y_1\theta+\frac{1}{2}\theta^2k_1s_1y_1+\frac{1}{2}\theta^2k_0s_1\left(y_2-\frac{y_1^2}{2}\right)+\theta^2\frac{1}{4}k_0s_2y_1^2+\cdots \tag{16}$$

since

$$\left(\frac{z\left(\mathfrak{D}_{s,a,\mu}^{\delta,\lambda}f(z)\right)'+\psi z^2\left(\mathfrak{D}_{s,a,\mu}^{\delta,\lambda}f(z)\right)''}{\mathfrak{D}_{s,a,\mu}^{\delta,\lambda}f(z)}\right)+\tau z\left(\mathfrak{D}_{s,a,\mu}^{\delta,\lambda}f(z)\right)''+\left(\mathfrak{D}_{s,a,\mu}^{\delta,\lambda}f(z)\right)'$$
$$=1+(3+2\psi+2ж)a_2\beta_{2,\mu}z+(5+6\psi+6\tau)a_3\beta_{3,\mu}z^2-(1+2\psi)a_2^2\beta_{2,\mu}z^2+ \tag{17}$$

$$\left(\frac{\theta\left(\mathfrak{D}_{s,a,\mu}^{\delta,\lambda}g(\theta)\right)'+\psi z^2\left(\mathfrak{D}_{s,a,\mu}^{\delta,\lambda}g(\theta)\right)''}{\mathfrak{D}_{s,a,\mu}^{\delta,\lambda}g(\theta)}\right)+\tau\theta\left(\mathfrak{D}_{s,a,\mu}^{\delta,\lambda}g(\theta)\right)''+\left(\mathfrak{D}_{s,a,\mu}^{\delta,\lambda}g(\theta)\right)'$$
$$=1+\left(-(3+2\psi+2\tau)\right)a_2\beta_{2,\mu}\theta+(5+6\psi+6\tau)a_3\beta_{3,\mu}\theta^2-(1+2\psi)a_2^2\beta_{2,\mu}\theta^2+\cdots. \tag{18}$$

With compensation (15) and (17) in equation (13) and in the same way (16) and (18) in (14), we get

$$(5+6\psi+6\tau)a_3\beta_{3,\mu}-(1+2\psi)a_2^2\beta_{2,\mu}=\frac{1}{2}k_1s_1x_1, \tag{19}$$

$$+\frac{1}{2}k_0s_1\left(x_2-\frac{x_1^2}{2}\right)+\frac{1}{4}k_0s_2x_1^2, \tag{20}$$

and

$$-(3+2\psi+2\sigma)a_2\beta_{2,\mu}=\frac{1}{2}k_0s_1y_1, \tag{21}$$

$$(5+6\psi+6\sigma)(2a_2^2-a_3)\beta_{3,\mu}-(1+2\psi)a_2^2\beta_{2,\mu}=\frac{1}{2}k_1s_1y_1+\frac{1}{2}k_0s_1\left(y_2-\frac{y_1^2}{2}\right)+\frac{1}{4}k_0s_2y_1^2. \tag{22}$$

From (19) and (21), we find that

$$a_2=\frac{k_0s_1x_1}{(6+4\psi+4\sigma)\beta_{2,\mu}}=-\frac{k_0s_1y_1}{(6+4\psi+4\sigma)\beta_{2,\mu}} \tag{23}$$

$$x_1=-y_1 \tag{24}$$

and

$$(24+16\psi+16\tau)^2\beta_{2,\mu}^2 a_2^2 = \psi^2 k_0^2 s_1^2 (x_1^2+y_1^2). \quad (25)$$

Applying (20) and (22), by using (24) and (25), we have

$$(40+48\psi+48\tau)a_2^2\beta_{3,\mu} - (8+16\psi)a_2^2\beta_{3,\mu} = 2k_0 s_1(x_2+y_2) + k_0(x_1^2+y_1^2)(s_2-s_1),$$

which implies that

$$a_2^2 = \frac{2k_0 s_1(x_2+y_2) + k_0(x_1^2+y_1^2)(s_2-s_1)}{(40+48\psi+48\tau)\beta_{3,\mu} - (8+16\psi)\beta_{3,\mu}}\,. \quad (27)$$

By Lemma (1) in (27), we get (3).

We find the bound on the coefficient $|a_3|$, by subtracting (20) and (22), we get

$$(40+48\psi+48\tau)a_3\beta_{3,\mu} - 8(5+6\psi+6\tau)a_2^2\beta_{3,\mu} = 2k_1 s_1 x_1 + k_0 s_1(x_2-y_2). \quad (28)$$

Now, substituting (22) from (20) and computation using (24) and (25), we obtain

$$a_3 = \frac{2k_1 s_1 x_1}{(40+48\psi+48\sigma)\beta_{3,\mu}} + \frac{k_0 s_1(x_2-y_2)}{(40+48\psi+48\tau)\beta_{3,\mu}} + a_2^2\,, \quad (29)$$

$$|a_3| \le \frac{|k_0^2| s_2}{(3+2\psi+2ж)^2\beta_{2,\mu}^2} + \frac{k_0 s_1}{(5+6\psi+6\tau)\beta_{3,\mu}}\,. \quad (30)$$

By application of Lemma1 in a monologue (30), we get (6).■

Count $\tau = 0$, in Theorem 1, we get corollary(1).

Corollary 1: Let f belong to the class $\mathcal{N}_{\Omega,b,\alpha,\gamma}^{\mu,\delta}(\psi,0,\varphi)$, and f be given by (1), then

$$|a_2| \le \min\{\frac{|k_0|s_1}{(3+2\psi)\beta_{2,\mu}}, \sqrt{\frac{|k_0|s_1 + |k_0|(s_2-s_1)}{(10+12\psi)\beta_{3,\mu} - (2+4\psi)\beta_{3,\mu}}}\},$$

$$|a_3| \le \min\{\frac{|k_0|s_1}{(5+6\psi)\beta_{2,\mu}} + \frac{|k_0^2|s_1}{(6+4\psi)\beta_{2,\mu}}, \frac{|k_0|s_1}{(5+6\psi)\beta_{2,\mu}} + \frac{|k_0|s_1 + |k_0|(s_2-s_1)}{(10+12\psi)\beta_{3,\mu} - (2+4\psi)\beta_{3,\mu}}\}.$$

Corollary 2: Let f belong to the class $\mathcal{N}_{\Omega}(\psi,\sigma,\varphi)$, and f be given by (1), then

$$|a_2| \le \min\{\frac{|k_0|s_1}{(3+2\psi+2\tau)}, \sqrt{\frac{|k_0|s_1 + |k_0|(s_2-s_1)}{(10+12\psi+12\tau) - 2(1+2\psi)}}\},$$

$$|a_3| \le \min\{\frac{|k_0|s_1}{(5+6\psi+6ж)} + \frac{|k_0^2|s_1}{2(3+2\psi+2\tau)}, \frac{|k_0|s_1}{(5+6\psi+6\tau)} + \frac{|k_0|s_1 + |k_0|(s_2-s_1)}{(10+12\psi+12\tau) - (2+4\psi)}\}.$$

Definition 2: A function $f \in \mathcal{N}$ is in the class $C_{\Omega,b,\alpha,\gamma}^{\mu,\delta}(\tau,a,\psi,\varphi)$ $(0 < \tau \le 1)$, where $\psi \in \mathbb{C}\backslash\{0\}, 0 \le a \le 1,$ if it satisfies a quasi-subordination:

$$\frac{1}{\psi}\left\{\tau z\left(\frac{\mathfrak{d}_{s,a,\mu}^{\delta,\lambda} f(z)}{z}\right)' + \left(\frac{z\left(\mathfrak{d}_{s,a,\mu}^{\delta,\lambda} f(z)\right)'}{\mathfrak{d}_{s,a,\mu}^{\delta,\lambda} f(z)}\right) + az^2\left(\mathfrak{d}_{s,a,\mu}^{\delta,\lambda} f(z)\right)'''\right\} \prec_q (\varphi(z)-1), \quad (31)$$

$$\frac{1}{\psi}\left\{\tau\theta\left(\frac{\mathfrak{d}_{s,a,\mu}^{\delta,\lambda} g(\theta)}{\theta}\right)' + \left(\frac{\theta\left(\mathfrak{d}_{s,a,\mu}^{\delta,\lambda} g(\theta)\right)'}{\mathfrak{d}_{s,a,\mu}^{\delta,\lambda} g(\theta)}\right) + a\theta^2\left(\mathfrak{d}_{s,a,\mu}^{\delta,\lambda} g(\theta)\right)'''\right\} \prec_q (\varphi(\theta)-1), \quad (32)$$

such that g is the inverse of f and $z, \theta \in \upsilon$.

Properties

1-For $\tau = 0$ and $0 \leq \delta \leq 1, \psi \in \mathbb{C}\backslash\{0\}, 0 \leq a \leq 1$. A function $f \in \Omega$ is in the class $C_{s,a,\mu}^{\delta,\lambda}(0, a, \psi, \varphi)$, if the following quasi-subordination conditions are satisfied and f defined in (1).

$$\frac{1}{\psi}\left\{\left(\frac{z\left(\mathfrak{D}_{s,a,\mu}^{\delta,\lambda} f(z)\right)'}{\mathfrak{D}_{s,a,\mu}^{\delta,\lambda} f(z)}\right) + az^2\left(\mathfrak{D}_{s,a,\mu}^{\delta,\lambda} f(z)\right)'''\right\} \prec_q (\varphi(z) - 1),$$

$$\frac{1}{\psi}\left\{\left(\frac{\theta\left(\mathfrak{D}_{s,a,\mu}^{\delta,\lambda} g(z)\right)'}{\mathfrak{D}_{s,a,\mu}^{\delta,\lambda} g(z)}\right) + a\theta^2\left(\mathfrak{D}_{s,a,\mu}^{\delta,\lambda} g(\theta)\right)'''\right\} \prec_q (\varphi(\theta) - 1).$$

2- For all $a = 0$ and $(0 \leq \delta \leq 1, \psi \in \mathbb{C}\backslash\{0\}, 0 \leq \tau \leq 1)$. A function $f \in \Omega$ is in the class $C_{s,a,\mu}^{\delta,\lambda}(\tau, 0, \psi, \varphi)$, if quasi-subordination conditions are satisfied when f defined in (1).

$$\frac{1}{\psi}\left\{\tau z\left(\frac{\mathfrak{D}_{s,a,\mu}^{\delta,\lambda} f(z)}{z}\right)' + \left(\frac{z\left(\mathfrak{D}_{s,a,\mu}^{\delta,\lambda} f(z)\right)'}{\mathfrak{D}_{s,a,\mu}^{\delta,\lambda} f(z)}\right)\right\} \prec_q (\varphi(z) - 1),$$

$$\frac{1}{\psi}\left\{\tau \theta\left(\frac{\mathfrak{D}_{s,a,\mu}^{\delta,\lambda} g(\theta)}{\theta}\right)' + \left(\frac{w\left(\mathfrak{D}_{s,a,\mu}^{\delta,\lambda} g(\theta)\right)'}{\mathfrak{D}_{s,a,\mu}^{\delta,\lambda} g(\theta)}\right)\right\} \prec_q (\varphi(\theta) - 1).$$

3- For $\mu = 0, (b = 0, \alpha = 0, \gamma = 0, \delta = 0)$ and $(0 \leq \tau \leq 1, \psi \in \mathbb{C}\backslash\{0\}, 0 \leq a \leq 1)$, a function $f \in \Omega$ in the class $C_{\Omega}(\tau, a, \psi, \varphi)$, if quasi-subordination conditions are satisfied when f defined in (1).

$$\frac{1}{\psi}\left\{\tau z\left(\frac{f(z)}{z}\right)' + \left(\frac{z\big(f(z)\big)'}{f(z)}\right) + az^2\big(f(z)\big)'''\right\} \prec_q (\varphi(z) - 1),$$

$$\frac{1}{\psi}\left\{\tau \theta\left(\frac{g(\theta)}{\theta}\right)' + \left(\frac{\theta\big(g(\theta)\big)'}{g(\theta)}\right) + a\theta^2\big(g(\theta)\big)'''\right\} \prec_q (\varphi(\theta) - 1).$$

Theorem 2: If f is given by (1) belongs to on the subclass $C_{s,a,\mu}^{\delta,\lambda}(\tau, a, \psi, \varphi)$, then

$$|a_2| \leq \frac{\lambda|k_0|s_1\sqrt{s_1}}{\sqrt{2(2\tau + 2 + 6a)\psi k_0 s_1^2 \beta_{2,\mu} - \psi k_0 s_1^2 \beta_{2,\mu}^2 - (\tau + 1)^2 \beta_{2,\mu}(s_2 - s_1)}}, \tag{33}$$

$$|a_3| \leq \frac{2\psi k_1 s_1}{(4\tau + 4 + 24a)} + \frac{\psi^2 k_0^2 s_1^2}{2(\tau + 1)^2 \beta_{2,\mu}^2}. \tag{34}$$

Proof: If $f \in C_{s,a,\mu}^{\delta,\lambda}(\tau, a, \psi, \varphi)$ and $g = f^{-1}$, then, where φ, F are analytic functions in υ and $\varphi, F: \upsilon \to \upsilon$, with $|\varphi(z)| \leq 1$, such that $\varphi(0) = F(0) = 0$ and $|f(0)| < 1$, satisfied:

$$\frac{1}{\psi}\left\{\tau z\left(\frac{\mathfrak{D}_{s,a,\mu}^{\delta,\lambda} f(z)}{z}\right)' + \left(\frac{z\left(\mathfrak{D}_{s,a,\mu}^{\delta,\lambda} f(z)\right)'}{\mathfrak{D}_{s,a,\mu}^{\delta,\lambda} f(z)}\right) + az^2\left(\mathfrak{D}_{s,a,\mu}^{\delta,\lambda} f(z)\right)'''\right\} = \phi(z)\big(\varphi\big(F(z)\big) - 1\big), \tag{35}$$

$$\frac{1}{\psi}\left\{\tau \theta\left(\frac{\mathfrak{D}_{s,a,\mu}^{\delta,\lambda} g(\theta)}{\theta}\right)' + \left(\frac{\theta\left(\mathfrak{D}_{s,a,\mu}^{\delta,\lambda} g(\theta)\right)'}{\mathfrak{D}_{s,a,\mu}^{\delta,\lambda} g(z)}\right) + a\theta^2\left(\mathfrak{D}_{s,a,\mu}^{\delta,\lambda} g(\theta)\right)'''\right\} = \phi(\theta)\big(\varphi\big(N(\theta)\big) - 1\big). \tag{36}$$

The function $x(z)$ and $y(z)$ define by (9) and (10), respectively proceeding similarly as in Theorem 1, we get

$$\frac{1}{\psi}\left\{\tau z\left(\frac{\mathfrak{D}^{\delta,\lambda}_{s,a,\mu}f(z)}{z}\right)'+\left(\frac{z\left(\mathfrak{D}^{\delta,\lambda}_{s,a,\mu}f(z)\right)'}{\mathfrak{D}^{\delta,\lambda}_{s,a,\mu}f(z)}\right)+az^2\left(\mathfrak{D}^{\delta,\lambda}_{s,a,\mu}f(z)\right)'''\right\} = \phi(z)\left(\varphi\left(\frac{x(z)-1}{x(z)+1}\right)-1\right), \quad (37)$$

$$\frac{1}{\psi}\left\{\tau\theta\left(\frac{\mathfrak{D}^{\delta,\lambda}_{s,a,\mu}g(\theta)}{\theta}\right)'+\left(\frac{\theta\left(\mathfrak{D}^{\delta,\lambda}_{s,a,\mu}g(\theta)\right)'}{\mathfrak{D}^{\delta,\lambda}_{s,a,\mu}g(\theta)}\right)+a\theta^2\left(\mathfrak{D}^{\delta,\lambda}_{s,a,\mu}g(\theta)\right)'''\right\} = \phi(\theta)\left(\varphi\left(\frac{y(\theta)-1}{y(\theta)+1}\right)-1\right), \quad (38)$$

$$\frac{1}{\psi}\left\{\tau z\left(\frac{\mathfrak{D}^{\delta,\lambda}_{s,a,\mu}f(z)}{z}\right)'+\left(\frac{z\left(\mathfrak{D}^{\delta,\lambda}_{s,a,\mu}f(z)\right)'}{\mathfrak{D}^{\delta,\lambda}_{s,a,\mu}f(z)}\right)+az^2\left(S^{\mu,\delta}_{b,\alpha,\gamma}f(z)\right)'''\right\} = \frac{1}{\psi}\left\{(\tau+1)a_2\beta_{2,\mu}z + (2\tau+2+6a)a_3\beta_{3,\mu}z^2 - a_2^2\beta_{2,\mu}z^2+\cdots\right\}, \quad (39)$$

$$\frac{1}{\psi}\left\{\tau\theta\left(\frac{\mathfrak{D}^{\delta,\lambda}_{s,a,\mu}g(\theta)}{\theta}\right)'+\left(\frac{\theta\left(\mathfrak{D}^{\delta,\lambda}_{s,a,\mu}g(\theta)\right)'}{\mathfrak{D}^{\delta,\lambda}_{s,a,\mu}g(\theta)}\right)+a\theta^2\left(\mathfrak{D}^{\delta,\lambda}_{s,a,\mu}g(\theta)\right)'''\right\} = \frac{1}{\psi}\left\{-(\tau+1)a_2\beta_{2,\mu}\theta + (2\tau+2+6a)a_3\beta_{3,\mu}\theta^2 - a_2^2\beta_{2,\mu}\theta^2+\cdots\right\}, \quad (40)$$

Comparing the coefficients of (39) with (15) and (40) with (16), then we have

$$\left(\frac{\tau+1}{\psi}\right)a_2\beta_{2,\mu} = \frac{1}{2}k_0s_1x_1, \quad (41)$$

$$\frac{1}{\psi}\left\{(2\tau+2+6a)a_3\beta_{3,\mu} - a_2^2\beta_{2,\mu}\right\} = \frac{1}{2}k_1s_1x_1 + \frac{1}{2}k_0s_1\left(x_2-\frac{x_1^2}{2}\right)+\frac{1}{4}k_0s_2x_1^2, \quad (42)$$

$$\frac{1}{\psi}\left\{(2\tau+2+6a)(2a_2^2-a_3)\beta_{3,\mu} - a_2^2\beta_{2,\mu}\right\} = \frac{1}{2}k_1s_1y_1 + \frac{1}{2}k_0s_1\left(y_2-\frac{y_1^2}{2}\right)+\frac{1}{4}k_0s_2y_1^2, \quad (43)$$

$$a_2 = \frac{\psi k_0s_1x_1}{2(\tau+1)\beta_{2,\mu}} = -\frac{\psi k_0s_1y_1}{2(\tau+1)\beta_{2,\mu}}.$$

From (39) and (41), we find that

$$x_1 = -y_1 \quad (44)$$

and

$$8(\tau+1)^2\beta_{2,\mu}^2a_2^2 = \lambda^2k_0^2s_1^2(x_1^2+y_1^2). \quad (45)$$

Applying (40) and (43), we get

$$a_2^2 = \frac{2\psi^2k_0^2s_1^3(x_2+y_2)}{8\left((2\tau+2+6a)\psi k_0s_1^2\beta_{3,\mu} - \psi k_0s_1^2\beta_{3,\mu} - (\tau+1)^2\beta_{2,\mu}\right)}. \quad (46)$$

Applying lemma (1) in (48), we get (33)

Now, find the bound on the coefficient $|a_3|$, by subtracting (40) and (43) we get,

$$(16\tau + 16 + 48a)a_3\beta_{3,\mu} - (16\tau + 16 + 48a)a_2^2\beta_{3,\mu} = 2\psi k_1 s_1 x_1 + k_0\psi s_1(x_2 - y_2). \quad (47)$$

Substituting (44) from (42), further computation using (44) and (45), we get

$$a_3 = \frac{2\lambda k_1 s_1 x_1}{(16\tau + 16 + 48a)\beta_{3,\mu}} + \frac{k_0\lambda s_1(x_2 - y_2)}{(16\tau + 16 + 48a)\beta_{3,\mu}} + a_2^2 \ . \quad (48)$$

Applying Lemma (1) in (48), we get (34). ■

Taking $\tau = 0$, in Theorem (2), we get the following corollary.

Corollary 3: Let f belong to the class $C_{s,a,\mu}^{\delta,\lambda}(0, a, \lambda, \varphi)$, when f given by (1). Then

$$|a_2| \le \frac{\psi|k_0|s_1\sqrt{s_1}}{\sqrt{(4 + 24a)\psi k_0 s_1^2\beta_{2,\mu} - \psi k_0 s_1^2\beta_{2,\mu}^2 - \beta_{2,\mu}(s_2 - s_1)}},$$

and

$$|a_3| \le \frac{\psi k_1 s_1}{(4 + 24a)} + \frac{\psi^2|k_0^2|s_1^2}{2\beta_{2,\mu}^2}.$$

Corollary 4: Let f belong to the class $C_{s,a,\mu}^{\delta,\lambda}(\tau, 0, \lambda, \varphi)$, and f given by (1). Then

$$|a_2| \le \frac{\psi|k_0|s_1\sqrt{s_1}}{\sqrt{(4\tau + 4)\psi k_0 s_1^2\beta_{2,\mu} - \psi k_0 s_1^2\beta_{2,\mu}^2 - (\tau + 1)^2\beta_{2,\mu}(s_2 - s_1)}},$$

and

$$|a_3| \le \frac{\psi k_1 s_1}{(4\tau + 4)} + \frac{\psi^2 k_0^2 s_1^2}{2(\tau + 1)^2\beta_{2,\mu}^2}.$$

Corollary 5: Let f belong to the class $C_\Omega(\tau, a, \psi, \varphi)$ and f given by (1). Then

$$|a_2| \le \frac{\psi|k_0|s_1\sqrt{s_1}}{\sqrt{(4\tau + 4 + 24a)\psi k_0 s_1^2 - \lambda k_0 s_1^2\beta_{2,\mu}^2 - (\tau + 1)^2(s_2 - s_1)}},$$

$$|a_3| \le \frac{2\psi k_1 s_1}{(4\tau + 4 + 24a)} + \frac{\psi^2 k_0^2 s_1^2}{2(\tau + 1)^2}$$

4. Conclusions

In this research and through suggested some classes $\mathcal{N}_{\Omega,b,\alpha,\gamma}^{\mu,\delta}(\psi, \tau, \varphi)$ and $C_{\Omega,b,\alpha,\gamma}^{\mu,\delta}(\tau, a, \psi, \varphi)$ of bi-univalent functions in the unit disk υ, we found estimates the first two Taylor-Maclaurin coefficients $|a_2|$ and $|a_3|$ by convolution of the operators $R_q^\delta f(z)$ and $I_{s,a,\mu}^\lambda f(z)$.

References

[1] P. L. Duren, "Univalent functions," In: Grundlehen der Mathematischen Wissenschaften," *Springer- Verlag*, 1983.

[2] A. H. M. Srivastava, "Applications of the q-Srivastava Attiya operator involving a certain family of bi-univalent functions associated with the Horadam polynomials," *Symmetry*, vol. 13, no. 7, 2021.

[3] L. M., "On a coefficient problem for bi-univalent," *Amer. Math.Soc.*, vol. 18, pp. 63-68, 1967.

[4] D. A. Brannan and J. G. Clunie, Aspect of contemporary complex analysis, London: Academic press, 1980.

[5] E. Netanyahu, "The minimal distance of the image boundary from the origin and the second coefficient of a univalent function in |Z|," *Arch. Rational Mech. Anal.,* vol. 32, pp. 100-112, 1969.

[6] D. Styer and J. Wright, "Result on bi-univalent functions," *Proc.Amer.Math.Soc.,* pp. 243-248, 1981.

[7] D. L. Tan, "Coefficient estimates for bi-univalent functions," *Chinese Ann.Math.Ser.A,* vol. 5, pp. 559-568, 1984.

[8] H. M. Srivastava, S. Gaboury and F. Ghanim, "Coefficient estimates for some general subclasses of analytic and bi-univalent functions," vol. 28, pp. 693-706, 2017.

[9] S. A. AL-Ameede, W. G. Atshan and F. A. AL-Maamori, "Results on Coefficients Estimates of Bi-Univalent Functions Defined by New Subclass Function," *Journal of physics:Conference Series,* vol. 1530, 2020.

[10] H. Aldweby and M. Darus, "some subordination results on q-analogue of Ruscheweyh differential operator," *Abstr.Appl.Anal.,* pp. 1-6, 2014.

[11] y. Komatu, "on analytic prolongation of family of integral operators," *Mathematics(CLUJ),* vol. 32, pp. 141-145, 1990.

[12] B. K. Mihsin, W. G. Atshan and S. S. Alhily, "On New sandwich Rsults of univalent function defined by a Liner operator," *Iraqi Journal of Science,* vol. 63, pp. 5467-5475, 2021.

[13] L. A. Majed and A. K. Abdul-Rahman, "Negative Coefficients Sub class of Multivalent Functions," *Iraqi Journal of Science,* vol. 63, pp. 3925-3930, 2022.