Coefficients Estimates for New Subclasses of Bi-univalent Functions Using Convolution Operator

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Received: 10/2/2023 Accepted: 4/7/2023 Published: 30/7/2024

Abstract

We offered and suggested some classes \(\mathcal{N}_{\Omega}^\mu,\delta(\psi, \tau, \varphi)\) and \(\mathcal{C}_{\Omega}^{\mu,\delta}(\tau, a, \psi, \varphi)\) of bi-univalent functions in the unit disk \(\Upsilon\), to apply the effects to the classes through convolution of the operators \(R^\delta_q f(z)\) and \(I_{s,a,\mu}^\lambda f(z)\). Which satisfies the condition quasi-subordination. We got estimates the first two Taylor-Maclaurin coefficients \(|a_2|\) and \(|a_3|\) by convolution of the operators \(R^\delta_q f(z)\) and \(I_{s,a,\mu}^\lambda f(z)\).

Keywords: normalized analytic, bi-univalent function, estimate, integral operator.

1. Introduction

In an open unit disk \(\Upsilon = \{z: |z| < 1\}\) let \(\mathcal{N}\) be the class of normalized analytic function \(f\) defined with Taylor series

\[ f(z) = z + \sum_{k=2}^{\infty} a_k z^k. \]
In addition to, \( \mathcal{N} \) which are univalent and normalized by the conditions in and suppose \( \mathcal{D} \) denote the class of all functions in \( \mathcal{N} \). According to the Koebe One-Quarter Theorem [1] such that the inverse \( f^{-1} \) of every \( f \in \mathcal{D} \) satisfies:

\[
f^{-1}(f(z)) = z, \quad (z \in \nu),
\]

and

\[
f^{-1}(f(\theta)) = \theta, \quad (|\theta| < r_0(f); r_0(f) \geq \frac{1}{4}),
\]

where

\[
f^{-1}(\theta) = g(\theta) = \theta - a_2\theta^2 + (2a_2^2 - a_3)\theta^3 - (5a_2^3 - 5a_2a_3 + a_4)\theta^4 + \ldots.
\]

(2)

Let \( \Omega \) the class of all bi-univalent function \( f \in \mathcal{N} \) in \( \nu \).

Some similar studies have been published but with different operators they are as follows see [2-9].

2. Preliminary

For functions \( f, g \in \mathcal{N} \) depending on the equation (1). Now the function \( g(z) \)

\[
g(z) = z + \sum_{k=2}^{\infty} b_k z^k \quad (z \in \nu).
\]

The convolution of the function \( f \) and \( K \) is defined as:

\[
(f * g)(z) = z + \sum_{k=2}^{\infty} a_k b_k z^k.
\]

In [10] the Rusyeywewh type q-analogue operator \( R_\delta^q \) is defined by

\[
R_\delta^q f(z) = z + \sum_{k=2}^{\infty} \frac{[k + \delta - 1]q^k - 1}{[\delta]q^k} a_k b_k z^k,
\]

where \( \delta \geq 0 \). Also, as \( q \to 1^- \), we have

\[
\lim_{q \to 1^-} R_\delta^q f(z) = z + \lim_{q \to 1^-} \left( \sum_{k=2}^{\infty} \frac{[k + \delta - 1]q^k - 1}{[\delta]q^k} a_k b_k z^k \right).
\]

(3)

Komatu [11] introduced and investigated a family of integral operator \( f_\mu^\lambda : \Omega \to \Omega \). That is obtained as follows

\[
f_\mu^\lambda f(z) = z + \sum_{k=1}^{\infty} \left( \frac{\mu}{\mu + k - 1} \right) a_k b_k z, \quad z \in \nu, k > 1, \lambda \geq 0.
\]


\[
\phi(z, s, \alpha) = \sum_{k=0}^{\infty} \frac{z^k}{(t + a)^{s}} \quad \alpha \in \mathbb{C} - z_0, s \in \mathbb{C} \text{ when } 0 < |z| < 1.
\]

Interims of convolution. Where \( G_{s,a}(z) \) is given by

\[
G_{s,a}(z) = [1 + a]s(\phi(z, s, a) - a^{-s}), \quad z \in \nu.
\]

The linear operator \( L_{s,a,\mu} f(z) : \sum \to \sum \) is defined

\[
L_{s,a,\mu} f(z) = G_{s,a}(z) * f_\mu^\lambda f(z) = z + \sum_{k=1}^{\infty} \left[ \frac{1+a}{s} \right]^{s} \left( \frac{\mu}{\mu + k - 1} \right)^{\lambda} a_k b_k z^k.
\]

The convolution of the operator \( R^S f(z) \) and \( L_{s,a,\mu}^i f(z) \) are defined by

\[
\mathcal{O}^{\delta, \lambda}_{s,a,\mu} f(z) = R^S f(z) * f_\mu^\lambda f(z).
\]

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\[ \mathcal{D}_{s,a,\mu}^0 f(z) = z + \sum_{k=2}^{\infty} [k + \delta - 1]_q! \left[ \frac{1 + a}{t + a} \right] \left[ \frac{\mu}{\mu + k - 1} \right] a_k z^k. \]
\[ \mathcal{D}_{s,a,\mu}^k f(z) = z + \sum_{k=2}^{\infty} \theta_{k,\delta} a_k z^k. \]
Where
\[ \theta_{k,\delta} = \left[ \frac{[k + \delta - 1]_q!}{\delta q! [k - 1]_q!} \right] \left[ \frac{1 + a}{t + a} \right] \left[ \frac{\mu}{\mu + k - 1} \right]^3. \]

Now we can say there exists analytic functions \( \phi \) and \( F \), with \( |\phi(z)| \leq 1, F(0) = 0 \) and \( |F(z)| < 1 \), then function \( f \) is a quasi-subordinate to \( g \) in \( v \), such that \( f(z) = \phi(z) g(F(z)) \), so
\[ f(z) < q g(z), \quad (z \in v). \]

We denote this a quasi-subordination by [12], as follows:

We will notes if \( \phi(z) = 1 \), then \( f(z) = g(F(z)) \), hence \( f(z) < g(z) \) in [7] and [8]. Furthermore, if \( F(z) = z \), then \( f(z) = \phi(z) g(z) \), and in case, \( f \) is majorized by \( g \), written \( f(z) \ll g(z) \) in \( v \). By the method of subordination we defined and studied classes \( \mathcal{D}^*(\phi) \) and \( G^*(\phi) \) of starlike functions by:
\[ \mathcal{D}^*(\phi) = \left\{ f \in \mathcal{N}: \frac{z f'(z)}{f(z)} < \phi(z), z \in v \right\}, \]
and
\[ G^*(\phi) = \left\{ f \in \mathcal{N}: 1 + \frac{z f''(z)}{f(z)} < \phi(z), z \in v \right\}, \]
where
\[ \phi(z) = k_0 + k_1 z + k_2 z^2 + \cdots, \quad (z \in v). \]
Consider
\[ \varphi(z) = 1 + s_1 z + s_2 z^2 + s_3 z^3 + \cdots, \quad s \in v, \]
and univalent analytic function with a positive real part in \( v \), symmetric with respect to the real axis, and starlike with respect to \( \varphi(\text{zero}) = 1 \) and \( \varphi'(\text{zero}) > 0 \). By \( \mathcal{D}^*_\Omega(\phi) \) and \( G^*_\Omega(\phi) \), we introduce and study here certain subclasses of the class \( \Omega [13] \).

**Definition 1:** Let \( f \) belong to \( \mathcal{N} \) is called in the class \( \mathcal{N}_0^{\mu,\delta} \) \((0 \leq \psi, \leq 1, 0 \leq \tau \leq 1) \), if satisfied the quasi-subordination:
\[ \begin{align*}
\theta \left( \mathcal{D}_{s,a,\mu}^\delta g(\theta) \right)' + \psi \theta^2 \left( \mathcal{D}_{s,a,\mu}^\delta g(\theta) \right)'' + \tau \theta \left( \mathcal{D}_{s,a,\mu}^\delta g(\theta) \right)'' & + \left( \mathcal{D}_{s,a,\mu}^\delta f(z) \right) \left( \mathcal{D}_{s,a,\mu}^\delta f(z) \right)' \mathcal{D}_{s,a,\mu}^\delta f(z) \\
& < q \varphi(z) - 1,
\end{align*} \]
where \( g \) is the inverse function of \( f \) and \( z, \theta \in v \).

**Properties**

1. For all \( \tau = \text{zero} \) and \((0 \leq \psi \leq 1)\) a function \( f \in \Omega \) defined (1) is called in the class \( \mathcal{N}_0^{\mu,\delta} \) \((0 \leq \psi, \leq 1, 0 \leq \tau \leq 1) \) if satisfied the condition quasi-subordination:
\[ z \left( \mathcal{G}_{s,a,\mu}^\delta \phi(z) \right)' + \psi z^2 \left( \mathcal{G}_{s,a,\mu}^\delta \phi(z) \right)'' + \left( \mathcal{G}_{s,a,\mu}^\delta \phi(z) \right)' \mathcal{G}_{s,a,\mu}^\delta \phi(z) - 1 < q \varphi(z) - 1, \]

\[ \theta \left( \mathcal{G}_{s,a,\mu}^{\delta,\lambda} \phi(z) \right)' + \psi \theta^2 \left( \mathcal{G}_{s,a,\mu}^{\delta,\lambda} \phi(z) \right)'' + \left( \mathcal{G}_{s,a,\mu}^{\delta,\lambda} \phi(z) \right)' \mathcal{G}_{s,a,\mu}^{\delta,\lambda} \phi(z) - 1 < q \varphi(\theta) - 1. \]

2- For all \( \delta = \text{zero}, (\lambda, s, a, \mu) = 0 \) and \((0 \leq \psi \leq 1)\) a function \( f \in \Omega \) defined (1) is called in the class \( \mathcal{N}_b(\psi, o, \varphi) \) if satisfied the quasi-subordination:

\[ \frac{z(f(z))'}{f(z)} + \psi z^2 \left( f(z) \right)'' + \tau z(f(z))' f(z) + (f(z))' f(z) - 1 < q \varphi(z) - 1, \]

\[ \frac{\theta(g(\theta))'}{g(\theta)} + \psi \theta^2 (g(\theta))'' + \tau \theta(g(\theta))' + (g(\theta))' - 1 < q \varphi(\theta) - 1. \]

**Lemma 1:** [1] If \( p \in P \), then \( |p_i| \leq 2 \forall i \), where \( P \) is the family of all analytic \( p \), for which \( Re \{ p(z) \} > 0, z \in v \), where

\[ p(z) = 1 + p_1 z + p_2 z^2 + p_3 z^3 + \ldots, \quad (z \in v). \]

3. **Main results**

**Theorem 1:** If \( f \in \mathcal{N}_{\Omega,b,a,y}^{\mu,\delta}(\psi, \tau, \varphi) \), then

\[ |a_2| \leq \min \left\{ \frac{|k_0| s_1}{(3 + 2\psi + 2\tau) \beta_{2,\mu}}, \left( \frac{|k_0| s_1 + |k_0|(s_2 - s_1)}{(10 + 12\psi + 12\tau) \beta_{3,\mu} - 2(1 + 2\psi) \beta_{3,\mu}} \right)^{\frac{1}{2}} \right\}, \]

\[ |a_3| \leq \min \left\{ \frac{|k_0| s_1}{(5 + 6\psi + 6\sigma) \beta_{2,\mu}} + \frac{|k_0| s_1}{(6 + 4\psi + 4\tau) \beta_{2,\mu}} \left( \frac{30 + 24\psi + 24\sigma) \beta_{3,\mu} - 2(1 + 2\psi) \beta_{3,\mu}}{(|k_0| s_1 + |k_0|(s_2 - s_1))} \right) \right\}. \]

Proof: Since \( f \in \mathcal{N}_{\Omega,b,a,y}^{\mu,\delta}(\psi, \tau, \varphi) \), then there exists analytic function, \( F, \varphi \) in \( v \) and \( \varphi, F: v \to v \), with \(|\varphi(z)| \leq 1\), such that \( \varphi(o) = F(o) = 0 \) and \(|F(z)| \leq 1\), satisfied:

\[ z \left( \mathcal{G}_{s,a,\mu}^{\delta,\lambda} f(z) \right)' + \psi z^2 \left( \mathcal{G}_{s,a,\mu}^{\delta,\lambda} f(z) \right)'' + \tau z \left( \mathcal{G}_{s,a,\mu}^{\delta,\lambda} f(z) \right)' \mathcal{G}_{s,a,\mu}^{\delta,\lambda} f(z) + \left( \mathcal{G}_{s,a,\mu}^{\delta,\lambda} f(z) \right)' \mathcal{G}_{s,a,\mu}^{\delta,\lambda} f(z) - 1 = \phi(z)(\varphi(F(z)) - 1), \]

\[ \theta \left( \mathcal{G}_{s,a,\mu}^{\delta,\lambda} g(\theta) \right)' + \psi \theta^2 \left( \mathcal{G}_{s,a,\mu}^{\delta,\lambda} g(\theta) \right)'' + \tau \theta \left( \mathcal{G}_{s,a,\mu}^{\delta,\lambda} g(\theta) \right)' \mathcal{G}_{s,a,\mu}^{\delta,\lambda} g(\theta) + \left( \mathcal{G}_{s,a,\mu}^{\delta,\lambda} g(\theta) \right)' \mathcal{G}_{s,a,\mu}^{\delta,\lambda} g(\theta) - 1 = \phi(\theta)(\varphi(V(\theta)) - 1), \]

so \( g \) is the inverse of \( f \) and \( z \in v \).

\[ x(z) = \frac{1 + F(z)}{1 - F(z)} = 1 + x_1 z + x_2 z^2 + x_3 z^3 + \ldots, \]

\[ y(\theta) = \frac{1 + V(\theta)}{1 - V(\theta)} = 1 + y_1 \theta + y_2 \theta^2 + y_3 \theta^3 + \ldots, \]

or

\[ F(z) = \frac{x(z) - 1}{x(z) + 1} = \frac{1}{2} \left( x_1 z + \left( x_2 - \frac{x_1^2}{2} \right) z^2 + \ldots, \right) \]
\[
V(\theta) = \frac{y(z) - 1}{y(z) + 1} = \frac{1}{2} y_1 \theta + \left( y_2 - \frac{y_1^2}{2} \right) \theta^2 + \ldots. \tag{12}
\]
i.e., the analytic function \( x(z) \) and \( y(\theta) \) are in \( v \) with \( x(0) = y(0) = 1 \), since \( F, V: v \rightarrow v, x(z) \) and \( y(\theta) \) have a positive real in \( v \), and \( |x_i| \leq 2 \) and \( |y_i| \leq 2 \), \( i = 1, 2 \)
\[
z \left( \delta_{s,a,\mu}^\lambda f(z) \right)' + \psi z^2 \left( \delta_{s,a,\mu}^\lambda f(z) \right)'' + \tau z \left( \delta_{s,a,\mu}^\lambda f(z) \right)' + s \left( \delta_{s,a,\mu}^\lambda g(z) \right)' \]
\[
\delta_{s,a,\mu}^\lambda f(z) - 1 = \phi(z) \left( \frac{y(\theta) - 1}{y(\theta) + 1} - 1 \right), \tag{13}
\]
we can write (13) and (14) of the form (15) and (16)
\[
k_0 s_1 x_1 z + \frac{1}{2} z^2 k_1 s_1 x_1 + \frac{1}{2} z^2 k_0 s_1 \left( x_2 - \frac{x_1^2}{2} \right) + \frac{1}{4} k_0 s_2 x_1^2 z^2 + \ldots, \tag{15}
\]
\[
\frac{1}{2} k_0 s_1 y_1 \theta + \frac{1}{2} \theta^2 k_1 s_1 y_1 + \frac{1}{2} \theta^2 k_0 s_1 \left( y_2 - \frac{y_1^2}{2} \right) + \theta^2 \frac{1}{4} k_0 s_2 y_1^2 + \ldots \tag{16}
\]
since
\[
\left( z \left( \delta_{s,a,\mu}^\lambda f(z) \right)' + \psi z^2 \left( \delta_{s,a,\mu}^\lambda f(z) \right)'' + \tau z \left( \delta_{s,a,\mu}^\lambda f(z) \right)' + s \left( \delta_{s,a,\mu}^\lambda g(z) \right)' \right) = 1 + (3 + 2 \psi + 2 \psi) a_2 \beta_{2,\mu} z + (5 + 6 \psi + 6 \alpha) a_3 \beta_{3,\mu} z^2 - (1 + 2 \psi) a_2^2 \beta_{2,\mu} z^2 + \ldots \tag{17}
\]
\[
\left( \theta \left( \delta_{s,a,\mu}^\lambda g(z) \right)' + \psi z^2 \left( \delta_{s,a,\mu}^\lambda g(z) \right)'' + \tau \theta \left( \delta_{s,a,\mu}^\lambda g(z) \right)' + s \left( \delta_{s,a,\mu}^\lambda g(z) \right)' \right) = 1 + (-3 + 2 \psi + 2 \psi) a_2 \beta_{2,\mu} \theta + (5 + 6 \psi + 6 \alpha) a_3 \beta_{3,\mu} \theta^2 - (1 + 2 \psi) a_2^2 \beta_{2,\mu} \theta^2 + \ldots \tag{18}
\]
With compensation (15) and (17) in equation (13) and in the same way (16) and (18) in (14), we get
\[
(5 + 6 \psi + 6 \alpha) a_3 \beta_{3,\mu} - (1 + 2 \psi) a_2^2 \beta_{2,\mu} = \frac{1}{2} k_1 s_1 x_1, \tag{19}
\]
\[
\frac{1}{2} k_0 s_1 \left( x_2 - \frac{x_1^2}{2} \right) + \frac{1}{4} k_0 s_2 x_1^2, \tag{20}
\]
and
\[
- (3 + 2 \psi + 2 \psi) a_2 \beta_{2,\mu} = \frac{1}{2} k_0 s_1 y_1, \tag{21}
\]
\[
(5 + 6 \psi + 6 \alpha) (2a_2^2 - a_3) \beta_{3,\mu} - (1 + 2 \psi) a_2^2 \beta_{2,\mu}
\]
\[
= \frac{1}{2} k_1 s_1 y_1 + \frac{1}{2} k_0 s_1 \left( y_2 - \frac{y_1^2}{2} \right) + \frac{1}{4} k_0 s_2 y_1^2. \tag{22}
\]
From (19) and (21), we find that
\[
a_2 = \frac{k_0 s_1 x_1}{(6 + 4 \psi + 4 \alpha) \beta_{2,\mu}} = - \frac{k_0 s_1 y_1}{(6 + 4 \psi + 4 \alpha) \beta_{2,\mu}}, \tag{23}
\]
\[
x_1 = -y_1 \tag{24}
\]
and

\[(24 + 16\psi + 16\tau)^2 \beta_{2,\mu} a_2^2 = \psi^2 k_0 s_1^2 (x_1^2 + y_1^2).\]  

Applying (20) and (22), by using (24) and (25), we have

\[(40 + 48\psi + 48\tau) a_2^2 \beta_{3,\mu} - (8 + 16\psi) a_2^2 \beta_{3,\mu} = 2k_0s_1(x_2 + y_2) + k_0(x_1^2 + y_1^2)(s_2 - s_1),\]

which implies that

\[a_2^2 = \frac{2k_0s_1(x_2 + y_2) + k_0(x_1^2 + y_1^2)(s_2 - s_1)}{(40 + 48\psi + 48\tau) \beta_{3,\mu} - (8 + 16\psi) \beta_{3,\mu}}.\]  

By Lemma (1) in (27), we get (3).

By application of Lemma 1 in monologue (30), we get (6).

**Corollary 1:** Let \( f \) belong to the class \( \mathcal{N}_{\Omega, \sigma}^{\mu, \delta} (\psi, 0, \varphi), \) and \( f \) be given by (1), then

\[|a_2| \leq \min\left\{ \frac{|k_0|s_1}{(3 + 2\psi)\beta_{2,\mu}}, \sqrt{\frac{|k_0|s_1 + |k_0|(s_2 - s_1)}{(10 + 12\psi)\beta\beta_{3,\mu} - (2 + 4\psi)\beta_{3,\mu}}} \right\},\]

\[|a_3| \leq \min\left\{ \frac{|k_0|s_1}{(5 + 6\psi)\beta_{2,\mu}} + \frac{|k_0|s_1}{(5 + 6\psi)\beta_{2,\mu}} + \frac{|k_0|s_1}{(5 + 6\psi)\beta_{2,\mu}} \right\},\]

\[+ \frac{|k_0|s_1}{(10 + 12\psi)\beta_{3,\mu} - (2 + 4\psi)\beta_{3,\mu}}.\]

**Corollary 2:** Let \( f \) belong to the class \( \mathcal{N}_{\Omega}^{\psi, \sigma} (\psi, \varphi, \alpha), \) and \( f \) be given by (1), then

\[|a_2| \leq \min\left\{ \frac{|k_0|s_1}{(3 + 2\psi + 2\tau)^2}, \frac{|k_0|s_1 + |k_0|(s_2 - s_1)}{(10 + 12\psi + 12\tau) - 2(1 + 2\psi)} \right\},\]

\[|a_3| \leq \min\left\{ \frac{|k_0|s_1}{(5 + 6\psi + 6\alpha)^2} + \frac{|k_0|s_1}{(5 + 6\psi + 6\alpha)^2} \right\},\]

\[+ \frac{|k_0|s_1}{(10 + 12\psi + 12\tau) - (2 + 4\psi)}.\]

**Definition 2:** A function \( f \in \mathcal{N} \) is in the class \( \mathcal{C}_{\Omega, \psi, \alpha, \gamma}^{\mu, \delta} (\psi, \alpha, \psi, \varphi) (0 < \tau \leq 1), \) where \( \psi \in \mathbb{C}\setminus\{0\}, \) \( 0 \leq \alpha < 1, \) if it satisfies a quasi-subordination:

\[\frac{1}{\psi} \left\{ \tau z \left( \frac{\gamma^{\delta, \lambda}_{s, a, u} f(z)}{z} \right) + z \left( \frac{\gamma^{\delta, \lambda}_{s, a, u} f(z)}{z} \right) \right\} < q (\varphi(z) - 1),\]

\[\frac{1}{\psi} \left\{ \tau \theta \left( \frac{\gamma^{\delta, \lambda}_{s, a, u} g(\theta)}{\theta} \right) + \theta \left( \frac{\gamma^{\delta, \lambda}_{s, a, u} g(\theta)}{\theta} \right) \right\} < q (\varphi(\theta) - 1),\]

where \( g \) is the inverse of \( f \) and \( z, \theta \in \mathbb{D}.\)
Properties

1- For $\tau = 0$ and $0 \leq \delta \leq 1$, $\psi \in \mathbb{C}\backslash\{0\}$, $0 \leq a \leq 1$. A function $f \in \Omega$ is in the class $C_{s,a,\mu}^{\delta,\lambda}(0, a, \psi, \phi)$, if the following quasi-subordination conditions are satisfied and $f$ defined in (1).

\[
\frac{1}{\psi}\left\{\left(\frac{z}{s^s a^s \mu^s f(z)}\right)\phi + az^2\left(\frac{s^s a^s \mu^s f(z)}{s^s a^s \mu^s f(z)}\right)\right\} < _q (\phi(z) - 1),
\]

\[
\frac{1}{\psi}\left\{\left(\frac{g(z)}{s^s a^s \mu^s g(z)}\right)\phi + a\theta^2\left(\frac{s^s a^s \mu^s g(z)}{s^s a^s \mu^s g(z)}\right)\right\} < _q (\phi(\theta) - 1).
\]

2- For all $a = 0$ and $(0 \leq \delta \leq 1, \psi \in \mathbb{C}\backslash\{0\})$, $0 \leq \tau \leq 1$. A function $f \in \Omega$ is in the class $C_{s,a,\mu}^\delta(\tau, a, \psi, \phi)$, if quasi-subordination conditions are satisfied when $f$ defined in (1).

\[
\frac{1}{\psi}\left\{\left(\frac{z}{s^s a^s \mu^s f(z)}\right)\phi + az^2\left(\frac{s^s a^s \mu^s f(z)}{s^s a^s \mu^s f(z)}\right)\right\} < _q (\phi(z) - 1),
\]

\[
\frac{1}{\psi}\left\{\left(\frac{g(z)}{s^s a^s \mu^s g(z)}\right)\phi + a\theta^2\left(\frac{s^s a^s \mu^s g(z)}{s^s a^s \mu^s g(z)}\right)\right\} < _q (\phi(\theta) - 1).
\]

3- For $\mu = 0, (b = 0, \alpha = 0, \gamma = 0, \delta = 0)$ and $(0 \leq \tau \leq 1, \psi \in \mathbb{C}\backslash\{0\}), 0 \leq a \leq 1$, a function $f \in \Omega$ in the class $C_{s,\phi,\mu}^\delta(\tau, a, \psi, \phi)$, if quasi-subordination conditions are satisfied when $f$ defined in (1).

\[
\frac{1}{\psi}\left\{\left(\frac{z}{s^s a^s \mu^s f(z)}\right)\phi + az^2\left(\frac{s^s a^s \mu^s f(z)}{s^s a^s \mu^s f(z)}\right)\right\} < _q (\phi(z) - 1),
\]

\[
\frac{1}{\psi}\left\{\left(\frac{g(z)}{s^s a^s \mu^s g(z)}\right)\phi + a\theta^2\left(\frac{s^s a^s \mu^s g(z)}{s^s a^s \mu^s g(z)}\right)\right\} < _q (\phi(\theta) - 1).
\]

Theorem 2: If $f$ is given by (1) belongs to on the subclass $C_{s,a,\mu}^\delta(\tau, a, \psi, \phi)$, then

\[
|a_2| \leq \frac{\lambda|k_0|s_1^2s_1^{\phi}}{\sqrt{2(2\tau + 2 + 6\alpha)\psi k_0 s_2^2 s_2^2 - \psi k_0 s_2^2 s_2^2}} \frac{(\tau + 1)^2}{2(\tau + 1)^2 s_2^2}, \quad (33)
\]

\[
|a_3| \leq \frac{2\psi k_1 s_1}{4(4\tau + 4 + 24\alpha)} + \frac{\psi^2 k_0 s_1^2}{2(\tau + 1)^2 s_2^2}. \quad (34)
\]

Proof: If $f \in C_{s,a,\mu}^\delta(\tau, a, \psi, \phi)$ and $g = f^{-1}$, then, where $\phi, F: v \rightarrow v$, with $|\phi(z)| \leq 1$, such that $\phi(0) = F(0) = 0$ and $|f(0)| < 1$, satisfied:

\[
\frac{1}{\psi}\left\{\left(\frac{z}{s^s a^s \mu^s f(z)}\right)\phi + az^2\left(\frac{s^s a^s \mu^s f(z)}{s^s a^s \mu^s f(z)}\right)\right\} = \phi(z)(\phi(F(z)) - 1), \quad (35)
\]

\[
\frac{1}{\psi}\left\{\left(\frac{g(z)}{s^s a^s \mu^s g(z)}\right)\phi + a\theta^2\left(\frac{s^s a^s \mu^s g(z)}{s^s a^s \mu^s g(z)}\right)\right\} = \phi(\theta)(\phi(N(\theta)) - 1). \quad (36)
\]
The function $x(z)$ and $y(z)$ define by (9) and (10), respectively proceeding similarly as in Theorem 1, we get

$$
\frac{1}{\psi} \left( \tau \left( \frac{\delta_{s,a,\mu} f(z)}{z} \right)' + \left( \frac{z \delta_{s,a,\mu} f(z)}{z} \right)'' + az^2 \left( \frac{\delta_{s,a,\mu} f(z)}{z} \right)''' \right)
= \phi(z) \left( \phi \left( \frac{x(z) - 1}{x(z) + 1} \right) - 1 \right), \quad (37)
$$

$$
\frac{1}{\psi} \left( \tau \left( \frac{\delta_{s,a,\mu} g(\theta)}{\theta} \right)' + \left( \frac{z \delta_{s,a,\mu} g(\theta)}{z} \right)'' + a\theta^2 \left( \frac{\delta_{s,a,\mu} g(\theta)}{z} \right)''' \right)
= \phi(\theta) \left( \phi \left( \frac{\gamma(\theta) - 1}{\gamma(\theta) + 1} \right) - 1 \right), \quad (38)
$$

Comparing the coefficients of (39) with (15) and (40) with (16), then we have

$$
\left( \tau + 1 \right)a_2 \beta_{2,\mu} = \frac{1}{2} k_0 s_1 x_1, \quad (41)
$$

$$
\frac{1}{\psi} \left\{ \left( 2\tau + 2 + 6a \right) a_3 \beta_{3,\mu} - a_2^2 \beta_{2,\mu} \right\} = \frac{1}{2} k_1 s_1 x_1 + \frac{1}{2} k_0 s_1 \left( x_2 - \frac{x_1^2}{2} \right) + \frac{1}{4} k_0 s_2 x_1^2, \quad (42)
$$

$$
\frac{1}{\psi} \left\{ \left( 2\tau + 2 + 6a \right) \left( 2a_2^2 - a_3 \right) \beta_{3,\mu} - a_2^2 \beta_{2,\mu} \right\}
= \frac{1}{2} k_1 s_1 y_1 + \frac{1}{2} k_0 s_1 \left( y_2 - \frac{y_1^2}{2} \right) + \frac{1}{4} k_0 s_2 y_1^2, \quad (43)
$$

$$
a_2 = \frac{\psi k_0 s_1 x_1}{2(\tau + 1) \beta_{2,\mu}} = - \frac{\psi k_0 s_1 y_1}{2(\tau + 1) \beta_{2,\mu}}. \quad (44)
$$

From (39) and (41), we find that

$$
x_1 = -y_1 \quad (45)
$$

and

$$
8(\tau + 1) \beta_{2,\mu} a_2^2 = \lambda^2 k_0^2 s_1^2 \left( x_1^2 + y_1^2 \right). \quad (46)
$$

Applying (40) and (43), we get

$$
a_2^2 = \frac{2\psi^2 k_0^2 s_1^3 (x_2 + y_2)}{8 \left( (2\tau + 2 + 6a) \psi k_0 s_1^2 \beta_{3,\mu} - \psi k_0 s_1^2 \beta_{3,\mu} - (\tau + 1) \beta_{2,\mu} \right)}. \quad (47)
$$

Applying lemma (1) in (48), we get (33)

Now, find the bound on the coefficient $|a_3|$, by subtracting (40) and (43) we get,
(16\tau + 16 + 48a)a_3\beta_{3,\mu} - (16\tau + 16 + 48a)a_2^2\beta_{3,\mu} \\
= 2\psi k_1 s_1 x_1 + k_0 \psi s_1 (x_2 - y_2).

Substituting (44) from (42), further computation using (44) and (45), we get

\[ a_3 = \frac{2\lambda k_1 s_1 x_1}{(16\tau + 16 + 48a)\beta_{3,\mu}} + \frac{k_0 \lambda s_1 (x_2 - y_2)}{(16\tau + 16 + 48a)\beta_{3,\mu}} + a_2^2. \tag{48} \]

Applying Lemma (1) in (48), we get (34). ■

Taking \( \tau = 0 \), in Theorem (2), we get the following corollary.

**Corollary 3:** Let \( f \) belong to the class \( C_{\delta,\alpha,\mu}^\delta (0, a, \lambda, \varphi) \), when \( f \) given by (1). Then

\[ |a_2| \leq \frac{\psi |k_0| s_1 \sqrt{s_1}}{\sqrt{(4 + 24\alpha)\psi k_0 s_1^2 \beta_{2,\mu} - \psi k_0 s_1^2 \beta_{2,\mu}^2 - \beta_{2,\mu}(s_2 - s_1)}} \]

and

\[ |a_3| \leq \frac{\psi k_1 s_1}{(4 + 24\alpha)} + \frac{\psi^2 |k_0|^2 s_1^2}{2\beta_{2,\mu}^2}. \]

**Corollary 4:** Let \( f \) belong to the class \( C_{\delta,\alpha,\mu}^\delta (\tau, 0, \lambda, \varphi) \), and \( f \) given by (1). Then

\[ |a_2| \leq \frac{\psi |k_0| s_1 \sqrt{s_1}}{\sqrt{(4\tau + 4)\psi k_0 s_1^2 \beta_{2,\mu} - \psi k_0 s_1^2 \beta_{2,\mu}^2 - (\tau + 1)^2 \beta_{2,\mu}(s_2 - s_1)}} \]

and

\[ |a_3| \leq \frac{\psi k_1 s_1}{(4\tau + 4)} + \frac{\psi^2 k_0^2 s_1^2}{2(\tau + 1)^2 \beta_{2,\mu}^2}. \]

**Corollary 5:** Let \( f \) belong to the class \( C_{\alpha,\psi,\varphi} (\tau, a, \psi, \varphi) \) and \( f \) given by (1). Then

\[ |a_2| \leq \frac{\psi |k_0| s_1 \sqrt{s_1}}{\sqrt{(4\tau + 4 + 24\alpha)\psi k_0 s_1^2 - \lambda k_0 s_1^2 \beta_{2,\mu}^2 - (\tau + 1)^2 (s_2 - s_1)}} \]

\[ |a_3| \leq \frac{2\psi k_1 s_1}{(4\tau + 4 + 24\alpha)} + \frac{\psi^2 k_0^2 s_1^2}{2(\tau + 1)^2}. \]

4. **Conclusions**

In this research and through suggested some classes \( N_{\Omega,\beta,\alpha,\gamma}^\mu (\psi, \tau, \varphi) \) and \( C_{\Omega,\beta,\alpha,\gamma}^\mu (\tau, a, \psi, \varphi) \) of bi-univalent functions in the unit disk \( U \), we found estimates the first two Taylor-Maclaurin coefficients \( |a_2| \) and \( |a_3| \) by convolution of the operators \( R_q^\delta f(z) \) and \( I_{s,a,\mu}^2 f(z) \).

**References**


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