DOI: 10.24996/ijs.2024.65.5.24

ISSN: 0067-2904

On A Generalization of Small-Injective Modules

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Received: 9/2/2023 Accepted: 12/5/2023 Published: 30/5/2024

Abstract
Let R be a ring. JS-injective right R-modules are introduced and studied in this paper as a generalization of small-injective right R-modules. Let N and M be right R-modules. A module M is said to be JS-N-injective if every R-homomorphism from a submodule of $\mathrm{J}(N)\mathrm{J}(R_R)$ into M extends to N. If a module M is JS-R-injective, then M is called JS-injective. Many characterizations and properties of JS-injective right R-modules are obtained. Rings over which every right module is JS-injective are characterized. We study quotients of JS-injective right modules. Then we give conditions under which the class of JS-injective right modules is closed under direct sums.

Keywords: JS-injective module; small-injective module; Noetherian module; projective module.

حول تعميم المقاسات الاغمارية الصغيرة

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الخلاصة
لتكن R حلقة. المقاسات الاغمارية من النمط- JS اليمنى على الحلقة R قد قدمت ودرست في هذا البحث كتعميم للمقاسات الاغمارية الصغيرة. ليكن N و M مقاسان ايمنان على الحلقة R. المقاس M يدعى بالمقاس الاغماري بالنسبة الى N من النمط-JS اذا كان كل تماثل مقاسي على الحلقة R من مقاس جزئي من $\mathrm{J}(N)\mathrm{J}(R)$ إلى M يتوسع إلى N. إذا كان المقاس M هو مقاس اغماري بالنسبة الى R من النمط-JS، فان M يسمى مقاس اغماري من النمط-JS. تم الحصول على العديد من تشخيصات وخصائص المقاسات الاغماري من النمط-JS. تم تشخيص الحلقات التي تكون جميع المقاسات المعرفة عليها هي مقاسات اغماري من النمط-JS. تم دراسة مقاسات القسمة للمقاسات الاغماري من النمط-JS. بعد ذلك تم اعطاء شروطاً يكون بموجبها صنف جميع المقاسات الاغماري من النمط-JS مغلق تحت الجمع المباشر.

1. Introduction

Let R be a ring with identity 1. Throughout this paper, all modules are unitary and by a module (resp. homomorphism) we mean a right R-module (resp. right R-homomorphism), if not otherwise specified. The class of right R-modules is denoted by Mod-R. We write $\mathrm{J}(M)$ and $\mathrm{soc}(M)$ for the Jacobson radical and the socle of a right R-module M, respectively. We write Z_r for the right singular ideal of a ring R. We denote to $\mathrm{J}(M)\mathrm{J}(R_R)$ by $\mathrm{JS}(M)$ for any

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right R-module M. For any $a \in M$, we use $\mathrm{r}(a)$ to denote the right annihilator of a in R. Throughout this paper, R is an associative ring. We refer the reader to [1-6], for general background materials.

Injective modules play an important role in module theory, and extensively many authors are studied their generalizations (see, for example, [7-12]). A right R-module M is called small injective if every R-homomorphism from a small right ideal of R into M can be extended to R_R [7, p.2160]. A right R-module M is called soc-injective if any homomorphism $f: \mathrm{soc}(R_R) \to M$ can be extended to R_R [10].

In this article we introduce the concept of JS-injective modules. Let N and M be modules. We say that M is a JS-N-injective if every homomorphism $f: K \to M$ extends to N, where K is a submodule of $\mathrm{J}(N)\mathrm{J}(R_R)$. If M is a JS-R-injective, then M is called a JS-injective. First, we give an example to clarify that the notions JS-injectivity and small-injectivity are deferent. Some properties of JS-injective modules are obtained. We prove that this class of modules is closed under isomorphic copies, direct products, summands and finite direct sums. Some characterizations of JS-injective modules are given, for example we prove that a module M is JS-injective if and only if $\mathrm{Ext}^1(R/K, M) = 0$, for any submodule K of $\mathrm{JS}(R_R)$, where $\mathrm{Ext}^1(A, B)$ is defined as the first right derived functor of $\mathrm{Hom}_R(A, B)$, for any two right R-modules A, B (see [5, Ch. III] for more details). We characterize rings over which all modules are JS-injective. We prove the equivalence of the following statements: (1) $\mathrm{JS}(R_R) = 0$; (2) All modules are JS-injective; (3) All submodules of $\mathrm{JS}(R_R)$ are JS-injective; (4) All submodules of $\mathrm{JS}(R_R)$ are direct summand of R_R. We study quotients of JS-injective modules. For instance, we prove that the equivalence of the following: (1) The class of JS-injective right R-modules (JSI_R) is closed under quotient; (2) Sums of any two JS-injective submodules of any right R-module is JS-injective; (3) All submodules of $\mathrm{JS}(R_R)$ are projective. Finally, we give conditions such that the class JSI_R is closed under direct sums. For instance, we prove that $\mathrm{JS}(R_R)$ is Noetherian if and only if any direct sum of JS-injective right R-modules is JS-injective.

2. JS-Injective Modules

In this section, we introduce and study the concept of JS-injective modules.

Definition 2.1. Let N and M be modules. A module M is said to be JS-N-injective, if any homomorphism $f: K \to M$ extends to N, where K is a submodule of $\mathrm{JS}(N)$=$\mathrm{J}(N)\mathrm{J}(R_R)$. If M is a JS-R-injective, then a module M is called a JS-injective.
The class of JS-injective right R-modules is denoted by JSI_R.

Examples 2.2.

1- Clearly, every small-injective module is a JS-injective, but the converse is not true in general, for example: let $\mathbb{Z}_2$ be the field of two elements and let $R = \mathbb{Z}_2[x_1, x_2, ...]$ with $x_i^2 = x_j^2 \neq 0$ for all i, $x_i x_j = 0$ for all $i \neq j$ and $x_i^3 = 0$ for all i. If $m = x_i^2$, then $J(R_R) = \mathrm{span}\{m, x_1, x_2, ...\}$, $(J(R_R))^2 = \mathrm{soc}(R_R) = \mathbb{Z}_2 m$ and R_R is a soc-injective module (see [10, Example 5.7]) and hence R_R is a JS-injective module. By [10, Example 5.7], the R-homomorphism $\gamma: J(R_R) \longrightarrow R_R$ which is given by $\gamma(a) = a^2$ for all $a \in J(R_R)$ can not extend to R, then R is not small injective. Hence JS-injectivity is a proper generalization of small-injectivity.

2- If N is a right R-module with $\mathrm{JS}(N) = 0$ (in particular, if $\mathrm{J}(N) = 0$ or $\mathrm{J}(R_R) = 0$), then every module is JS-N-injective.

3- All $\mathbb{Z}$-modules are JS-N-injective, for any module N, and hence all $\mathbb{Z}$-modules are JS-injective.

Proposition 2.3. Let N, M and K be right R-modules. Then the following statements hold:

(1) The class of JS-N-injective modules is closed under isomorphic copies, direct products, direct summands and finite direct sums.

(2) For any submodule K of N, if M is JS-N-injective module, then M is JS-K-injective.

(3) If M is JS-N-injective module, then M is JS-K-injective, for any module K isomorphic to N.

Proof. Clear. □

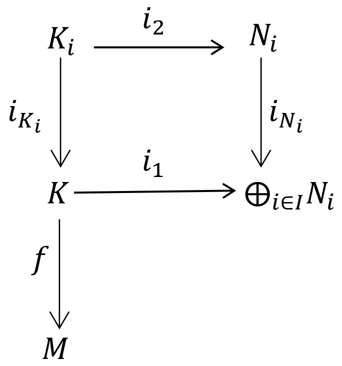
From Proposition 2.3, we get directly the following result.

Corollary 2.4. The class JSI_R is closed under isomorphic copies, direct products, summands and finite direct sums.

Proposition 2.5. Let $\{N_i : i \in I\}$ be a family of modules. If JS$(\oplus_{i\in I} N_i)$ is a multiplication module, then a module M is JS-$\oplus_{i\in I} N_i$-injective if and only if it is a JS-N_i-injective, for each $i \in I$.

Proof. ($\Rightarrow$) By Theorem 2.3 ((2) and (3)).

($\Leftarrow$) Suppose that M is JS-N_i-injective, for each $i \in I$. Let K be a submodule of JS$(\oplus_{i\in I} N_i)$. By [3, Corollaries 9.1.5(c), p.215] J$(\oplus_{i\in I} N_i)$= $\oplus_{i\in I}$J(N_i) and hence JS$(\oplus_{i\in I} N_i)$ =J$(\oplus_{i\in I} N_i)$J(R_R) = $\oplus_{i\in I}$(J$(N_i)J(R_R)$ = $\oplus_{i\in I}$JS(N_i). Since JS$(\oplus_{i\in I} N_i)$ is a multiplication module (by hypothesis) and K a submodule of JS$(\oplus_{i\in I} N_i)$, we have from [13, Theorem 2.2, p. 3844] that K=$\oplus_{i\in I} K_i$ with K_i is a submodule of JS(N_i). For each $i \in I$, consider the following diagram:



where i_{K_i}, i_{N_i} are injection homomorphisms and i_1, i_2 are inclusion homomorphisms. Since M is a JS-N_i-injective, there exists a homomorphism $h_i : N_i \longrightarrow M$ such that $h_i i_2 = f i_{K_i}$. By [3, Theorem 4.1.6 (2), p.83], there exists one homomorphism $h: \oplus_{i\in I} N_i \longrightarrow M$ satisfying $h_i = h i_{N_i}$. Thus $f i_{K_i} = h_i i_2 = h i_{N_i} i_2 = h i_1 i_{K_i}$ for all $i \in I$. Let $(a_i)_{i\in I} \in \oplus_{i\in I} K_i$, thus $a_i \in K_i$, for all $i \in I$ and $f((a_i)_{i\in I}) = f(\sum_{i\in I} i_{K_i}((a_i)_{i\in I})) = (h i_1)((a_i)_{i\in I})$. Thus $f = h i_1$ and this implies that M is a JS-$\oplus_{i\in I} N_i$-injective module. □

If each right ideals of a ring R is an ideal, then R is called right invariant [13, p.3839].

Corollary 2.6. Let R be a right invariant ring with JS(R_R) be a cyclic ideal in R and let $1 = r_1 + r_2 + \cdots + r_n$ in R, where the r_i are orthogonal idempotents. Then a right R-module M is JS-injective if and only if M is JS-$r_i R$-injective for every $i = 1,2, \cdots ,n$.

Proof. By [1, Corollary 7.3, p.96], we have $R = \oplus_{i=1}^{n} r_i R$. Since R is a right invariant ring and JS$(\mathrm{R_R})$ is a cyclic ideal in R, we get from [13, Proposition 3.1, p. 3855], that JS(R_R) is a right multiplication module and hence from Proposition 2.5 implies that M is a JS-injective if and only if M is a JS-$r_i R$-injective. □

In the next result, we give some characterizations of JS-injective modules.

Proposition 2.7. The following statements are equivalent for a right R-module M:

(1) M is JS-injective;

(2) $\mathrm{Ext}^1(R/K, M) = 0$, for any submodule of $\mathrm{JS}(R_R)$;

(3) For each submodule K of $\mathrm{JS}(R_R)$ and for each R-homomorphism $f: K \longrightarrow M$, there is $m \in M$ with $f(r) = mr$ for any $r \in K$.

Proof. **(1)⇒(2)** Since

$$0 \longrightarrow K \xrightarrow{i} R \xrightarrow{\pi} R/K \longrightarrow 0$$

is an exact sequence, where i and π are the inclusion and canonical homomorphisms, respectively, it follows from [14, Theorem 4.4(3), p.491] that there exists an exact sequence

$$0 \longrightarrow \mathrm{Hom}_R(R/K, M) \xrightarrow{\pi^*} \mathrm{Hom}_R(R, M) \xrightarrow{i^*} \mathrm{Hom}_R(K, M)$$

$$\longrightarrow \mathrm{Ext}^1(R/K, M) \longrightarrow \mathrm{Ext}^1(R, M) \longrightarrow \mathrm{Ext}^1(K, M) \longrightarrow \ldots\ldots\ldots .$$

Since R_R is projective, it follows from [14, Theorem 4.4(1), p. 491] that $\mathrm{Ext}^1(R, M) = 0$ and hence the sequence $0 \longrightarrow \mathrm{Hom}_R(R/K, M) \xrightarrow{\pi^*} \mathrm{Hom}_R(R, M) \xrightarrow{i^*} \mathrm{Hom}_R(K, M) \longrightarrow \mathrm{Ext}^1(R/K, M) \rightarrow 0$ is exact. By hypothesis, the sequence $0 \rightarrow \mathrm{Hom}_R(R/K, M) \xrightarrow{\pi^*} \mathrm{Hom}_R(R, M) \xrightarrow{i^*} \mathrm{Hom}_R(K, M) \rightarrow 0$ is exact and hence $\mathrm{Ext}^1(R/K, M) = 0$.

(2)⇒(1) Let K be a submodule of $\mathrm{JS}(R_R)$. As the proof of (1) ⇒(2) we have that the sequence

$$0 \longrightarrow \mathrm{Hom}_R(R/K, M) \xrightarrow{\pi^*} \mathrm{Hom}_R(R, M) \xrightarrow{i^*} \mathrm{Hom}_R(K, M) \longrightarrow \mathrm{Ext}^1(R/K, M) \longrightarrow 0$$

is exact. By hypothesis, $\mathrm{Ext}^1(R/K, M) = 0$ and hence the sequence

$$0 \longrightarrow \mathrm{Hom}_R(R/K, M) \xrightarrow{\pi^*} \mathrm{Hom}_R(R, M) \xrightarrow{i^*} \mathrm{Hom}_R(K, M) \longrightarrow 0$$ is exact.

(1)⇔(3) It is clear. □

Proposition 2.8. For a module M, the following conditions are equivalent:

(1) Every module is JS-M-injective;

(2) All submodules of $\mathrm{JS}(N)$ are JS-M-injective, where N is any right R-module;

(3) All submodules of $\mathrm{JS}(M)$ are JS-M-injective;

(4) All submodules of $\mathrm{JS}(M)$ are summand of M;

(5) $\mathrm{JS}(M) = 0$.

Proof. **(1)⇒(2)⇒(3)** and **(5)⇒(1)** are clear.

(3)⇒(4) Let K be a submodule of $\mathrm{JS}(M)$. Let $i: K \longrightarrow M$ and $I_K: K \longrightarrow K$ be the inclusion and the identity homomorphisms, respectively. By hypothesis, K is JS-M-injective and so there is a homomorphism $g: M \longrightarrow K$ with $gi = I_K$. Then a monomorphism i is split and this implies that K is a summand of M.

(4)⇒ (5) Let $x \in \mathrm{JS}(M)$, thus $x \in \mathrm{J}(M)\mathrm{J}(R_R) \subseteq \mathrm{J}(M)$. By [3, Corollary 9.1.3(a), p.214], xR is a small submodule of M. By hypothesis, xR is a summand of M and hence $M = xR \oplus K$ for some submodule K of M. Since xR is a small submodule of M, $K = M$ and hence $xR = 0$ So, $x = 0$ and hence $\mathrm{JS}(M) = 0$. □

Corollary 2.9. The following statements are equivalent for a ring R:

(1) Every right R-module is JS-injective;

(2) Every submodule of $\mathrm{JS}(N)$ is JS-injective, where N is any right R-module;

(3) Every submodule of $\mathrm{JS}(R_R)$ is JS-injective;
(4) Every submodule of $\mathrm{JS}(R_R)$ is a direct summand of R_R;
(5) $\mathrm{JS}(R_R) = 0$.
Proof. By taking $M = R_R$ and applying Proposition 2.8. □

Proposition 2.10. Let M be a right R-module. Then $\mathrm{JS}(M)$ is a semisimple direct summand of M if and only if all modules are JS-M-injective.
Proof. (⇒) Let $\mathrm{JS}(M)$ be a semisimple direct summand of M. Let K be submodule of $\mathrm{JS}(M)$. By hypothesis, $M = \mathrm{JS}(M) \oplus W$ for some submodule W of M. Since $\mathrm{JS}(M)$ is semisimple, $\mathrm{JS}(M) = K \oplus H$, for some submodule H. We obtain $M = K \oplus H \oplus W$ and hence every submodule of $\mathrm{JS}(M)$ is a summand of M. Thus Proposition 2.8 implies that all modules are JS-M-injective.
(⇐) Suppose that every right R-module is JS-M-injective. By Proposition 2.8, $\mathrm{JS}(M) = 0$ and hence $\mathrm{JS}(M)$ is a semisimple summand of M. □

Definition 2.11. [15] A ring R is called zero insertive if for any $a, b \in R$ such that $ab = 0$, then $aRb = 0$.

Lemma 2.12. [15, Lemma 2.11] Let R be a zero insertive ring, then $RaR + r(a) \subseteq^{ess} R_R$, for every $a \in R$.
Proposition 2.13. If all simple singular right modules over a zero insertive ring R are JS-injective, then $\mathrm{JS}(R_R) = 0$.
Proof. Assume that $\mathrm{JS}(R_R) \neq 0$. Thus there is $0 \neq a \in \mathrm{JS}(R_R)$, and hence RaR is a small right ideal in R. If $RaR + r(a) \subsetneq R$, then $RaR + r(a) \subseteq K$ for some maximal right ideal K of R. By Lemma 2.12, we have $RaR + r(a)$ is an essential in R_R and hence K is an essential in R_R and so R/K is a simple singular right R-module (by [4, Example 7.6(3) p. 247]). By hypothesis, R/K is a JS-injective module. Consider the mapping $f: aR \longrightarrow R/K$ defined by $f(ar) = r + K$ for all $r \in R$. Thus f is a well-defined right R-homomorphism. Since aR is a right ideal of R with $aR \subseteq \mathrm{JS}(R_R)$, it follows from JS-injectivity of R/K, there is a homomorphism $g: R \longrightarrow R/K$ with $g(x) = f(x)$ for all $x \in aR$. Thus $1 + K = f(a) = g(a) = g(1)a = (c + K)a = ca + K$, for some $c \in R$ and hence $1 - ca \in K$. Since $ca \in RaR \subseteq K$, it follows that $1 \in K$ and hence $K = R$ and this is a contradiction. Thus, $RaR + r(a) = R$. Since RaR is a small ideal in R_R which implies that $r(a) = R$ and so $a = 0$ and this is a contradiction. Thus $\mathrm{JS}(R_R) = 0$. □

Corollary 2.14. If all simple singular right modules over a zero insertive ring R (in particular, over a commutative ring R) are JS-injective, then Mod-$R = JSI_R$.
Proof. By Proposition 2.13 and Corollary 2.9. □

Theorem 2.15. The following conditions are equivalent for a ring R:
(1) $\mathrm{JS}(R_R) = 0$;
(2) Every right R-module is JS-injective;
(3) All simple modules are JS-injective.
Proof. Clearly, from Corollary 2.9, we have (1)⟹(2)⟹(3).
(3)⟹(1). Assume that $\mathrm{JS}(R_R) \neq 0$. Thus there is $0 \neq a \in \mathrm{JS}(R_R)$, and hence aR is a small right ideal in R. If $\mathrm{JS}(R_R) + r(a) \subsetneq R$, then $\mathrm{JS}(R_R) + r(a) \subseteq K$ for some maximal right ideal K of R, by [3, Theorem 2.3.11, p. 28]. Since R/K is a simple module, it follows from hypothesis that R/K is a JS-injective. Consider the mapping $f: aR \longrightarrow R/K$ defined by $f(ar) = r + K$ for all $r \in R$. Thus f is a well-defined right R-homomorphism. Since aR is a

right ideal of R with $aR \subseteq \mathrm{JS}(R_R)$ it follows from JS-injectivity of R/K, there is a right R-homomorphism $g: R \longrightarrow R/K$ such that $g(x) = f(x)$ for all $x \in aR$. Thus $1 + K = f(a) = g(a) = g(1)a = (c + K)a = ca + K$, for some $c \in R$ and hence $1 - ca \in K$. Since $ca \in \mathrm{JS}(R_R) \subseteq K$, it follows that $1 \in K$ and hence $K = R$ and this is a contradiction. Therefore, $\mathrm{JS}(R_R) + r(a) = R$. Since $\mathrm{JS}(R_R)$ is a small ideal in R_R which implies that $r(a) = R$ and so $a = 0$ and this is a contradiction. Thus $\mathrm{JS}(R_R) = 0$. □

Proposition 2.16. If all simple singular right R-modules are JS-injective, then $r(a)$ is a summand of R_R and aR is projective, for every $a \in \mathrm{JS}(R_R)$.

Proof. For every $a \in \mathrm{JS}(R_R)$, let $L = RaR + r(a)$. There exists $K \leq R_R$ such that $L \oplus K \subseteq^{ess} R_R$. Assume that $L \oplus K \neq R$, then $L \oplus K \subseteq I$ for some maximal right ideal I of R and so $I \subseteq^{ess} R_R$. Therefor R/I is simple singular and by hypothesis, R/I is JS-injective. We define $f: aR \longrightarrow R/I$ by $f(ar) = r + I$ for all $r \in R$. Then f is a well-defined R-homomorphism. Since aR is a right ideal of R with $aR \subseteq \mathrm{JS}(R_R)$, it follows from JS-injectivity of R/K, there is a homomorphism $g: R \longrightarrow R/I$ with $g(x) = f(x)$ for all $x \in aR$. Thus $1 + I = f(a) = g(a) = g(1)a = (c + I)a = ca + I$, for some $c \in R$ and hence $1 - ca \in I$. Since $ca \in RaR \subseteq I$ it follows that $1 \in I$ and hence $I = R$ and this is a contradiction. Thus $L \oplus K = R$ or $RaR + (r(a) \oplus K) = R$ which implies that $r(a) \oplus K = R$ (since $RaR \ll R_R$). Now, we will prove that aR is a projective module. Since $r(a)$ is a summand of R_R, it follows that there exists an idempotent element, say e in R with $r(a) = (1 - e)R$ (by [6, 2.3(3), p.8]) with $R = eR \oplus (1 - e)R$. Define $\lambda: eR \longrightarrow aeR$ by $\lambda(er) = aer$, for all $r \in R$. It is clear that λ is an epimorphism. Let $x \in \ker(\lambda)$, thus $\lambda(x) = 0$ and so $x = er$ for some $r \in R$ and $aer = 0$. Hence $er \in r(a)$ and $er \in eR$, and this implies that $x \in eR \cap r(a)$ and so $\ker(\lambda) \subseteq eR \cap r(a)$. Let $y \in R \cap r(a)$, thus $y = er$ and $ay = 0$. So $aer = 0$ and hence $\lambda(y) = 0$. Thus $y \in \ker(\lambda)$ and so $eR \cap r(a) \subseteq \ker(\lambda)$. Thus $\ker(\lambda) = eR \cap r(a)$. Since $R = eR \oplus (1 - e)R$, we have $eR \cap (1 - e)R = 0$. Since $r(a) = (1 - e)R$, we have $eR \cap r(a) = 0$. Since $\ker(\lambda) = eR \cap r(a)$, we have $\ker(\lambda) = 0$. Thus $\lambda: eR \longrightarrow aeR$ is an isomorphism. Clearly $aR = aeR$, since $aeR \subseteq aR$ and if $x \in aR$, then $x = ar$ for some $r \in R$. So, $x = ar = aer + a(1 - e)r$. Since $r(a) = (1 - e)R$, we have $a(1 - e)r = 0$ and so $x = aer \in aeR$. Thus $aR \subseteq aeR$ and hence $aR = aeR$. Since $R = eR \oplus (1 - e)R$, we have eR is projective. Since $eR \cong aeR$, we have aeR is projective. Since $aR = aeR$,we have aR is a projective module. □

Corollary 2.17. If all simple singular right R-modules are JS-injective, then $Z_r \cap \mathrm{JS}(R_R) = 0$.

Proof. Assume that $Z_r \cap \mathrm{JS}(R_R) \neq 0$, then there exists $0 \neq a \in Z_r \cap \mathrm{JS}(R_R)$. Since $a \in Z_r$, we have $r(a) \subseteq^{ess} R_R$. By Proposition 2.16, $r(a) \subseteq^{\oplus} R_R$ and so $r(a) \cap K = 0$ and $r(a) + K = R$, for some $K \leq R_R$. Since $r(a) \subseteq^{ess} R_R$, which implies that $K = 0$ and so $r(a) = R$ and hence $a = 0$ but this a contradiction. Thus $Z_r \cap \mathrm{JS}(R_R) = 0$. □

If M is a projective right R-module, then it is not necessary that all submodules of $\mathrm{JS}(M)$ are projective, for example $M = \mathbb{Z}_8$ as $\mathbb{Z}_8$-module, then $\mathrm{JS}(M) = \mathrm{JS}(\mathbb{Z}_8) = <\bar{4}> = \{\bar{0}, \bar{4}\}$ is not projective $\mathbb{Z}_8$-module. Assume that $<\bar{4}>$ is a projective $\mathbb{Z}_8$-module. Since $<\bar{4}>$ is a local $\mathbb{Z}_8$-module, we get from [1, Corollary 26.7, p.300] that a finitely generated $\mathbb{Z}_8$-module $<\bar{4}>$ is a free $\mathbb{Z}_8$-module and hence $<\bar{4}> = \{\bar{0}, \bar{4}\}$ is isomorphic to $(\mathbb{Z}_8)^n$ for some positive integer n and this is a contradiction. Thus $\mathrm{JS}(\mathbb{Z}_8) = <\bar{4}>$ is not projective.

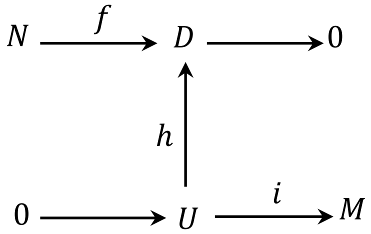
Theorem 2.18. The following statements are equivalent for a projective module M:

(1) The class of JS-M-injective modules is closed under quotient;

(2) All quotients of an injective module are JS-M-injective;
(3) The sum of any two JS-M-injective submodules of any module is a JS-M-injective;
(4) The sum of any two injective submodules of any module is a JS-M-injective;
(5) All submodules of JS(M) are projective.

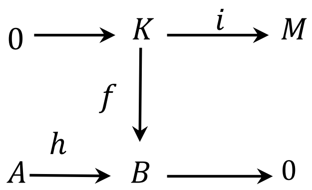
Proof. Clearly, we have **(1)⇒(2)** and **(3)⇒(4)**.

(2)⇒(5) Let D and N be modules and consider the following diagram



where U is a submodule of JS(M), f epimorphism, h is a homomorphism, and i is the inclusion homomorphism. By Proposition 5.2.10 in [2, p. 148], we can take N to be an injective R-module. By JS-M-injectivity of D, we have $\alpha i = h$ for some homomorphism $\alpha: M \longrightarrow D$. By projectivity of M, we get that α can be lifted to an R-homomorphism $\tilde{\alpha}: M \longrightarrow N$ with $f\tilde{\alpha} = \alpha$. Let $\tilde{h}: U \longrightarrow N$ be the restriction of $\tilde{\alpha}$ over U. It is clear that $f\tilde{h} = h$ and hence U is projective.

(5)⇒(1) Let A and B be modules with A is JS-M-injective and let $h: A \longrightarrow B$ be an epimorphism. Consider the following diagram:



where K is a submodule of JS(M), i is the inclusion map and $f: K \longrightarrow B$ is a homomorphism. By (5), K is projective and hence there exists a homomorphism $g: K \longrightarrow A$ with $hg = f$. Since A is JS-M-injective, there exists a homomorphism $\tilde{g}: M \longrightarrow A$ with $\tilde{g}i = g$. Put $\alpha = h\tilde{g}: M \longrightarrow B$. Thus $\alpha i = h\tilde{g}i = hg = f$ and hence B is JS-M-injective.

(1)⇒(3) Let K be a module and K_1 and K_2 be JS-M-injective submodules of it. Clearly, there is an epimorphism form $K_1 + K_2$ onto $K_1 \oplus K_2$. Since $K_1 \oplus K_2$ is JS-M-injective (by Corollary 2.4), it follows from hypothesis that $K_1 + K_2$ is JS-M-injective.

(4)⇒(2) Let N be a submodule of an injective module E. Let $Q = E \oplus E$, $K = \{(n,n) |\ n \in N\}$, $\bar{Q} = Q/K$, $H_1 = \{y + K \in \bar{Q} |\ y \in E \oplus 0\}$ and $H_2 = \{y + K \in \bar{Q} |\ y \in 0 \oplus E\}$. Then $\bar{Q} = H_1 + H_2$. Since $(E \oplus 0) \cap K = 0$ and $(0 \oplus E) \cap K = 0$, it follows that $E \cong H_i$, $i = 1, 2$. Clearly, $H_1 \cap H_2 \cong N$ under $y \mapsto y + K$ for all $y \in N \oplus 0$. By hypothesis, $\bar{Q}$ is JS-M-injective. Since H_1 is injective, it follows that $\bar{Q} = H_1 \oplus A$ for some submodule A of $\bar{Q}$ and hence $A \cong (H_1 + H_2)/H_1 \cong H_2/(H_1 \cap H_2) \cong E/N$. By Theorem 2.3 ((4),(5)), E/N is JS-M-injective. □

Corollary 2.19. For a ring R, the following conditions are equivalent:
(1) The class JSI_R is closed under quotient;
(2) All quotients of small-injective modules are JS-injective;
(3) All quotients of injective modules are JS-injective;
(4) The sum of any two JS-injective submodules of any module is a JS- injective;
(5) The sum of any two small-injective submodules of any module is a JS-injective;
(6) The sum of any two injective submodules of any module is a JS-injective.

(7) All submodules of $\mathrm{JS}(R_R)$ are projective.

Proof. The equivalence of (1),(3),(4),(6) and (7) are clear, by taking $M = R_R$ and applying Theorem 2.18. Also, (1)⟹(2)⟹(3) and (4)⟹(5)⟹(6) are clear. □

Let N be a right R-module. A right R-module M is called a rad-N-injective, if for any submodule K of $J(N)$, any right R-homomorphism $f: K \to M$ extends to N [16, p.412].

Theorem 2.20. The following conditions are equivalent for a finitely generated module M:
(1) $\mathrm{JS}(M)$ is a Noetherian R-module;
(2) The class of JS-M-injective modules is closed under a direct sums;
(3) All direct sums of rad-M-injective modules are JS-M-injective;
(4) All direct sums of small-M-injective modules are JS-M-injective;
(5) All direct sums of injective modules are JS-M-injective;
(6) $K^{(L)}$ is JS-M-injective, for any injective module K and for any index set L;
(7) $K^{(\mathbb{N})}$ is JS-M-injective, for any injective right R-module K.

Proof. (2)⇒(3)⇒ (4) ⇒(5) ⇒(6)⇒(7) are clear.

(1)⇒(2) Let $E = \oplus_{i\in I} M_i$, where M_i are JS-M-injective modules. Let K be a submodule of $\mathrm{JS}(M)$ and $f: K \longrightarrow E$ be a homomorphism. Since $\mathrm{JS}(M)$ is a Noetherian module (by hypothesis), K is finitely generated and hence $f(K) \subseteq \oplus_{j\in J} M_j$, for some finite subset J of I. Since a finite direct sum of JS-M-injective modules is a JS-M-injective (by Corollary 2.4), we have $\oplus_{j\in J} M_j$ is JS-M-injective. Define $\alpha: K \longrightarrow \oplus_{j\in J} M_j$ by $\alpha(x) = f(x)$, for every $x \in K$. It is clear that α is a right R-homomorphism. By JS-M-injectivity of $\oplus_{j\in J} M_j$, we have $g i_K = \alpha$ for some homomorphism $g: M \longrightarrow \oplus_{j\in J} M_j$, where $i_K: K \to M$ is the inclusion map. Let $\tau: \oplus_{j\in J} M_j \longrightarrow \oplus_{i\in I} M_i$ be the inclusion homomorphism. Define $h: M \longrightarrow E = \oplus_{i\in I} M_i$ by $h(x) = (\tau \circ g)(x)$ for every $x \in M$. Since τ and g are right R-homomorphisms, we have that h is a right R-homomorphism. Thus, for all $a \in K$, we have that $(h i_K)(a) = ((\tau g) i_K)(a) = (g i_K)(a) = \alpha(a) = f(a)$ and hence E is JS-M-injective.

(7)⇒(1) Let $K_1 \subseteq K_2 \ldots$ be a chain of submodules of $\mathrm{JS}(M)$. For each $i \geq 1$, let $E_i = E(M/K_i)$ and $E = \oplus_{i=1}^{\infty} E_i$. For every $i \geq 1$, we put $M_i = \prod_{j=1}^{\infty} E_j = E_i \oplus \left(\prod_{\substack{j=1 \\ i\neq j}}^{\infty} E_j \right)$, then M_i is injective. By hypothesis, $\oplus_{i=1}^{\infty} M_i = (\oplus_{i=1}^{\infty} E_i) \oplus \left(\oplus_{i=1}^{\infty} \prod_{\substack{j=1 \\ i\neq j}}^{\infty} E_j \right)$ is JS-M-injective. By using Theorem 2.3(5), we obtain that E is JS-M-injective. Define $f: H = \bigcup_{i=1}^{\infty} K_i \longrightarrow E$ by $f(x) = (x + K_i)_i$. Obviously, f is a well-defined right R-homomorphism. Since K_i are submodules of $\mathrm{JS}(M)$, so $\bigcup_{i=1}^{\infty} K_i$ is a submodule of $\mathrm{JS}(M)$. By JS-M-injectivity of E, there exists a right R-homomorphism $g: M \longrightarrow E = \oplus_{i=1}^{\infty} E_i$ such that $g \circ i_H = f$, where $i_H: H \longrightarrow M$ is the inclusion homomorphism. Since M is finitely generated, $g(M) \subseteq \oplus_{i=1}^{n} E(M/K_i)$ for some n and hence $f(H) \subseteq \oplus_{i=1}^{n} E(M/K_i)$. Let $\pi_i: \oplus_{j=1}^{\infty} E(M/K_j) \longrightarrow E(M/K_i)$ be the projection homomorphism. Thus $\pi_i f(x) = \pi_i((x + K_j)_{j\geq 1}) = x + K_i$ for all $x \in H$ and $i \geq 1$ and hence $\pi_i f(H) = H/K_i$ for all $i \geq 1$. Since $f(H) \subseteq \oplus_{i=1}^{n} E(M/K_i)$, we have that $H/K_i = \pi_i f(H) = 0$ for all $i \geq n+1$. So $H = K_i$ for all $i \geq n+1$ and hence the chain $K_1 \subseteq K_2 \subseteq \cdots$ stops at K_{n+1} and so $\mathrm{JS}(M)$ is Noetherian. □

Corollary 2.21. If N is a finitely generated module, then the following statements are equivalent:
(1) $\mathrm{JS}(N)$ is a Noetherian module;

(2) For any index set L, $M^{(L)}$ is a JS-N-injective, for each rad-N-injective module M;
(3) For any index set L, $M^{(L)}$ is a JS-N-injective, for each small-N-injective module M;
(4) For any index set L, $M^{(L)}$ is a JS-N-injective, for each JS-N-injective module M;
(5) $M^{(\mathbb{N})}$ is a JS-N-injective, for each rad-N-injective module M;
(6) $M^{(\mathbb{N})}$ is a JS-N-injective, for each small-N-injective module M;
(7) $M^{(\mathbb{N})}$ is a JS-N-injective, for each JS-N-injective module M.
Proof. By Theorem 2.20. □

Corollary 2.22. The following statements are equivalent for a ring R:
(1) $\mathrm{JS}(R_R)$ is a Noetherian module;
(2) The class JSI_R is closed under direct sums;
(3) The direct sums of a small-injective modules are JS-injective;
(4) The direct sums of an injective modules are JS-injective;
(5) For any index set L, $M^{(L)}$ is a JS-injective, for any injective module M;
(6) For any index set L, $M^{(L)}$ is a JS-injective, for any small-injective module M;
(7) For any index set L, $M^{(L)}$ is a JS-injective, for any JS-injective module M;
(8) $M^{(\mathbb{N})}$ is a JS-injective, for any injective module M;
(9) $M^{(\mathbb{N})}$ is a JS-injective, for any small-injective module M;
(10) $M^{(\mathbb{N})}$ is a JS-injective, for any JS-injective module M;
Proof. By using Theorem 2.20 and Corollary 2.21. □

3. Conclusions

A JS-injective right R-module is an introduced and studied in this paper as a generalization of small-injective right R-module. We say that a right R-module M is a JS-injective if every right R-homomorphism $\mathrm{f}: \mathrm{K} \to \mathrm{M}$ extends to R, where K is a submodule of $\mathrm{J(R_R)J(R_R)}$. We prove that the class JS-injective modules is closed under isomorphic copies, direct products, summands and finite direct sums. Some characterizations of JS-injective modules are given. We characterize rings over which all modules are JS-injective, for example we prove that $\mathrm{JS}(R_R) = 0$ if and only if all modules are JS-injective if and only if all submodules of a $\mathrm{JS}(R_R)$ are direct summand of R_R. We study quotients and direct sums of JS-injective modules. We prove that the class of a JS-injective right R-modules is closed under quotients if and only if all submodules of $\mathrm{JS}(R_R)$ are projective. Also, we prove that the class of JS-injective right R-modules is closed under direct sums if and only if $\mathrm{JS}(R_R)$ is a Noetherian module.

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