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On A Modified SP-Iterative Scheme for Approximating Fixed Point of A Contraction Mapping

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Abstract
In this paper, we will show that the Modified SP iteration can be used to approximate fixed point of contraction mappings under certain condition. Also, we show that this iteration method is faster than Mann, Ishikawa, Noor, SP, CR, Karahan iteration methods. Furthermore, by using the same condition, we shown that the Picard S- iteration method converges faster than Modified SP iteration and hence also faster than all Mann, Ishikawa, Noor, SP, CR, Karahan iteration methods. Finally, a data dependence result is proven for fixed point of contraction mappings with the help of the Modified SP iteration process.

Keywords: Modified SP-iterative, contraction mapping, data dependence result.

حول تكرارات SP المحسنة لتقارب النقاط الصامدة للمؤثرات الانكماشية

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الخلاصة
في هذا البحث سوف نبين بأن تكرارات SP المحسنة ممكن استخدامها لتقريب النقطة الصامدة للمؤثرات الانكماشية تحت شرط معين كذلك بينا بأن طريقة التكرار هذه اسرع من الطرق التكرارية الاتية مان, ايشي كاوا, نور, SP , CR , كروان , واكثر من ذلك بينا تحت نفس الشرط بينا ان طريقة بيكارد من النمط S اسرع من تكرار SP المحسن وبالتالي اسرع من التكرارات اعلاه واخيرا نتيجة البيانات المعتمدة برهنت في ظل تكرار SP المحسن للمؤثرات الانكماشية.

1. Introduction

In Agarwal et al [1] showed that by using certain condition, S-iteration method converges at a same rate as Picard iteration and faster than Mann iterative method. After, Khan [2] showed that normal S-iteration faster than all of Picard. Mann and Ishikawa Iterative methods for contractions. Recently in [3] Kadioglu and Yildirim introduce a new iterative without name, both of Kadioglu and Yildirim are used different condition to show that this iterative converges faster than S-iteration iterative and the normal S-Iteration. Çeliker in [4] give a name to this process, she called it Modified SP iterative.

In a paper of Soltuz [5] establish the data dependence result of Ishikawa method for contraction mappings. In [6] Soltuz and Grosan used contractive-like operators to establish a data dependence result of Ishikawa iterative process. Recently [7] Asaduzzaman and Zulfikar established a data dependence result of Noor iterative for contractives like operoters.

This paper consists of three sections: In section 1, we used the same condition in [3] to study rate of converge of Modified SP iteration with various iterations schemas. In section 2, we give an example

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explain the rate of convergence. Finally in section 3, we prove the data dependence result of Modified SP iterative scheme using contraction operator and certain condition.
Throughout this paper, $\mathbb{N}$ is the set of all non-negative numbers, and F_T is the set of all fixed points of an operator T.

2. Preliminaries:

In this section we will recall definitions that we need them in the rest of our paper.

Definition2.1: [8]

Let E be a normed space and C be a nonempty closed convex subset of E, and $T: C \longrightarrow C$ be a mapping, T is called contraction mapping if their exist $L \in (0,1)$ such that

$$\|Tx - Ty\| \leq L\,\|x - y\| \quad \text{For all } x, y \in C \tag{1.1}$$

Definition 2.2: [8] Let E be a normed space $T, \widetilde{T}: E \to E$ be two operators, we say that $\widetilde{T}$ is an approximate operator of T if for all $x \in E$ and for fixed $\varepsilon > 0$ we have

$$\|Tx - \widetilde{T}x\| \leq \varepsilon \tag{1.2}$$

Now we introduce some iteration methods that we will used them in our paper:

Definition 2.3:

Let $\{\alpha_n\}_0^\infty, \{\beta_n\}_0^\infty, \{\gamma_n\}_0^\infty$ be real sequences in $\in (0,1)$. The following iteration processes are referred to as Picard [9], Mann [10], Ishikawa [11], Noor [12], SP [13], S-iteration [14], normal S-iteration [15], CR [16], Picard-S iteration [17], Modified SP iteration [3, 4], Karahan iteration [18], respectively

$$\begin{cases} \breve{a}_n \in C\,, \\ \breve{a}_{n+1} = T\breve{a}_n\,,\ n \in \mathbb{N} \end{cases} \tag{1.3}$$

$$\begin{cases} \hat{t}_0 \in C\,, \\ \hat{t}_{n+1} = (1-\alpha_n)\,\hat{t}_n + \alpha_n\, T\hat{t}_n\,,\ n \in \mathbb{N} \end{cases} \tag{1.4}$$

$$\begin{cases} a_0 \in C\,, \\ a_{n+1} = (1-\alpha_n)a_n + \alpha_n\, Tb_n\,, \\ b_n = (1-\beta_n)a_n + \beta_n\, Ta_n\,,\ n \in \mathbb{N} \end{cases} \tag{1.5}$$

$$\begin{cases} w_0 \in C\,, \\ w_{n+1} = (1-\alpha_n)w_n + \alpha_n\, Tu_n\,, \\ u_n = (1-\beta_n)w_n + \beta_n\, Tv_n\,, \\ v_n = (1-\gamma_n)w_n + \gamma_n\, Tw_n\,,\ n \in \mathbb{N} \end{cases} \tag{1.6}$$

,

$$\begin{cases} \hat{a}_0 \in C, \\ \hat{a}_{n+1} = (1-\alpha_n)\hat{b}_n + \alpha_n\, T\hat{b}_n\,, \\ \hat{b}_n = (1-\beta_n)\hat{c}_n + \beta_n T\hat{c}_n\,, \\ \hat{c}_n = (1-\gamma_n)\hat{a}_n + \gamma_n\, T\hat{a}_n\,,\ \in n\ \mathbb{N} \end{cases} \tag{1.7}$$

$$\begin{cases} \tilde{a}_0 \in C, \\ \tilde{a}_{n+1} = (1-\alpha_n)\tilde{a}_n + \alpha_n\, T\tilde{b}_n\,,, \\ \tilde{b}_n = (1-\beta_n)\tilde{a}_n + \beta_n\, T\tilde{a}_n\,, n \in \mathbb{N} \end{cases} \tag{1.8}$$

$$\begin{cases} t_0 \in C, \\ t_{n+1} = T\big((1-\alpha_n)t_n + \alpha_n\, Tt_n\big)\,, n \in \mathbb{N} \end{cases} \tag{1.9}$$

$$\begin{cases} \widehat{w}_0 \in C, \\ \widehat{w}_{n+1} = (1-\alpha_n)\hat{u}_n\,, + \alpha_n T\hat{u}_n\,, \\ \hat{u}_n = (1-\beta_n)T\widehat{w}_n + \beta_n\, T\hat{v}_n\,, \\ \hat{v}_n = (1-\gamma_n)\widehat{w}_n + \gamma_n\, T\widehat{w}_0\,, n \in \mathbb{N} \end{cases} \tag{1.10}$$

$$\begin{cases} \hat{x}_n \in C, \\ \hat{x}_{n+1} = T\hat{y}_n, \\ \hat{y}_n = (1-\alpha_n)T\hat{x}_n + \alpha_n T\hat{z}_n, \\ \hat{z}_n = (1-\beta_n)\hat{x}_n + \beta_n T\hat{x}_n, n \in \mathbb{N} \end{cases} \tag{1.11}$$

$$\begin{cases} x_0 \in C, \\ x_{n+1} = Ty_n, \\ y_n = (1-\alpha_n)z_n + \alpha_n Tz_n, \\ z_n = (1-\beta_n)x_n + \beta_n Tx_n, n \in \mathbb{N} \end{cases} \tag{1.12}$$

$$\begin{cases} \widetilde{w}_0 \in C, \\ \widetilde{w}_{n+1} = (1-\alpha_n)T\widetilde{w}_n + \alpha_n T\tilde{u}_n, \\ \tilde{u}_n = (1-\beta_n)\widetilde{w}_n + \beta_n T\tilde{v}_n, \\ \tilde{v}_n = (1-\gamma_n)\widetilde{w}_n + \gamma_n T\widetilde{w}_n, n \in \mathbb{N} \end{cases} \tag{1.13}$$

If we want to compare speeds of above iteration methods we need the following definition about the rate of convergence which is due to Brind [19].

Definition 2.4 [19] Let $\{a_n\}_0^\infty, \{b_n\}_0^\infty$ be two sequences of real numbers that converge to a and b, respectively, and assume that there exists

$$l = lim_{n\to\infty} \frac{|a_n - a|}{|b_n - b|}$$

(a) If $l = 0$, then it can be said that $\{a_n\}_0^\infty$ converges to a faster than $\{b_n\}_0^\infty$ to b.

(b) If $0 < l < \infty$, then it can be said that $\{a_n\}_0^\infty$ and $\{b_n\}_0^\infty$ have the same rate of convergence.

3. Rate of convergence

In this section, we will show that Modified SP-iteration process is faster than all of Mann (1.4), Ishikawa (1.5), Noor (1.6), Karahan (1.13), SP-iteration (1.7), CR (1.10) processes. Also we show that Picard- S iteration (1.11) converges faster than Modified SP- iteration and thus Picard S -iteration converges faster than all above process.

The following theorem shows that the three-step iterative Method Modified SP-iteration (1.12) faster than two-step iterative method Mann (1.4) and three-step iterative method Ishikawa (1.5).

Theorem 3.1 Let C be a nonempty closed convex subset of normed space E, and T be a contraction of C into itself. Suppose that each of iterative processes of Mann (1.4) iterative and Ishikawa (1.5) iterative and Modified SP-iteration (1.12) converge to the same fixed point p of T where $\{\alpha_n,\}, \{\beta_n\}, \{\gamma_n\}$ be sequences such that for some λ, $0 < \lambda \le \gamma_n, \alpha_n, \beta_n < 1$ for all $n \in \mathbb{N}$ then the Modified SP-iteration (1.12) converges faster than Mann(1.4) and Ishikawa(1.5) iterative.

Proof. For Modified SP-iteration (1.12), we obtain

$$\begin{aligned} \|x_{n+1} - p\| &= \|Ty_n - p\| \\ &\le L\|y_n - p\| \\ &= L\|(1-\alpha_n)z_n + \alpha_n Tz_n - p\| \\ &= L\|(1-\alpha_n)(z_n - p) + \alpha_n(Tz_n - p)\| \\ &\le L[(1-\alpha_n)\|z_n - p\| + L\alpha_n\|z_n - p\|] \\ &= L[1-\alpha_n + L\alpha_n]\|z_n - p\| \\ &= L[1-\alpha_n(1-L)]\|(1-\beta_n)x_n + \beta_n Tx_n - p\| \\ &\le L\big[(1-\alpha_n(1-L))\big((1-\beta_n)\|(x_n - p)\| + L\beta_n\|x_n - p\|\big)\big] \\ &= L\big[(1-\alpha_n(1-L))\big(1-\beta_n(1-L)\big)\big]\|x_n - p\| \\ &\le L\big(1-\lambda(1-L)\big)^2\|x_n - p\| \\ &\vdots \\ &\le \Big[L\big(1-\lambda(1-L)\big)^2\Big]^n\|x_0 - p\| \end{aligned} \tag{1.14}$$

$$Let\ M_n = \Big[L\big(1-\lambda(1-L)\big)^2\Big]^n\|x_0 - p\| \tag{1.15}$$

From Mann iteration (1.4), and by induction, we obtain that

$$\begin{aligned} &\|\hat{t}_{n+1} - p\| = \|(1-\alpha_n)\hat{t}_n + \alpha_n T\hat{t}_n - p\| \\ &= \|(1-\alpha_n)(\hat{t}_n - p) + \alpha_n(T\hat{t}_n - p)\| \\ &\le (1-\alpha_n)\|\hat{t}_n - p\| + \alpha_n\|T\hat{t}_n - p\| \end{aligned}$$

$\leq (1-\alpha_n)\,\|\hat{t}_n - p\| + L\,\alpha_n\,\|\hat{t}_n - p\|$
$\leq [1-\lambda(1-L)]\|\hat{t}_n - p\|$
$\vdots$
$\leq [1-\lambda(1-L)]^{\,n}\ \|\ \hat{t}_0 - p\|$
$Let\ MA_n = [1-\lambda(1-L)]^{\,n}\ \|\ \hat{t}_0 - p\|$
From Ishikawa (1.5), process, we obtain that
$\|a_{n+1} - p\| = \|(1-\alpha_n)\,a_n + \alpha_n\,Tb_n - p\|$
$= \|(1-\alpha_n)\,(a_n - p) + \alpha_n\,(Tb_n - p)\|$
$\leq (1-\alpha_n)\,\|a_n - p\| + \alpha_n\,\|Tb_n - p\|$
$\leq (1-\alpha_n)\,\|a_n - p\| + \alpha_n L\,\|b_n - p\|$
$= (1-\alpha_n)\,\|a_n - p\| + L\alpha_n\|(1-\beta_n)\,(a_n - p) + \beta_n(Ta_n - p)\,\|$
$\leq (1-\alpha_n)\,\|a_n - p\| + L\alpha_n(1-\beta_n)\|a_n - p\| + L^2\alpha_n\,\beta_n\,\|a_n - p\|$
$\leq (1-\alpha_n)\,\|a_n - p\| + L\alpha_n(1-\beta_n)\|a_n - p\| + L^2\alpha_n\,\beta_n\,\|a_n - p\|$
$= [1-\alpha_n(1-L) - L\alpha_n\,\beta_n(1-L)]\ \|a_n - p\|$
$= [1-\lambda(1-L) - L\lambda^2(1-L)]\ \|a_n - p\|$
$\vdots$
$= [1-\lambda(1-L) - L\lambda^2(1-L)]^n\,\|a_0 - p\|$
Put $I_n = [1-\lambda(1-L) - L\lambda^2(1-L)]^{\,n}\,\|a_0 - p\|$
Now after simple compute we get
$$\frac{M_n}{I_n} = \frac{[L(1-\lambda(1-L)^2]^n}{[1-\lambda(1-L)-L\lambda^2(1-L)]^{\,n}}\,\frac{\|(x_0-p)\|}{\|a_0 - P\|} \longrightarrow 0\ \ as\ \ n\longrightarrow\infty$$
and
$$\frac{M_n}{MA_n} = \frac{[L\,(1-\lambda(1-L))^2\,]^n}{[\,1-\alpha_n(1-L)]^n}\,\frac{\|(x_0-p)\|}{\|\hat{t}_0 - P\|} \longrightarrow 0\ \ as\ \ n\longrightarrow\infty$$
Hence $\{x_n\}$ converges to p faster than $\{\hat{t}_n\}$, $\{a_n\}$.

The following theorem shows that the three-step iterative methods Modified SP iterative (1.12) faster than Karahan (1.13), Noor (1.6) .

Theorem 3.2 Let C be a nonempty closed convex subset of normed space *E*, and T be a contraction of C into itself. Suppose that each of iterative processes Karahan iterative (1.13) and Noor iterative (1.6)and Modified SP-iteration (1.12) converge to the same fixed point p of T where $\{\alpha_n\}$,$\{\beta_n\}$,$\{\gamma_n\}$ be sequencese such that for some λ, $0 < \lambda \leq \gamma_n, \alpha_n, \beta_n < 1$ for all $n \in \mathbb{N}$ then the Modified SP-iteration (1.12) converge faster than Karhan (1.13) and Noor iterative(1.6).

Proof:

From Karahan iterative (1.13), we obtain that
$\|\tilde{w}_{n+1} - p\| = \|(1-\alpha_n)\,T\tilde{w}_n + \alpha_n\,T\tilde{u}_n - p\|$
$\leq L(1-\alpha_n)\,\|\tilde{w}_n - p\| + L\alpha_n\,\|\tilde{u}_n - p\|$
$= L[(1-\alpha_n)\,\|\tilde{w}_n - p\| + \alpha_n\,\|(1-\beta_n)\,T\tilde{w}_n + \beta_n\,T\tilde{v}_n - p\|\,]$
$\leq L[(1-\alpha_n)\,\|\tilde{w}_n - p\| + \alpha_n\ (L\,(1-\beta_n)\,\|\tilde{w}_n - p\| + L\,\beta_n\|\tilde{v}_n - p\|\,)\,]$
$= L[(1-\alpha_n)\,\|\tilde{w}_n - p\| + \alpha_n\ (L(1-\beta_n)\,\|\tilde{w}_n - p\| + L\,\beta_n\|(1-\gamma_n)\tilde{w}_n + \gamma_n\,T\tilde{w}_n - p\|\,)\ ]$
$\leq L[(1-\alpha_n)\,\|\tilde{w}_n - p\| + \alpha_n\ (L(1-\beta_n)\,\|\tilde{w}_n - p\| + L\,\beta_n(1-\gamma_n)\|\tilde{w}_n - p\| +$
$L\,\beta_n\gamma_n\,\|T\tilde{w}_n - p\|\,)]$
$\leq L[(1-\alpha_n)\,\|\tilde{w}_n - p\| + \alpha_n\,(L(1-\beta_n)\,\|\tilde{w}_n - p\| + L\,\beta_n(1-\gamma_n)\|\tilde{w}_n - p\|$
$+ L^2\,\beta_n\gamma_n\,\|\tilde{w}_n - p\|\,)]$
$\leq L[1-\alpha_n(1-L) - L\,\beta_n\gamma_n\alpha_n(1-L)]\,\|\tilde{w}_n - p\|$
$\leq L[1-\lambda(1-L) - L\lambda^3(1-L)]\,\|\tilde{w}_n - p\|$
$\vdots$
$\leq \left[L\big(1-\lambda(1-L) - L\lambda^3(1-L)\big)\right]^n\,\|\tilde{w}_1 - p\|$
Let $K_n = \left[L\big(1-\lambda(1-L) - L\lambda^3(1-L)\big)\right]^n\,\|\tilde{w}_1 - p\|$

From Noor iterative (1.6), we obtain that
$\|w_{n+1} - p\| = \|(1-\alpha_n)\,w_n + \alpha_n\,Tu_n - p\|$
$\leq (1-\alpha_n)\|w_n - p\| + L\alpha_n\|u_n - p\|$
$\leq (1-\alpha_n)\|w_n - p\| + L\alpha_n\|(1-\beta_n)\,w_n + \beta_n\,Tv_n - p\|$
$\leq (1-\alpha_n)\|w_n - p\| + L\alpha_n\,\big((1-\beta_n)\,\|w_n - p\| + L\,\beta_n\,\|v_n - p\|\big)$

$\leq (1-\alpha_n)\|w_n - p\| + L\alpha_n(1-\beta_n)\|w_n - p\| + L^2\alpha_n\beta_n\|v_n - p\|$
$\leq (1-\alpha_n)\|w_n - p\| + L\alpha_n(1-\beta_n)\|w_n - p\| + L^2\alpha_n\beta_n\|(1-\gamma_n)w_n + \gamma_n Tw_n - p\|$
$= (1-\alpha_n)\|w_n - p\| + L\alpha_n(1-\beta_n)\|w_n - p\| + L^2\alpha_n\beta_n\|(1-\gamma_n)(w_n - p) + \gamma_n(Tw_n - p)\|$
$\leq (1-\alpha_n)\|w_n - p\| + L\alpha_n(1-\beta_n)\|w_n - p\| + L^2\alpha_n\beta_n(1-\gamma_n)\|w_n - p\|$
$+ L^3\alpha_n\beta_n\gamma_n\|w_n - p\|$
$\leq [1-\alpha_n + L\alpha_n(1-\beta_n) + L^2\alpha_n\beta_n(1-\gamma_n) + L^3\alpha_n\beta_n\gamma_n]\|w_n - p\|$
$\leq [1-\lambda(1-L) - L\lambda^2(1-L) - L^2\lambda^3(1-L)]\|w_n - p\|$
$\vdots$
$\leq [1-\lambda(1-L) - L\lambda^2(1-L) - L^2\lambda^3(1-L)]^n\|w_0 - p\|$

Let $N_n = [1-\lambda(1-L) - L\lambda^2(1-L) - L^2\lambda^3(1-L)]^n\|w_0 - p\|$

by (1.15) we have $M_n = \left[L(1-\lambda(1-L))^2\right]^n\|x_0 - p\|$

Now note that:

$$\frac{M_n}{N_n} = \frac{[L(1-\lambda(1-L))^2]^n}{[1-\lambda(1-L) - L\lambda^2(1-L) - L^2\lambda^3(1-L)]^n}\frac{\|x_0 - p\|}{\|w_0 - p\|} \longrightarrow 0 \text{ as } n \longrightarrow \infty$$

and

$$\frac{M_n}{K_n} = \frac{[L(1-\lambda(1-L))^2]^n}{[L(1-\lambda(1-L) - L\lambda^3(1-L))]^n}\frac{\|x_0 - p\|}{\|\tilde{w}_0 - P\|} \longrightarrow 0 \text{ as } n \longrightarrow \infty$$

Hence $\{x_n\}$ converges to p faster than $\{w_n\}, \{\tilde{w}_n\}$.

■

The following theorems show that the following three-step iterative methods Modified SP-iteration (1.12) faster than SP (1.7), CR (1.10).

Theorem 3.3 Let C be a nonempty closed convex subset of normed space E, and T be a contraction of C into itself. Suppose that each of iterative processes CR iterative(1.10), SP- iteration (1.7) and Modified SP-iteration (1.12) converges to the same fixed point P of T where $\{\alpha_n\},\{\beta_n\},\{\gamma_n\}$ be sequences such that for some λ, $0< \lambda \leq \alpha_n, \beta_n, \gamma_n <1$ for all $n \in \mathbb{N}$ then the Modified SP-iteration(1.12) converges faster than CR (1.10)and SP-iteration (1.7).

Proof:

From CR (1.10) iterative, we obtain that

$\|\hat{w}_{n+1} - p\| = \|(1-\alpha_n)\hat{u}_n + \alpha_n T\hat{u}_n - p\|$
$= (1-\alpha_n(1-L))\|(1-\beta_n)T\hat{w}_n + \beta_n T\hat{v}_n - p\|$
$\leq (1-\alpha_n(1-L))[L(1-\beta_n)\|\hat{w}_n - p\| + \beta_n\|T\hat{v}_n - p\|]$
$= (1-\alpha_n(1-L))[L(1-\beta_n)\|\hat{w}_n - p\| + L\beta_n\|(1-\gamma_n)\hat{w}_n + \gamma_n T\hat{w}_n - p\|]$
$\leq (1-\alpha_n(1-L))[L(1-\beta_n)\|\hat{w}_n - p\|$
$+ L\beta_n((1-\gamma_n)\|\hat{w}_n - p\| + L\gamma_n\|\hat{w}_n - p\|)]$
$= (1-\alpha_n(1-L))(L - L\beta_n\gamma_n(1-L))\|\hat{w}_n - p\|$
$\leq L(1-\lambda(1-L))(1-\lambda^2(1-L))\|\hat{w}_n - p\|$
$\vdots$
$\leq [L(1-\lambda(1-L))(1-\lambda^2(1-L))]^n\|\hat{w}_0 - p\|$

Let $CR_n = [L(1-\lambda(1-L))(1-\lambda^2(1-L))]^n\|\hat{w}_0 - p\|$

From SP-iteration (1.7), we obtain that

$\|\hat{a}_{n+1} - p\| = \|(1-\alpha_n)\hat{a}_n + \alpha_n T\hat{a}_n - p\|$
$= \|(1-\alpha_n)(\hat{a}_n - p) + \alpha_n(T\hat{a}_n - p)\|$
$\leq (1-\alpha_n(1-L))\|\hat{a}_n - p\|$
$= (1-\alpha_n(1-L))\|(1-\beta_n)(\hat{b}_n - p) + \beta_n(T\hat{b}_n - p)\|$
$\leq (1-\alpha_n(1-L))(1-\beta_n(1-L))\|\hat{b}_n - p\|$
$= (1-\alpha_n(1-L))(1-\beta_n(1-L))\|(1-\gamma_n)\hat{a}_n + \gamma_n T\hat{a}_n - p\|$
$\leq (1-\alpha_n(1-L))(1-\beta_n(1-L))(1-\gamma_n(1-L))\|\hat{a}_n - p\|$
$\leq (1-\lambda(1-L))^3\|\hat{a}_n - p\|$
$\vdots$
$\leq \left[(1-\lambda(1-L))^3\right]^n\|\hat{a}_0 - p\|$

Let $SP_n = \left[(1-\lambda(1-L))^3\right]^n\|\hat{a}_0 - p\|$

by (1.15) we have $M_n = \left[L\left(1-\lambda(1-L)\right)^2\right]^n \|x_0-p\|$

Finally we get

$$\frac{M_n}{SP_n} = \frac{\left[\left(1-\lambda(1-L)\right)^2\right]^n}{\left[\left(1-\lambda(1-L)\right)^3\right]^n} \frac{\| x_0-p \|}{\|a\hat{}_0 - P\|} \longrightarrow 0 \text{ as } n \longrightarrow \infty.$$

and

$$\frac{M_n}{CR_n} = \frac{[L(1-\lambda(1-L))^2]^n}{[L(1-\lambda(1-L)(1-\lambda^2(1-L)]^n} \frac{\| x_0-p \|}{\|w\hat{}_0 - P\|} \longrightarrow 0 \text{ as } n \longrightarrow \infty.$$

Hence $\{x_n\}$ converges faster than $\{w\hat{}_n\},\{a\hat{}_n\}$ to p. ■

The following theorem explain that the three-step iterative methods Picard S-iteration (1.11) faster than Modified SP-iteration (1.12).

Theorem 3.4 Let C be a nonempty closed convex subset of normed space *E*, and *T* be a contraction of C into itself. Suppose that each of iterative processes Picard S-iteration (1.11) and Modified SP-iteration (1.12) converges to the same fixed point p of T where $\{\alpha_n\},\{\beta_n\},\{\gamma_n\}$ be sequences such that for some λ, $0<\lambda\le\alpha_n,\beta_n<1$ for all $n\in\mathbb{N}$,Then the Picard S-iteration (1.11) converges faster than modified SP-iteration (1.12).

Proof.

For Picard S-iteration (1.11) we obtain that

$$\begin{aligned}
\|x\hat{}_{n+1}-p\| &= \|Ty\hat{}_n-p\| \\
&\le L\|y\hat{}_n-p\| \\
&\le L\|(1-\alpha_n)(Tx\hat{}_n-p)+\alpha_n(Tz\hat{}_n-p)\| \\
&\le L\left[(1-\alpha_n)\left\|Tx\hat{}_n-p\right\|+\alpha_n\left\|Tz\hat{}_n-p\right\|\right] \\
&\le L^2[(1-\alpha_n)\|x\hat{}_n-p\|+\alpha_n\|z\hat{}_n-p\|] \\
&\le L^2[(1-\alpha_n)\|x\hat{}_n-p\|+\alpha_n\|(1-\beta_n)x\hat{}_n+\beta_n Tx\hat{}_n-p\|] \\
&\le L^2[(1-\alpha_n)\|x\hat{}_n-p\|+\alpha_n(1-\beta_n)\|x\hat{}_n-p\|+L\,\alpha_n\beta_n\|x\hat{}_n-p\|] \\
&= L^2[(1-\alpha_n)+\alpha_n(1-\beta_n)+L\,\alpha_n\beta_n]\|x\hat{}_n-p\| \\
&\le L^2[1-\lambda^2(1-L)]\|x\hat{}_n-p\| \\
&\vdots \\
&\le [L^2(1-\lambda^2(1-L))]^n\|x\hat{}_0-p\|
\end{aligned}$$

Let $p_n = [L^2(1-\lambda^2(1-L))]^n\|x\hat{}_0-p\|$

For Modified SP-iteration and by (1.15) we have $M_n = \left[L\left(1-\lambda(1-L)\right)^2\right]^n \|x_0-p\|$

Now

$$\frac{p_n}{M_n} = \frac{[L^2(1-\lambda^2(1-L))]^n}{[L(1-\lambda(1-L))^2]^n} \frac{\| x\hat{}_0-p \|}{\|x_0 - p\|} \longrightarrow 0 \text{ as } n \longrightarrow \infty$$

Thus $\{x\hat{}\}$ converges to p faster than $\{x_n\}$.

■

4. Application

We use the following example to backup our above analytical proof. This example shows that All the iterative methods converge to the same fixed point 2 . Also Modified Sp iteration converges faster than all Mann, Ishikawa, SP, Noor, CR, Karahan and Picard-S iteration method converge faster than Modified Sp iteration, so Picard-S iteration still faster iterative.

Example 4.1. Let E = ℝ and $C=[0,\infty)$. Let $T: C \to C$ be a mapping defined by $Tx=\sqrt[3]{2x+4}$ for all $x\in C$. it is easily seen that the mapping T is contraction mapping with the unique fixed point $P=2$, take $\{\alpha_n\}=\frac{1}{2}$, $\{\beta_n\}=\frac{5}{7}$, $\{\gamma_n\}=\frac{3}{4}$, $\lambda=\frac{1}{2}$ with initial value 10.

Table 1- Compare the speed of above processes

No of it	Picard	Modify SP	CR	Karahan	SP	Noor	Ishikawa	Mann
1	10	10	10	10	10	10	10	10
2	2.1142	2.3518	2.4196	2.7168	3.4278	6.2746	6.2827	6.4422
3	2.0025	2.0196	2.0313	2.0881	2.2754	4.3012	4.3119	4.5038
4	2.0001	2.0011	2.0024	2.0113	2.0543	3.2455	3.2557	3.4278
5	2.0000	2.0001	2.0002	2.0015	2.0108	2.6764	2.6848	2.821
6	2.0000	2.0000	2.0000	2.0002	2.0021	2.3681	2.3744	2.4747
7	2.0000	2.0000	2.0000	2.0000	2.0004	2.2006	2.205	2.2754
8		2.0000	2.0000	2.0000	2.0001	2.1094	2.1123	2.1602
9		2.0000	2.0000	2.0000	2.0000	2.0597	2.0616	2.0933
10				2.0000	2.0000	2.0325	2.0338	2.0543
11				2.0000	2.0000	2.0178	2.0185	2.0317
12					2.0000	2.0097	2.0102	2.0185
13					2.0000	2.0053	2.0056	2.0108
14					2.0000	2.0029	2.0031	2.0063
15						2.0016	2.0017	2.0037
16						2.0009	2.0009	2.0021
17						2.0005	2.0005	2.0012
18						2.0003	2.0003	2.0007
19						2.0001	2.0002	2.0004
20						2.0001	2.0001	2.0002
21						2.0000	2.0000	2.0001
22						2.0000	2.0000	2.0001
23						2.0000	2.0000	2.0000

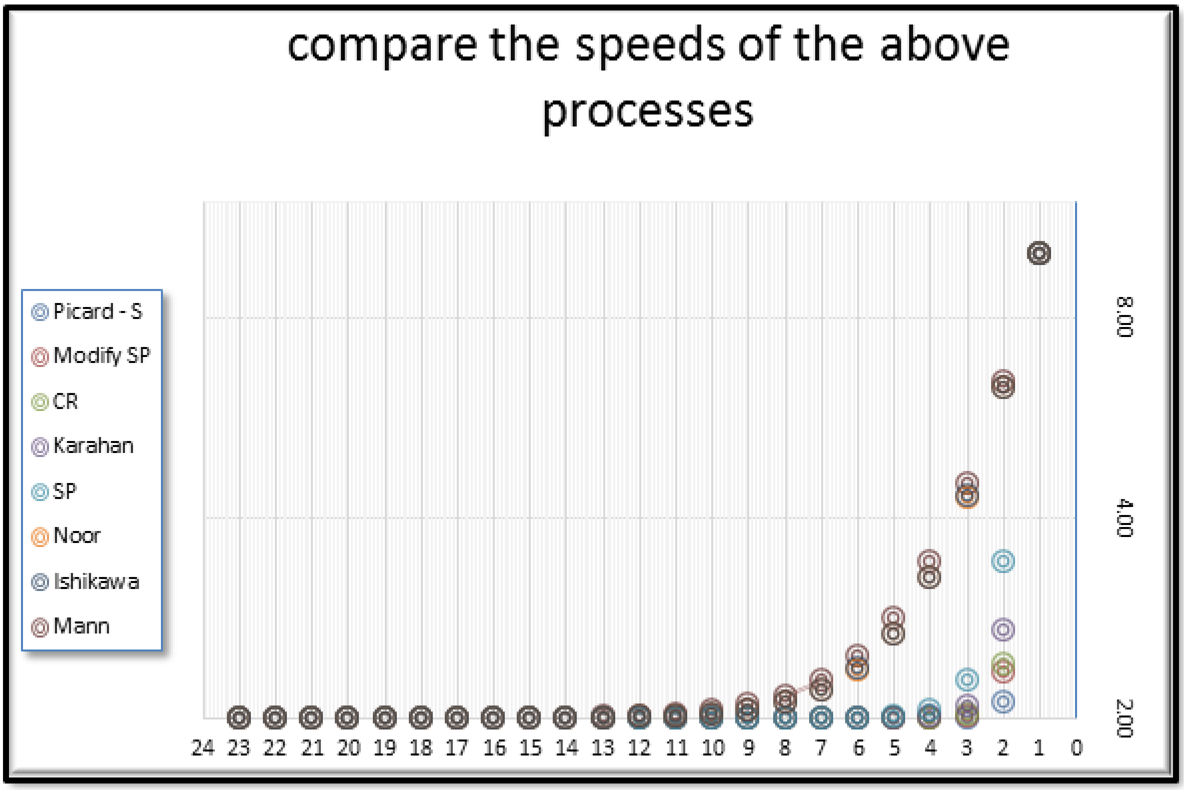

Figure 1- The above figure declares the different of speeds of the iterative methodes.

5. A Data Dependence Result

In this section we show that, if T is contraction mapping and $F_T \neq \emptyset$ and Modified SP iterative method converges to some fixed point $p \in F_T$, then we can computing p by using $\tilde{T}$ satisfying (1.2), $\tilde{p}$ the fixed point of the approximatet $\tilde{T}$.

But first we give need the following results:

Lemma5.1 Let $\{a_n\}$ nonnegative sequence suppose that there exist $n_0 \in \mathbb{N}$ such that $a_{n+1} \leq (1-\delta)\ a_n + \delta\,\sigma_n$ for all $n \geq n_0$ where $\delta \in (0,1)$ and $\sigma_n \geq 0$ for all $n \in \mathbb{N}$ then $0 \leq lim_{n\to\infty}\ sup\ a_n \leq lim_{n\to\infty}\ sup\ \sigma_n$

Proof:

It is clear that, There exist $n_1 \in \mathbb{N}$ such that $\sigma_n \leq lim_{n\to\infty}\ sup\ \sigma_n$ for all $n \geq n_1$

We claim that $a_{n+1} \leq (1-\delta)^{\ n-n_1+1}\ a_{n_1} + lim_{n\to\infty}\ sup\ \sigma_n$ for all $n > n_0$

By mathematical induction:

If $n = n_1$, then $a_{n+1} \leq (1-\delta)a_{n_1} + lim_{n\to\infty}\ sup\ \sigma_n$

Suppose that the statement true when $n = k$

i.e. $a_{k+1} \leq (1-\delta)^{\ k-n_1+1}\ a_{n_1} + lim_{n\to\infty}\ sup\ \sigma_n$

If $n = k+1$, then by our hypothesis we get

$$\begin{aligned} a_{k+2} &\leq (1-\delta)\ a_{k+1} + \delta\,\sigma_{k+1} \\ &\leq (1-\delta)\left[\ (1-\delta)^{\ k-n_1+1}\ a_{n_1} + lim_{n\to\infty}\ sup\ \sigma_n\right] + \delta\,\sigma_{k+1} \\ &\leq (1-\delta)^{\ k-n_1+2}\ a_{n_1} + (1-\delta)\lim_{n\to\infty} sup\ \sigma_k + \delta\,\sigma_{k+1} \\ &\leq (1-\delta)^{\ k-n_1+2}\ a_{n_1} + \lim_{n\to\infty} sup\ \sigma_n - \delta \lim_{n\to\infty} sup\ \sigma_n + \delta \lim_{n\to\infty} sup\ \sigma_n \\ &\leq (1-\delta)^{\ k-n_1+2}\ a_{n_1} + \lim_{n\to\infty} sup\ \sigma_n \end{aligned}$$

By Taking limit sup for both sides we get:

$lim_{n\to\infty}\ sup\ a_n \leq lim_{n\to\infty}\ sup\ \sigma_n$

■

From Picard-Banach theorem and by taking $n \to \infty$ to the inequality (1.14) we get the following proposition.

Proposition 5.1 Let C be a nonempty closed convex subset of normed space E , let T be a contraction of C into itself, let $\{x_n\}$ be an iterative sequence generated by (1.12) with real sequence $\{\alpha_n\},\{\beta_n\},\{\gamma_n\}$ for som λ where $0 < \lambda \leq \alpha_n, \beta_n < 1$ for all $n \in \mathbb{N}$ then $\{x_n\}$ converges to a unique fixed point of T say P.

Now we can establish the following data dependence result.

Theorem 5.1 let $\tilde{T}$ be an approximate of T satisfying (1.1). Let $\{x_n\}$ be an iterative sequence generated by Modified SP iteration for T and define an iterative sequence $\{\tilde{x}_n\}$ as follow:

$$\begin{cases} \tilde{x}_0 \in C \\ \tilde{x}_{n+1} = \tilde{T}\tilde{y}_n \\ \tilde{y}_n = (1-\alpha_n)\tilde{z}_n + \alpha_n\tilde{T}\tilde{z}_n \\ \tilde{z}_n = (1-\beta_n)\tilde{x}_n + \beta_n\tilde{T}\tilde{x}_n, n \in \mathbb{N} \end{cases} \tag{5.1}$$

Where $\{\beta_n\}$, $\{\alpha_n\}$be real sequence in [0,1] satisfying $0 < \lambda \leq \beta_n, \alpha_n < 1$ for some λ if $p \in F_T$ and $\tilde{p} \in F_{\tilde{T}}$ such that $\tilde{x}_n \longrightarrow \tilde{p}$ as $n \longrightarrow \infty$ then for $\varepsilon > 0$ and $L \in (0,1)$ we have $\| p - \tilde{p} \| \leq \frac{3\,\varepsilon}{\lambda\,(1-L)}$.

Proof:

$$\begin{aligned} \| x_{n+1} - \tilde{x}_{n+1} \| &= \| Ty_n - T\tilde{y}_n + T\tilde{y}_n - \tilde{T}\tilde{y}_n \| \\ &\leq \| Ty_n - T\tilde{y}_n \| + \| T\tilde{y}_n - \tilde{T}\tilde{y}_n \| \\ &\leq L \| y_n - \tilde{y}_n \| + \varepsilon \end{aligned} \tag{5.2}$$

On the other hand

$$\begin{aligned} \left\| y_n - \tilde{y}_n \right\| &= \left\| (1-\alpha_n)\,z_n + \alpha_n T z_n - (1-\alpha_n)\,\tilde{z}_n - \alpha_n\tilde{T}\tilde{z}_n \right\| \\ &\leq (1-\alpha_n) \| z_n - \tilde{z}_n \| + \alpha_n \left\| Tz_n - \tilde{T}\tilde{z}_n \right\| \\ &\leq (1-\alpha_n) \| z_n - \tilde{z}_n \| + \alpha_n (L \| z_n - \tilde{z}_n \| + \varepsilon) \end{aligned}$$

$$\leq (1-\alpha_n) \parallel z_n - \tilde{z}_n \parallel + L\ \alpha_n \parallel z_n - \tilde{z}_n \parallel + \alpha_n \varepsilon$$
$$\leq [1-\alpha_n + L\ \alpha_n] \parallel z_n - \tilde{z}_n \parallel + \alpha_n \varepsilon$$
$$= [1-\alpha_n(1-L)] \parallel z_n - \tilde{z}_n \parallel + \alpha_n \varepsilon$$
$$\leq [1-\lambda(1-L)] \parallel z_n - \tilde{z}_n \parallel + \alpha_n \varepsilon \qquad (5.3)$$

But,

$$\|z_n - \tilde{z}_n\| = \left\|(1-\beta_n)\, x_n + \beta_n T x_n - (1-\beta_n)\, \tilde{x}_n - \beta_n \tilde{T} \tilde{x}_n\right\|$$
$$\leq (1-\beta_n)\| x_n - \tilde{x}_n\| + \beta_n \|T x_n - \tilde{T}\tilde{x}_n\|$$
$$\leq (1-\beta_n)\| x_n - \tilde{x}_n\| + L\beta_n \|x_n - \tilde{x}_n\| + \beta_n \varepsilon$$
$$\leq [(1-\beta_n) + L\beta_n]\|x_n - \tilde{x}_n\| + \beta_n \varepsilon$$
$$\leq [(1-\beta_n) + L\beta_n]\|x_n - \tilde{x}_n\| + \varepsilon \qquad (5.4)$$

Combining $(4,2), (4,3), (4,4)$ and using the facts that L and $L^2 \in (0,1)$

$$\| x_{n+1} - \tilde{x}_{n+1} \| \leq L[1-\lambda(1-L)]^2 \|x_n - \tilde{x}_n\| + 2\varepsilon + \varepsilon$$
$$= L[1-\lambda(1-L)]^2 \|x_n - \tilde{x}_n\| + 3\varepsilon$$
$$\leq [1-\lambda(1-L)]\ \|x_n - \tilde{x}_n\| + 3\varepsilon$$
$$= [1-\lambda(1-L)]\|x_n - \tilde{x}_n\| + \lambda(1-L)\ \frac{3\varepsilon}{\lambda(1-L)}$$

Put $a_n := \| x_n - \tilde{x}_n \|$, $(1-L) \in (0,1)$, $\sigma_n := \frac{3\varepsilon}{\lambda\,(1-L)}$

In Lemma 5.1 we get:

$0 \leq lim_{n\to\infty}\, sup\ \| x_n - \tilde{x}_n \| \leq lim_{n\to\infty}\, sup\ \frac{3\,\varepsilon}{\lambda\,(1-L)}$

From proposition (5.1) we know that $\lim_{n\to\infty} x_n = p$. Thus, using this fact together with the assumption $\lim_{n\to\infty} \tilde{x}_n = \tilde{p}$ we obtain that

$\| p - \tilde{p} \| \leq \frac{3\,\varepsilon}{\lambda\,(1-L)}$

■

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