Iraqi Journal of Science
ISSN: 0067-2904
GIF: 0.851

Nearly Semiprime Submodules

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Abstract

Let R be a commutative ring with unity and let N be a submodule of anon zero left R-module M, N is called semiprime if whenever $r^n x \in N, r \in R, x \in M, n \in Z^+$, implies $rx \in N$. In this paper we say that N is nearly semiprime, if whenever $r^n x \in N, r \in R, x \in M, n \in Z^+$, implies $rx \in N + J(M)$,(in short N. semiprime),where $J(M)$is the Jacobson radical of M. We give many results of this type of submodules.

Keywords: semiprime submodule, nearly semiprime submodule, regular module, nearly regular module.

المقاسات شبه الاولية تقريبا

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الخلاصة

لتكن R حلقة ابدالية ذات عنصر محايد وليكن N مقاسا جزئيا من المقاسغير صفري الايسر M المعرف على R.يقال ان N مقاس جزئي شبه اولي اذا كان $r^n x \in N, r \in R, x \in M, n \in Z^+$. فان $rx \in N$.في هذا ألبحث نقول ان المقاس الجزئي N مقاس شبه اولي تقريبا، اذا كان $r^n x \in N, r \in R, x \in M, n \in Z^+$ يؤدي الى $rx \in N + J(M)$ بحيث ان $J(M)$ هو جذر جاكوبسن. لقد اعطينا العديد من النتائج على هذا النوع من المقاسات الجزئية.

Introduction :

A submodule of an R- module M which Dauns[1] was named semiprime submodules that they are generalized of semiprime ideals,which get big importance at last year, many studiesand searches are published about semiprime submodules by many people who care with the subject of commutative algebra and some of them are J.Dauns ,R.L.McCasland,C.P.LU, P.F.smith,M.E.Moore.The definition comes in [1] as following a proper submodule N of an $R-$module M is called semiprime submodule if whenever $r^n x \in N, r \in R, x \in M, n \in Z^+$, implies $rx \in N$. Let's show the most important results that studies get. If N is a proper submodule of an $R-$module M , then N be semiprim iff for each $r \in R$, $x \in M$ such that $r^2 x \in N$, then $rx \in N$. If N is a proper submodule of an $R-$module M , then the following statements are equivalent:N is semiprime submoduleof M , then $[N:K]$ is semiprime ideal of R[2], then $[N:\langle x \rangle]$ is semiprime ideal of $R, x \notin N$. If N is semiprime submodule, then $[N:M]$ is semiprime ideal of R[2], If $0 \neq M$is Z-regular module ,then every submodule is semiprime. If $0 \neq M$is a module over(P.I.R),thenMis$F-$regular[3] module iff every submodule is semiprime submodule.

1. preliminaries

Let R be a commutative ring with identity and M is a non-zero unitary left R-module M, and N is called semiprime if whenever $r^n x \in N, r \in R, x \in M, n \in Z^+$, implies $rx \in N$[2]equivalently N is called semiprime if whenever $r^2 x \in N, r \in R, x \in M$,, implies $rx \in N$. A submodule A of anR-module M is called *small* (for short $A \ll M$), if whenever $A + B = M$,for some submoduleB of M implies $B = M$[4].A proper submodule N of an R-module M is called *maximal* if whenever $N \subsetneq K \subseteq M$implies $K = M$[4].We know that Jacobson radical of M (for short $J(M)$) is defined by the

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intersection of all maximal submodules of M equivalantly the sum of all small submodule of M[4]. A ring R is said to be good ring if $J(M).R = J(M)$ for any submodule K of an R −module M[4]. If $\phi: M \longrightarrow M'$ is an R- homomorphism, then $\phi(J(M)) \subseteq J(M')$, If $\phi: M \longrightarrow M'$ is an R-epimorphism and $\ker\emptyset \ll M$,then $\phi(J(M)) = J(M')$, and $J(M).R \subseteq J(M)$,where R is a ring , if(M is projective module or R is good ring or R is local ring), then $J(M).R = J(M)$.[4].

2.Nearly Semiprime Submodules.

Recall that a proper submodule N of an R-module M is called semiprime submodule if whenever $r\in R$, $m \in M$, $k \in Z^+$ such that $r^k m \in M$, then $r m \in N$.

Proposition(2.1) :

If N is a proper submodule of an R-module M , then N is semiprime submodule of M if and only if whenever $r^2 m \in N$, where $r \in R$, $m \in M$, then $r m \in N$.[2]

Recall that a proper ideal I in a ring R is called semiprime ideal if $r^2 \in I$ implies that $r \in I$.

Proposition(2.2) :

A proper ideal I is semiprime if and only if $I = \sqrt{I} = \{r \in R: r \in I$ for some positie integer n$\}$

Proposition(2.3) :

If N is a semiprime submodule of an R – module M , then $[N:M]$ is a semiprime ideal of R but the converse is not true in general for example let $M = Z \oplus Z$ be a module over Z and N be a submodule generated be $\langle 4,0 \rangle$, then $0 = [N:M]$ is a semiprime ideal of Z but N is not semiprime submodule of Z . [2]

In this section we introduce the following:

Definetion(2.4)

Aproper submodule N of an R-module M is called nearly semiprime, if wheneever $r^n x \in N$, $r \in R, x \in M$, $n \in Z^+$, implies $rx \in N + J(M)$, where $J(M)$ is the Jacobson radical of M.

Remarks and examples (2.5)

1. It is clear that every semiprime submodule is nearly semiprime, but the converse is not true in general, for example in Z_8 as Z-module , the submodule $N = \langle \bar{4} \rangle$ is not semiprime, but it is nearly semiprime beccause $\bar{4} = 4.\bar{1} = 2^2.\bar{1} \in \langle \bar{4} \rangle$ but $2.\bar{1} = \bar{2} \in \langle \bar{4} \rangle + J(Z_8)$.
2. If $J(M)= 0$,then every nearly semiprime submodules is semiprime submodules.
3. If $J(M)$ is contained in every submodules of M ,then every nearly semiprime submodules is semiprime .
4. In Z as Z-module ,$J(Z)=0$, the submodule $N = \langle 4 \rangle$ is notnearly semiprime beccause $4 = 2^2.1 \in \langle 4 \rangle$, and $2.1 \notin \langle 4 \rangle + J(Z)$.
5. Every submodule in $Zp\infty$, Q as Z module is nearly seimiprime.

Preposition(2.6):

If N is nearly semiprime submodule of an R-module M, K is a proper submodule of M such that $K \nsubseteq N$ and $J(K) = J(M)$, then $N \cap K$ is nearly semiprime in K .

Proof: Since$K \nsubseteq N$, then $K \cap N$ is proper in K.. Let $r \in R, x \in K$, $n \in Z^+$ such that $r^n x \in N \cap K$, since N is nearly semiprime, then $rx \in N + J(M), rx \in N + J(K)$, but $x \in K$, then $rx \in K$, hence $rx \in (N + J(K)) \cap K.$, by modular low thus $rx \in (N \cap K) + J(K)$, whichimplies that $N \cap K$ is nearly semiprime in K.

R is said to be a good ring if $J(M) \cap K = J(K)$, for any submodule K of an R −module M[8].

Corollary(2.7):

Let R be good ring and N be nearly semiprime submodule of an R - module M , let K be a proper submodule of M such that $K \nsubseteq N$and $J(M) \subseteq K$, then $N \cap K$ is nearly semiprime in K.

Proof: Since $K \nsubseteq N$, then $K \cap N$ is proper in K. Let $r \in R, x \in K$, $n \in Z^+$ such that $r^n x \in N \cap K$, since N is nearly semiprime, then $rx \in N + J(M)$, but $x \in K$, then $rx \in (N + J(M)) \cap K)$, thus $rx \in (N \cap K) + J(M)) \cap K)$, since R is good ring , then $J(K) = J(M)$ by Preposition(2.6) $N \cap K$ is nearly semiprime in K.

Corollary(2.8):

Let N and K be nearly semiprime submodules of M such that $K \nsubseteq N$and either $J(M) \subseteq N$ or $J(M) \subseteq K$,then $N \cap K$ is nearly semiprime of M.

Proof: Since $N \cap K \subsetneq N$ and N is aproper submodule, then $K \cap N$ is propersubmodule in M .Let $r \in R$, $x \in K$, $n \in Z^+$ such that $r^n x \in N \cap K$, since N and K are nearly semiprime, then $rx \in N + J(M)$ and $rx \in$

$K+ J(M)$, hence $rx \in (N+J(M)) \cap (K+ J(M))$. If $J(M) \subseteq N$, then $rx \in N \cap (K+ J(M))$, then $rx \in N \cap K+J(M)$.If $J(M) \subseteq K$, then $rx \in (N+J(M)) \cap K$, hence $rx \in N \cap K+J(M)$, which implies that $N \cap K$ is nearly semiprime in M .

Corollary (2.9):If N is a maximal submoduleof an R - module Mand K is nearly semiprimesubmodulof M , then $N \cap K$ is nearly semiprime of M.

Proof: Let N be a maximal submodule of M, then $J(M) \subseteq N$, hence by Preposition (2.6) $N \cap K$ is nearly semiprime of M.

N is semiprimesubmodule of an R-modul M iff $I^2K \subseteq N$, for some ideal I of R and some submodule K of M, implies that $IK \subseteq N$[5], for nearly semiprime submodule we have the following:

Preposition(2.10):Let N be a proper submodule of an R-modul M**.** Nis nearly semiprime submodule of M iff $I^nK \subseteq N$,for some ideal I of R and some submodule K of M and $n \in Z^+$, then $IK \subseteq N+J(M)$.

Proof: Let N be a nearly semiprime submodule of M and $I^nK \subseteq N$ for some ideal I of R and some submodule K of M and $n \in Z^+$,we want to show that$IK \subseteq N+J(M)$,Let $x \in IK$,then $x = r_1x_1 + r_2x_2 \cdots\cdots + r_nx_n$,where$r_i \in I$, $x_i \in K$,$i = 1,2,3,\ldots\ldots n$,thus $r_ix_i \in IK$, for each $i = 1,2,\ldots\ldots, n$,then$r_i^nx_i \in I^nK \subseteq N$, but N is a nearly semiprime submodule,then $r_ix_i \in N+J(M)$ for each $i = 1,2,3,\ldots n$,thus $x \in N+J(M)$, which implies that $IK \subseteq N+J(M)$. For converse ,let $r^nm \in N$, then $\langle r^n\rangle Rm \subseteq N$,thus$\langle r\rangle^n Rm \subseteq N$ (by assumpition) $\langle r\rangle Rm \subseteq N+J(M)$,then $rm \in \langle r\rangle Rm \subseteq N+J(M)$,then $rm \in N+J(M)$, which implies that N is nearly semiprime in M

Recall that an R- module M is called fully semiprime if every submodule in M is semiprime [6] and an R-modulemodule M is called hollow module if every submodule in M is small[7]

We introduse the following:

Definetion(2.11) : An R- module M is called fully nearly semiprime if every submodule in M is nearly semiprime .

Example(2.12):

1. Z_6 and Q as Z −module is fully nearly semiprime.
2. Z as Z −module is not fully nearly semiprime, since $\langle 4\rangle$ is not nearly semiprime submodule of Z as Z −module (Remark(2.2) (4).

Preposition (2.13): If M is hollow R- module and $J(M)$ is a semiprime submodule of M ,then M is fully nearly semiprime.

Proof: Let N be a proper submodule of M .Let $r \in R$, $x \in K$, $n \in Z^+$ such that $r^nx \in N$ and , but $J(M)$ is semiprime, then $rx \in J(M) \subseteq N+J(M))$, which implies that N is nearly semiprime in M.

If M and M' are two R- modules, and $\emptyset: M \to M'$ is an epimorphism with $\ker\emptyset \subseteq N$. IfN is semiprime submodule in M, then $\emptyset(N)$ is semiprime in M'**,** and If N' is semiprime in M' with $\ker\emptyset \ll M$, then $\emptyset^{-1}(N')$ is semiprime in M[2].

Now,we have the following :

Preposition (2.14): If M and M' are two R- modules, and $\emptyset: M \to M'$ is an epimorphism with $\ker\emptyset \subseteq N$. If N is nearly semiprime submodule in M, then $\emptyset(N)$ is nearly semiprime in M'.

Proof:$\emptyset(N)$ is a proper submodule of M . suppose not , then $\emptyset(N) = M'$ $m\in M$ such that $\emptyset(m) \in M' = \emptyset(N)$,$\exists\, n \in N$ such that $\emptyset(n)= \emptyset(m)$, hence $\emptyset(n-m) = 0$, then $n-m \in \ker\emptyset \subseteq N$,then $m \in N$, hence N= M (contradition) , since $N \subsetneq M$. Let $r \in R$, $m' \in M'$ such that $r^nm' \in \emptyset(N)$ since $\emptyset$ is onto, then $\exists m \in M$ such that $\emptyset(m) = m'$,then $\emptyset(N) \ni r^nm' = r^n\emptyset(m)= \emptyset(r^nm)$, then$\exists\, y \in N$ such that $\emptyset(y)=\emptyset(r^nm)$ hence $\emptyset(y-r^nm) = 0$,then $y - r^nm \in \ker\emptyset \subseteq N$, then $r^nm \in N$, but N is nearly semiprime in M, then $rm \in N+J(M)$

,$\emptyset(rm) = r\emptyset(m) \in \emptyset(N) + \emptyset(J(M)) \subseteq \emptyset(N) + J(M')$, then $r\emptyset(m) = rm' \in \emptyset(N) + J(M')$. Thus $\emptyset(N)$ is nearly semiprime.

Preposition (2.15):Let M and M' be two R- modules and $\emptyset: M \to M'$ is an epimorphism . If N' is nearly semiprime in M' with $\ker\emptyset \ll M$, then $\emptyset^{-1}(N')$ is nearly semiprime in M.

Proof: It is clear that $\emptyset^{-1}(N') \subsetneq M$. Let $r \in R$, $m \in M$ such that $r^nm \in \emptyset^{-1}(N')$ then $\emptyset(r^nm) \in N'$, hence $r^nm' = r^n\emptyset(m) \in N'$, where , $m' \in M'$, but , N' is nearly semiprime in M'. Then $rm' \in N' + J(M')$, since $\emptyset$ is an epimorphism and by[4] , then $\emptyset(J(M)) = J(M')$, hense $rm' = r\emptyset(m) = \emptyset(rm) \in N' + \emptyset(J(M))$. Thus $rm \in \emptyset^{-1}(N') + J(M)$, which implies that $\emptyset^{-1}(N')$ is nearly semiprime in M.

Corollary (2.16): If M is hollow R- module M and $\frac{M}{N}$ is fully nearly semiprime, then N is nearly semiprime in M .

Preposition (2.17):

If M is a hollow R- module and $\phi: M \longrightarrow B$, where B is any R- module , is an epimorphism such that $\ker\emptyset \subseteq A_\alpha$,for each submodule A_α of $M,\alpha \in \Lambda$,then M is fully nearly semiprimeiff B is fully nearly semiprime.

Proof: Let L be a proper submodule of B ,then $\phi^{-1}(L) \subsetneq M$ and M is fully nearly semiprime ,then $\phi^{-1}(L)$ is nearly semiprime in M,by assumpitionker$\emptyset \subseteq \phi^{-1}(L)$,by **Preposition** (2.14) ,then L is nearly semiprime submodule in B ,thus B is fully nearly semiprime . For the convrse , let K be a proper submodule of M,then $\phi(K) \subsetneq B$,but B is fully nearly semiprime, then $\phi(K)$ is nearly semiprime of B andsince Mis hollow, thenker$\emptyset \ll M$,clearly$\phi^{-1}(\phi(K)) = K$,thus K is nearly semiprime, (by **Preposition** (2.15)) , then M is fully nearly semiprime.

Preposition (2.18):Let N be nearly a submodule of an R −module M such that $J(M) \subseteq N$, then N is nearly semiprime submodule of M iff $[N: M]$is a semiprime ideal in R .

Proof: Since $J(M) \subseteq N$, then N is semiprime , then $[N: M]$ is a smeiprime ideal in R . For the converse, let $a \in \sqrt{[N: M]}$,then $\exists n \in Z^+$ such that $a^n \in [N: M]$, thus$a^n M \subseteq N$, but Nis nearly semiprimesubmodule of M**,** then $aM \subseteq N + J(M)$ and $J(M) \subseteq N$, thus $aM \subseteq N$, then $a \in [N: M]$, which implies that $[N: M]$ is semiprime ideal in R .

Remark(2.19):If $N + J(M)$ is nearly semiprime submodule of an R-module M, then $[N + J(M): M]$ is smeiprime ideal of R .

Proof: : Let $a \in \sqrt{[N + J(M): M]}$,then $\exists n \in Z^+$ such that $a^n \in [N + J(M): M]$ thus$a^n M \subseteq N + J(M)$ but $N + J(M)$ is nearly semiprime submodule of M**,** then $aM \subseteq N + J(M)$, then $a \in [N + J(M): M]$ which implies that $[N + J(M): M]$ is semiprime ideal in R.

3.**Nearly regular moduls**.

Recall that an R- module M is called Z- regular if for each $m \in M$, $\exists f \in M^*$ such that $m = f(m). m$. [8]and R- module M is called Z –Nearly regular if for each $m \in M$, $\exists f \in M^*$ such that $m - f(m). m \in J(M)$ (in short Z –N.regular) [9] .

[2] show that if M is Z – regular R- module, then every submodule of M is semiprime.If M is Z –N. regular R- moduler we have the following :

Preposition (3.1): If M is Z –N. regular R- module, then every submodule of M is nearly semiprime**.**

Proof: Let N be a proper submodule of an R- module M and $r^n m \in N$, where $r \in R$, $m \in M$. Since M is Z – N. regular, then $\exists f \in M^* = Hom_R(M , R)$ such that $m - f(m). m \in J(M)$, then $m = f(m). m + s$,where $s \in J(M)$, then $rm = rf(m)rm + s_1 = f(m). r^2 m + s_1 \in N + J(M)$; $s_1 \in J(M)$. Thus $rm \in N + J(M)$ and Nis nearly semiprime.

Corollary (3.2): If M is Z –N. regular R- module, thenM is fully nearly semiprime .

Recall that an R- module M is called F- regular if for each $m \in M$, $r \in R$, $\exists t \in R$ such thatrm= rt rm . [10]and an R- module M is called F –N. regular if for each $m \in M$, $r \in R$, $\exists t \in R$ such that$rm - rt\, rm \in J(M)$[10]

Preposition (3.3): If M is F –N. regular R- module , then every submodule of M is nearly semiprime submodule.

Proof: Let N be a proper submodule of an R- module M and $r^n m \in N$, where $r \in R$, $m \in M$. Since M is F – N. regular, then for each $m \in M$, $r \in R$, $\exists t \in R$ such that$rm - rt\, rm \in J(M)$, $rm = tr^2 m + s$, where $s \in J(M)$,then $rm \in N + J(M)$, thus N is nearly semiprime.

Corollary (3.4): If M is F –N. regular R- module, thenM is fully nearly semiprime. Recall that A submodule Nof an R- module M is called pure if $IM \cap N = IM$[11], and a submodule Nof an R- module M is called nearly pure if$IM \cap N = IM + J(M) \cap IM \cap N$[9]**.**

Now we have the following:

Definetion(3.5): An R- module M is called F-N. regular if for eachsubmodule is nearly pure.

Theorem (3.6): Let M be an R- module ,if M is Z –N.regular ,then M is F –N.regular. the converse is true if M is projective .

Proof: Let K be a proper submodule of an R- module M**,**by theorem (2.4) [9] ,then K is nearly pure and by definetion of F-N. regular**,** thus M is F –N.regular, For the converse, since M is F –N.regular**,** then each submodule is nearly pure and since M is projective ,then M is Z –N.regular, [9]

Preposition (3.7): If $0 \neq M$ is a module over (P. I. D), then M is F-N. regular module iff every submodule is nearly semiprime .

Proof: Let K be a proper submodule of an R- module M and $r^n m \in N$, where $r \in R$, $m \in M$. Since M isF – N. regular, then for each $m \in M, r \in R$, $\exists\ t \in R$ such that $rm - rt\,rm \in J(M)$, $rm = tr^2m + s$, where s$\in J(M)$then $rm \in K + J(M)$, thus K is nearly semiprime.

For the converse , let K be a propernearly semiprime submodule of an R- module M, clearly sK+ $J(M) \cap (sM \cap K) \subseteq sM \cap K$.Let $x \in sM \cap K$, $x = sm$, $m \in M, sx = s^2m \in sK$but sK is nearly semiprime submodule of M, then $sm \in sK + J(M)$, thus $x \in sK + J(M)$ and $x \in sM \cap K$,then $x \in (sK + J(M)) \cap (sM \cap K)$, but $sK \subseteq sM \cap K$,by modular law , then $x \in sK + (J(M) \cap (sM \cap K))$,thus K is nearly pure submodule of M , thenM is F –N. regular.

Finaly ,we have the following preposition :

Preposition (3.8): Let M be an R- module .If $\frac{R}{ann(x)+J(R)}$ is a regular ring for each $x \in M$, then every submodul of M is nearly semiprime submodule .

Proof: Let N be a proper submodule of an R- module M. Let $r \in R, x \in M$, $n \in Z^+$ such that $r^n x \in N$, since $\frac{R}{ann(x)+J(R)}$is a regular ring ,then for each $r \in R$, $\exists t \in R$ such that $r + (ann(x) + J(R)) = rtr + (ann(x) + J(R))$,thus $(r - rtr) \in ann(x) + J(R)$, then $(r - rtr - s) \in ann(x)$,where $s \in J(R)$, then $(r - rtr - s)x = 0$, then $(r - rtr)x = sx \in J(R)M \subseteq J(M)$,thus $rx - rtrx \in J(M)$,then $rx - tr^2x \in J(M)$, $rx = tr^2x + a$,where a$\in J(M)$,but $tr^2x \in N$,then , $rx \in N + J(M)$. which implies that N is nearly semiprime in M.

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