Applying the Shrinkage Technique for Estimating the Scale Parameter of Weighted Rayleigh Distribution

Intesar Obeid Hassoun*, Adel Abdulkadhim Hussein

Department of Mathematics, College of Education for Pure Sciences/ Ibn Al – Haithem, University of Baghdad
Iraq

Abstract

This paper includes the estimation of the scale parameter of weighted Rayleigh distribution using well-known methods of estimation (classical and Bayesian). The proposed estimators were compared using Monte Carlo simulation based on mean squared error (MSE) criteria. Then, all the results of simulation and comparisons were demonstrated in tables.

Keywords: Weighted Rayleigh distribution, Moment Method, Maximum likelihood method Shrinkage Technique, Mean Squared Error (MSE).

استخدام طريقة التقلص لتقدير معلمة قياس توزيع رالي الموزون

إنتصار عبيد حسون*، عادل عبدالكاظم حسين

قسم الرياضيات ، كلية التربية للعلوم الصرفة ، أبن الهيثم ، جامعة بغداد ، العراق

الخلاصه

هذا البحث يتضمن تقدير معلمة القياس لتوزيع رالي الموزون باستعمال طرق التقدير المعروفة (الكلاسيكية والبيزية). وإن المقدرات المتكونه (المستحصلة) باستعمال محاكاة مونتي كارلو قورنت بمقياس متوسط مربعات الخطأ (MSE) وبالتالي فأن كل نتائج المحاكاة والمقارنات وضعت في جداول .

Introduction

Rayleigh distribution is one of the failure distributions which have been used in reliability, life-testing, and survival analysis. Being first presented by Lord Rayleigh [1], these statistical properties were originally derived in connection with a problem in acoustics. More details on the Rayleigh distribution can be found in Johnson *et al*. [2].

Weighted distributions are applied in research associated with reliability meta-analysis, bio – medicine, econometrics , renewal processes , physics, ecology, and branching processes. Research on such applications can be found in Zelen and Feinleib [3], Patil and Ord [4], Patil and Rao [5], Gupta and Keating [6], Gupta and Kirmani [7], and Oluyede [8]. The weighted Rayleigh distribution was published by Reshi *et al*. [9]. They presented a new class of size – biased generalized Rayleigh distribution and investigated the various structural and characterizing properties of that model, as also performed by Das and Roy [10]. Rashwan [11] introduced the double weighted Rayleigh distribution properties and estimation. AL-Kadim and Hussein [12] published a research on the estimation of the reliability of weighted Rayleigh distribution through five methods, using simulation and a comparison between the proposed estimators. Ahmed and Ahmed [13] presented the characterization and estimation of double weighted Rayleigh distribution. Salman and Ameen [14] estimated the shape

---

*Email: wafadream67@yahoo.com

parameter of generalized Rayleigh distribution using the Bayesian-Shrinkage technique. Ajami and Jahanshahi [15] introduced the parameter estimation in weighted Rayleigh distribution .

The aim of this paper is to estimate the scale parameter of weighted Rayleigh distribution using shrinkage estimation methods that depend on classical estimators, namely the moment (mom) and maximum likelihood (ML) methods.

Weighted Rayleigh Distribution (WRD)

To present the concept of a weighted distribution, suppose that T is a non – negative random variable that follows Rayleigh distribution with one parameter $\beta\{T \sim RD(\beta)\}$, then the (pdf) of T is given by

$$f(t, \beta) = 2t\beta e^{-t^2\beta} \quad ; t > 0 \ , \beta > 0 \qquad \qquad \dots(1)$$

and w (t) a non- negative weight function satisfying the condition $\mu_w = E[w(t)]$. Then the r v T **w,** which is defined on the interval (L,U), is having pdf as below :

$$f\omega(t) = \frac{\omega(t) f(t)}{E[w(t)]} \ , \ l < t < U \qquad \qquad \dots (2)$$

where, $\omega(t) = e^{t^2}$ and $E(\omega(t)) = \int_0^\infty \omega(t) f(t) dt$

From equations (1) and (2), we get that the probability density function of the random variable Tw will be

$$f\omega(t;\theta) = 2t(\beta - 1)e^{-t^2(\beta-1)} \ ; t > 0 \ , \beta > 1 \qquad \qquad \dots (3)$$

and the cumulative distribution function (cdf) of Tw will be

$$F\omega(t; \theta) = 1 - e^{-t^2(\beta-1)} \qquad \qquad \dots (4)$$

By putting $\beta - 1 = \theta > 0$ in equations (3) we get

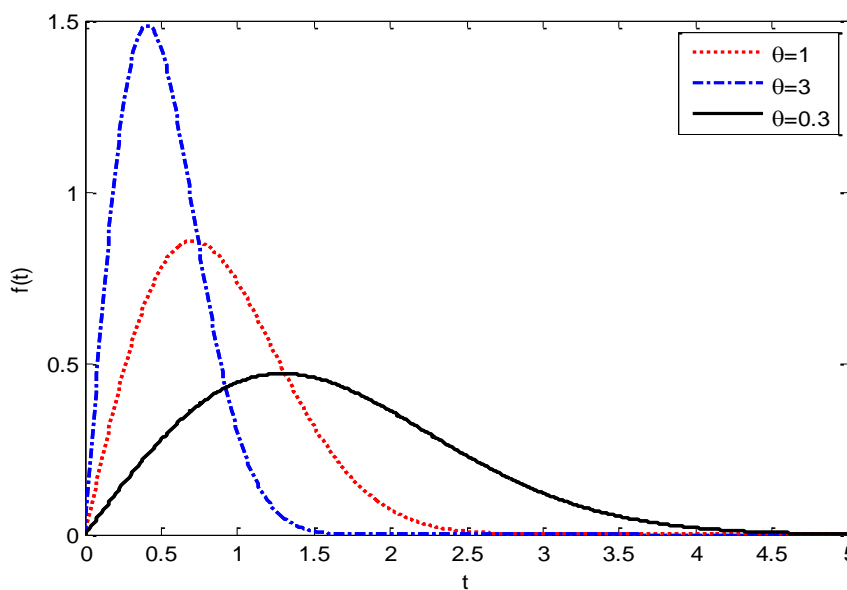$$f_w(t;\theta) = 2t\theta e^{-t^2\theta} \ ; t > 0 \ , \theta > 0 \qquad \qquad \dots (5)$$

and the cumulative distribution function (cdf) of T_W will be

$$F_W(t; \theta) = 1 - e^{-t^2\theta} \qquad \qquad \dots (6)$$

Accordingly, the reliability and hazard functions will be respectively as follows:

$$R\omega(t; \theta) = 1 - F\omega(t; \theta) = e^{-t^2\theta} \qquad \qquad \dots (7)$$

$$h\omega(t; \theta) = \frac{f_\omega(t;\theta)}{R_\omega(t; \theta)} = 2t\theta \qquad \qquad \dots (8)$$

figure(1): The Plot of Raleigh Distribution (p.d.f)
Figure 1-The Plot of Raleigh Distribution (p. d. f.)

Methods Estimation
Maximum Likelihood Estimation (MLE)
    Firstly , we find the likelihood function L(t1,t2,t3,…,tn; θ) based on the following

l =L(t$_1$,t$_2$,t$_3$,…,t$_n$; θ) = $\prod_{i=1}^{n} f$ ω (t, θ)

$l = \theta^n \prod_{i=1}^{n} 2t_i e^{-\theta \sum_{i=1}^{n} t_i^2}$

ln l = nlnθ + $\sum_{i=1}^{n} ln2t_i$ - θ $\sum_{i=1}^{n} t_i^2$

$\frac{\partial lnl}{\partial \theta} = \frac{n}{\theta}$ - $\sum_{i=1}^{n} t_i^2$

Put $\frac{\partial lnl}{\partial \theta}$ =0

$$\therefore \hat{\theta}_{mle} = \frac{n}{\sum_{i=1}^{n} t_i^2} \qquad \qquad …(9)$$

Method of Moments
    Let t1, t2,…, tn refer to a random sample of size n from the WRD with pdf (5). Then the moment estimator $\hat{\theta}$ of is obtained by setting the mean of the distribution to be equal to the sample mean ,
i.e., E(T^K)= $\sum_{i=1}^{n} t^k /n$

The moment estimator $\hat{\theta}_{mom}$ of θ is obtained as below

$\frac{\Gamma(1+\frac{k}{2})}{(\theta)^{k/2}} = \frac{\sum_i^n t_i^k}{n}$

For k = 1, we have

$\frac{\Gamma(1+\frac{1}{2})}{\theta^{1/2}} = \frac{\sum_{i=1}^{n} t_i}{n}$

$$\therefore \hat{\theta}_{mom} = \frac{\pi}{4(\bar{t})^2} \qquad \qquad …(10)$$

Shrinkage Method
    Thompson [16] studied the problem of the shrinkage of a usual estimator $\hat{\theta}$ of the parameter θ, depending on the observations of random samples along with prior studies and previous experiences. This was performed through merging the usual estimator $\hat{\theta}$mle and the initial estimate θo as a linear mixture, via the shrinkage weight factor Ø($\hat{\theta}$mle), 0 ≤ ($\hat{\theta}$mle) ≤ 1. The resulted estimator is so called the shrinkage estimator which has the form below

$$\hat{\theta} sh = Ø(\hat{\theta}mle)\hat{\theta}mle + (1 − Ø(\hat{\theta}mle))θo \qquad \qquad ...(11)$$

where Ø ($\hat{\theta}$mle) denotes the trust of $\hat{\theta}$mle and (1 − Ø ($\hat{\theta}$mle)) signifies the trust of θ$_o$, which might be constant or a function of $\hat{\theta}$mle function of sample size. It could be also found by reducing the mean square error for $\hat{\theta}$ sh . Thompson referred to the following significant reasons to use the initial value.
1 – Supposing that the initial value θo is near to the true value, then it is essential to use it.
2 – If the initial value θo is near to the actual value of the parameter θ , then we reach a bad situation [1, 5, 15, 17, 18, 19]. In this state, there is no doubt to take the moment method as an initial value. Consequently, equation (11) becomes :

$$\hat{\theta} sh = Ø\mathbf{1}(\hat{\theta}mle)\hat{\theta}mle + (1 − Ø\mathbf{1}(\hat{\theta}mle))\hat{\theta}mle \qquad \qquad …(12)$$

The shrinkage weight function (sh1)
  In this section, we claim the shrinkage weight factor as a function of sample size n, as below.
Ø1 ($\hat{\theta}$mle) = e^{-n}
Consequently , the shrinkage estimator of Ø , which is defined in (12), will be

$$\hat{\theta} sh1 = e^{-n} \hat{\theta}mle + (1- e^{-n}) \hat{\theta}mom \qquad \qquad ....(13)$$

Constant shrinkage weight factor (sh2)
In this subsection, we suggest a constant shrinkage weight factor as below.
Ø2 ($\hat{\theta}$mle) = 0.01
Accordingly , the shrinkage estimator of θ will be :

$$\hat{\theta} sh2 =(0.01) \hat{\theta}mle + (0.99) \hat{\theta}mom \qquad \qquad …(14)$$

Modified Thompson type shrinkage weight function (sh3)

In this subsection, we consider the modified shrinkage weight factor introduced by Thompson, as follows.

$$\emptyset_3(\hat{\theta}_{mle}) = \frac{(\hat{\theta}_{mle}-\hat{\theta}_{mom})^2}{(\hat{\theta}_{mle}-\hat{\theta}_{mom})^2+var(\hat{\theta}_{mle})} * (0.001) \quad\quad \dots (15)$$

where, $var(\hat{\theta}_{mle}) = \frac{n^2\theta^2}{(n-1)^2.(n-2)}$ \quad\quad \dots (16)

Hence, the shrinkage estimator of θ becomes:

$$\hat{\theta}_{sh3} = \emptyset_3(\hat{\theta}_{mle})\hat{\theta}_{mle} + \left(1 - \emptyset_3(\hat{\theta}_{mle})\right)\hat{\theta}_{mom} \quad\quad \dots (17)$$

Simulation Study

In this section, Monte Carlo simulation study was applied to compare the performance of the considered estimators for the scale parameter θ, which were obtained using different sample sizes (n=10,30,50,100), based on 1000 replications through MSE criteria, as in the steps below [6] .

Step 1: Generate random samples that follow the continuous uniform distribution from interval (0,1), say as u**1**, u**2**, …, u**n**.

Step 2: Transform the uniform random samples to random samples following the WRD and using the cumulative distribution function (c.d.f), as follows:

$u \sim Uinf(0,1)$
$F(t) = 1 - R(t)$
$u_i = 1 - e^{-t_i^2\theta}$
$t_i = \sqrt{-\frac{1}{\theta}\ln(1 - u_i)}$

Step 3: Calculate the maximum likelihood estimator of θ(t) from equation (9).

Step 4: Apply the moment method of θ(t) via equation (11).

Step 5: Compute the three shrinkage estimators of θ(t) by equations (13), (14), and (17).

Step 6: Based on (L=1000) replication, the MSE for all proposed estimation methods of θ(t) is utilized by:

$$\text{MSE}(\hat{\theta}_{mle}) = \frac{1}{L}\sum_{i=1}^{L}(\hat{\theta}_i - \theta)^2$$

where $\hat{\theta}_i$ denotes the suggested estimation method in iterative i for real θ.

Numerical results

We demonstrate all the results in the tables below.

Table 1-The estimation method of θ

N	θ	$\hat{\theta}_{mle}$	$\hat{\theta}_{mom}$	$\hat{\theta}$ sh1	sh $\hat{\theta}$ 2	sh $\hat{\theta}$ 3
10	2	2.2839	2.1672	2.1672	2.1683	2.2839
	3	3.4259	3.2508	3.2508	3.2525	3.4259
	4	4.5678	4.3344	4.3344	4.3367	4.5678
	5	5.7098	5.4179	5.4180	5.4209	5.7098
30	2	2.0135	2.0977	2.0977	2.0968	2.0135
	3	3.0203	3.1465	3.1465	3.1453	3.0203
	4	4.0270	4.1954	4.1954	4.1937	4.0270
	5	5.0338	5.2442	5.2442	5.2421	5.0338
50	2	2.2096	2.2609	2.2609	2.2604	2.2096
	3	3.3145	3.3913	3.3913	3.3906	3.3145
	4	4.4193	4.5218	4.5218	4.5207	4.4193
	5	5.5241	5.6522	5.6522	5.6509	5.5241
100	2	2.3025	2.2893	2.2893	2.2895	2.3025
	3	3.4537	3.4340	3.4340	3.4342	3.4537
	4	4.6050	4.5787	4.5787	4.5789	4.6050
	5	5.7562	5.7233	5.7233	5.7237	5.7562

Table 2-The MSE of the estimation method of θ.

n	θ	mle	mom	sh1	sh2	sh3	best
10	2	0.0006864	0.00075275	0.00075275	0.0007515	0.0007180	Sh1
	3	0.0014082	0.001400023	0.001400019	0.0013991	0.0013878	Sh1
	4	0.0031097	0.003108731	0.003108720	0.0031063	0.0013878	Sh1
	5	0.0045856	0.004380536	0.004380531	0.0043795	0.0030178	Sh1
30	2	0.00015622	0.000170968	0.000170968	0.00017068	0.0043464	Sh3
	3	0.00032290	0.0003292038	0.0003292038	0.00032885	0.00016431	Sh3
	4	0.00058794	0.0006288918	0.0006288918	0.00062799	0.00032322	Sh3
	5	0.00115660	0.0012120170	0.0012120170	0.00121060	0.00061422	Sh3
50	2	0.000088041	0.0000967384	0.0000967384	0.000096577	0.00093266	Sh3
	3	0.00020739	0.0002223561	0.0002223561	0.00022203	0.00021604	Sh3
	4	0.00036650	0.0003935831	0.0003935831	0.00039299	0.00038074	Sh3
	5	0.00060234	0.0006305347	0.0006305347	0.00062978	0.00061549	Sh3
100	2	0.000041813	0.00004597264	0.00004597264	0.000045893	0.000044647	Sh1&Sh2
	3	0.000094374	0.000103385	0.000103385	0.00010321	0.00010006	Sh1&Sh2
	4	0.00017602	0.0001863072	0.0001863072	0.00018606	0.00018206	Sh1&Sh2
	5	0.00024768	0.0002786088	0.0002786088	0.00027805	0.00026833	Sh1&Sh2

Results Analysis

1 - It is clear from Table-1 that MSE has a minimum value in the case of mle for all the values of θ = 2 , 3 , 4 , and 5, when the sizes of samples are 30 , 50 , and 100, which implies that MLE is the best.

2 - For n=10 (small sample size), the mean squared error (MSE) of the scale parameter $\widehat{\theta}$ sh3 is lower than the other estimators, followed by $\widehat{\theta}$ sh2 and $\widehat{\theta}$ sh1. Hence, the best estimator in this case is $\widehat{\theta}$ sh3 for all θ = 2, 3, 4, 5.

3 - For n = 30, 50, 100 (medium sample size), the MSE of the scale parameter $\widehat{\theta}$ sh3 has a lower value than the other estimators, followed by $\widehat{\theta}$ sh2 and $\widehat{\theta}$ sh1. Consequently, the best estimator in this case is $\widehat{\theta}$ sh3 for all θ = 2, 3, 4, 5.

4 - For all n, the MSE for all proposed estimators is approximately fixed with respect to θ.

Conclusions

From the results of the analysis, the maximum likelihood method was the best because it showed minimum MSE for all values of θ when the sizes of sample were 30 , 50 , and 100.

References

1. Rayleigh, J. w. **1880**. On the result large number of vibrations of the some pitch of arbitrary phase, *philosophical magazine 5th series*, **10**(60): 73 -78 doi : 10. 1080 /14786448008626893
2. Johnson, N.L, Kotz, S. and Balakrishnan, N. **1994**. *Continuous unvariate distribution*, vol, l, (2nd Ed) . Newyork : wiley.
3. Zelen, M. and Feinleib, M. **1969.** On the theory of chronic disease. *Bioment Rika*, **56**(3): 601- 614. doi: 10.2307/2334668.
4. Patil, G. P. and Ord, J.K. **1946**. On size - biased sampling and related form invariant weighted distribution sankhya , *the indian Journal of statistics B*, **38**: 48-61.
5. Patil, G.P. and Rao, C.R. **1978**. "Weighted Distributions and Size Biased Sampling Applications to Wildlife Populations and Human Families", *Biometrics*, **34**(2): 179-189. Doi : 102307/2530008
6. Gupta, R.C. and Keating, J.P. **1985**. "Relations for Reliability Measures under Length Biased Sampling", *Scan. Journal of statist*, **13**: 49-56.
7. Gupta, R.C. and Kirmani, S.N.U.A. 1990. "The Role of Weighted Distributions in Stochastic Modeling", *Commun. Statist*., **19**(9): 3147-3162.
8. Oluyede, B.O. **1999**. On inequalities and selection of experiments for length – biased distribution porbability in the engineering and informational sciences, **13**(2): 169-185 doi : 10 .1014 /0269 96 4999132030
9. Reshi, J. A, Ahmed, A. and Mir, k. A. **2014**. Characterizations and estimation in the length – biased generalized Rayleigh distribution, *mathematical theory and modeling*, **4**(6): 87- 98.

10. Das, K.K. and Roy, T.D. **2011**. "Applicability of Length Biased Weighted Generalized Rayleigh Distribution", *Advances in Applied Science Research*, **2**(4): 320-327.
11. Rashwan, N.I. **2013**. "The Double Weighted Rayleigh Distribution Properties and Estimation", *International Journal of Scientific & Engineering Research*, **4**(12): 2229-5518.
12. Al-Kadim, K.A. and Hussein, N.A. **2014**. "Comparison Between Five Estimation Methods for Reliability Function of Weighted Rayleigh Distribution by Using Simulation ".
13. Ahmed, S.P. Afaq and Ahmed, A. **2014**. "Characterization and Estimation of Double Weighted Rayleigh Distribution", *Journal of Agriculture and life Sciences*, **1**(2): 2375- 4222.
14. Salman, A.N. and Ameen, M.M. **2015**. "Estimate the Shape Parameter of Generalize Rayleigh Distribution Using Bayesion-Shrinkage Technique", *international Journal of Innovative Science, Engineering and Technology*, **2**: 675-683.
15. Ajami, M. and Jahanshahi, S. M. **2017**. Parameter estimation weighted Rayleigh distribution, *Journal of modern applied statistical methods November* , **16**(0.2): 256-276 doi: 10 .2223/Jmasm / 1509495240. 1SSN: 1538-9472 copyriyht@2017 jmasm, fac.
16. Thompson, J.R. **1968**. "Some shrinkage Techniques for Estimating The Mean", *J. Amer. Statist. Assoc.*, **63**: 113-122.
17. AL-Hemyari, Z.A, Hassan, I.H. and AL-Joboori, A.N. **2011**. "A class of efficient and Modified estimator for the mean of normal distribution using complete data", *International Journal of data Analysis Techniques and Strategies*, **3**(4): 406 – 425.
18. Al-Hemyari, Z.N. and Al-Joboori, A.N. **2009**. "On Thompson Type Estimators for the Mean of Normal Distribution", *Revista Investigacion Operacional, J.* **30**(2): 109-116.
19. Al-Joboori, A.N. **2014**. "Single and Double Stage Shrinkage Estimators for the Normal Mean with The Variance Cases", *International Journal of Statistic*, **38**(2): 1127-1134.