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Applications of Differential Inequalities Employing A New Convoluted Operator Constructed by The Supertrigonometric Function

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Abstract
In order to examine various geometric features, we expanded the supertrigonometric function (STF) and superhyperbolic function (SHF) into the open unit disk. A convolution differential operator of the STF provides the formulas. The suggested operator works with both integral and double differential inequalities. As a result, we present a collection of findings that includes recent works. The idea of subordination and superordination serves as a guide for our method, and we then developed the primary conclusion as a double side's theorem.

Keywords: Univalent functions, Hypergeometric supertrigonometric functions, Hadamard product, Differential Subordination, Differential superordination.

تطبيقات المتباينات التفاضلية التي تستخدم مؤثرًا جديدًا عقديًا تم تعريفه بواسطة الدالة المثلثية العليا

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الخلاصة
من أجل دراسة خواص هندسية مختلفة ، وسعنا الدالة المثلثية العليا (STF) والدالة الزائدية (SHF) في القرص المفتوح. ثم عرفنا صيغة المؤثرالتفاضلي العقدي باستخدام الدوال المثلثية .هذا المؤثر يطبق مع كل من المتباينات التكاملية والتفاضلية المزدوجة. استعرضنا مجموعة من النتائج التي تشمل نتائج قدمت في الوقت الحاضر. انشاء مجموعة من المبرهنات التابعية التفاضلية والتابعية التفاضلية العليا ومبرهنة الساندوج من صنف الدوال احادية التكافؤ.

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1. Introduction

Xiao-Jun [1, 2] has recently created the theories of the Wiman function, STFs, and SHFs. The Wiman function and connected functions integral representations are explored, and the generic fractional calculus operators are thoroughly covered as well. The integral equations and mathematical factors (model, design, formula, etc.) connected to the Wiman function are additionally in depth, along with the shortened Wiman function, STFs and SHFs. These functions produced the STFs and SHFs through the STFs and SHFs addressed the theory of the hypergeometric function. A depth discussion is provided on the integral formulas for the STFs and SHFs as well as the Laplace transfigures for the new special functions. The classical fractional calculus including the hypergeometric function is taken into consideration, along with the suggested truncated hypergeometric, STFs and SHFs. Additionally, the integral equations and mathematical models are included and connected to the hypergeometric function. Based on the presentation of STFs and SHFs, we extended these functions into the open unit disk to study some geometric properties. The formulations are suggested by a convolution differential operator of the STFs and SHFs. The proposed operator is involved in double differential inequalities, as well as integral. As a consequence, we introduce a set of results containing recent works. Our technique is indicated by the theory of subordination and superordination and then we formulated the main result as a double sides theorem.

2. Support results

Let H be the class of functions in the open unit disk $U:=\{\xi:\xi\in C \text{ and } |\xi|<1\}$ of the form:

$$f(\xi)=\xi+\alpha_2\xi^2+\alpha_3\xi^3+\cdots,$$

and let A denote the class of functions analytic in U, and usually defined by

$$f(\xi)=\xi+\sum_{m=2}^{\infty}\alpha_m\xi^m,\ \xi\in U. \tag{1}$$

Assume the analytic functions v and υ in U, then v is *subordinated* to υ if there exists a Schwarz function w, analytic in U with $w(0)=0$ and $|w|<1$ with $v(\xi)=\upsilon(w(\xi))$ for $\xi\in U$ satisfying the inequality $v(\xi)\prec\upsilon(\xi)$.

This *subordination* is equivalent to $v(0)=\upsilon(0)$ and $v(U)\subset\upsilon(U)$, specifically, if the function υ is univalent in U. Suppose that $p,\hbar\in H$ and $\Delta(r,s,t,\xi):C^3\times U\longrightarrow C$. If $p(\xi)$ and $\Delta(p(\xi),\xi p'(\xi),\xi^2p''(\xi);\ \xi)$ are univalent and if p admits the second inequality

$$\hbar(\xi)\prec\Delta(p(\xi),\xi p'(\xi),\xi^2p''(\xi);\ \xi), \tag{2}$$

consequently, p is the differential *superordination* equation's solution. Keep in mind that $\hbar$ is referred to as being superordinate to v if v is subordinate to $\hbar$.

For v,q and Δ Miller and Mocanu [3] proved the following implication:

$$\varphi(\xi)\prec\Delta(p(\xi),\xi p'(\xi),\xi^2p''(\xi);\ \xi)\Longrightarrow q(\xi)\prec p(\xi).$$

For $f\in A$ given by (1), Ali et al. [4] employed the outcomes of Bulboacă [5] found sufficient conditions on $v(\xi)\in A$ to get

$$q_1(\xi)\prec\frac{\xi v'(\xi)}{v(\xi)}\prec q_2(\xi).$$

where q_1 and q_2 are given univalent functions in U with $q_1(0)=q_2(0)=1$. A review of efforts on this direction can be found in [6-15].

We shall apply the following preliminaries in order to demonstrate our findings:

Lemma 1 [16]**:** Define the set Q of all functions f that are univalent on $\overline{U}-E(f)$, such that

$$E(f):=\left\{\vartheta\in\partial U:\lim_{\xi\to\vartheta}f(\xi)=\infty\right\}, \tag{3}$$

and are implies that, $f'(\vartheta)\neq 0$ for $\vartheta\in\partial U-E(f)$.

Lemma 2 [16]**:** Define the univalent function $q(\xi)$ in U and analytic functions θ and Δ in a domain D admitting $q(U)$ with $\Delta(w) \neq 0$ when $w \in q(U)$. Also, define

$$Q(\xi) := \xi q'(\xi)\Delta\big(q(\xi)\big) \text{ and } \hbar(z) := \theta(q(\xi)) + Q(\xi).$$

Consider that

(i) $Q(\xi)$ is starlike univalent in U, and

(ii) $Re(\frac{\xi \hbar'(\xi)}{Q(\xi)}) > 0$ for $\xi \in U$.

If

$$\theta(p(\xi)) + \xi p'(\xi)\Delta\big(p(\xi)\big) \prec \theta\big(q(\xi)\big) + \xi q'(\xi)\Delta\big(q(\xi)\big) \Rightarrow p(\xi) \prec q(\xi)$$

and $q(\xi)$ is the best dominant (BDT).

Lemma 3 [17]**:** Define a univalent function $q(\xi)$ in U and ϱ and Δ be two analytic functions in a domain D admitting $q(U)$. Let

(i) $Re(\frac{\varrho'(q(\xi))}{\Delta(q(\xi))}) > 0$ for $\xi \in U$ and

(ii) $\xi q'(\xi)\Delta(q(\xi))$ is starlike univalent in U.

If $p \in H[q(0), 1] \cap Q$, with $p(U) \subseteq D$, and $\varrho(p(\xi)) + \xi p'(\xi)\Delta(p(\xi))$ is univalent in U, also

$$\varrho\big(q(\xi)\big) + \xi q'(\xi)\Delta\big(q(\xi)\big) \prec \varrho(p(\xi)) + \xi p'(\xi)\Delta\big(p(\xi)\big) \Rightarrow q(\xi) \prec p(\xi)$$

and $q(\xi)$ is the best subordinate.

3. Convoluted operators

Let $\nu, c \neq 0 \in C$, $s \in N := \{1,2,\cdots\}$, $\nu_1, 2, \cdots, \nu_i, 2$ and $c_1, 2, \cdots, c_j, 2$ $(i, j \in N := \{1,2,\cdots\})$, the hypergeometric ${}_2Supercos_1$ function is defined by [1] and presented as:

$${}_iSupercos_j[(\nu_1, 2)\cdots,(\nu_i, 2);(c_1, 2),\cdots,(c_j, 2);\ \xi] = \sum_{m=0}^{\infty} \frac{(\nu_1)_{2m}\cdots(\nu_i)_{2m}}{(c_1)_{2m}\cdots(c_j)_{2m}} \frac{(-1)^m \xi^{2m}}{(2m)!},$$

where

$$(\partial)_m := \frac{\Gamma(\partial + m)}{\Gamma(\partial)} = \prod_{s=1}^{m} (\partial + s - 1) = \partial(\partial+1)(\partial+2),\cdots(\partial + s - 1).$$

By taking into account the ${}_iSupercos_j$ function; we present the following complicated operator. To accomplish our objective, we have used the $m-$ derivatives $(m \in \mathbb{Z} := \{\dots,-1,0,1,\dots\})$ as follows:

$$\left[{}_iSupercos_j[(\nu_1, 2)\cdots,(\nu_i, 2);(c_1, 2),\cdots,(c_j, 2);\ \xi]\right]^1 = \sum_{m=0}^{\infty} \frac{(\nu_1)_{2m}\cdots(\nu_i)_{2m}}{(c_1)_{2m}\cdots(c_j)_{2m}} \frac{2m(-1)^m}{(2m)!} \xi^{2m-1}$$

$$\left[{}_iSupercos_j[(\nu_1, 2)\cdots,(\nu_i, 2);(c_1, 2),\cdots,(c_j, 2);\ \xi]\right]^2 = \sum_{m=0}^{\infty} \frac{(\nu_1)_{2m}\cdots(\nu_i)_{2m}}{(c_1)_{2m},\cdots,(c_j)_{2m}} \frac{2m(2m-1)(-1)^m}{(2m)!} \xi^{2m-2}$$

$$\vdots$$

$$\left[{}_iSupercos_j[(\nu_1, 2)\cdots,(\nu_i, 2);(c_1, 2),\cdots,(c_j, 2);\ \xi]\right]^m = \sum_{m=0}^{\infty} \left(\frac{(\nu_1)_{2m}\cdots(\nu_i)_{2m}}{(c_1)_{2m}\cdots(c_j)_{2m}}\right) \times \left(\frac{2m(2m-1)\dots(2m-(m-1))(-1)^m}{(2m)!}\right) \xi^m.$$

Given two functions f of the form (1) and $\hbar$ defined by

$$\hbar(\xi) = \xi + \sum_{m=2}^{\infty} \beta_m \xi^m,$$

the Hadamard product (or *convolution*) of $f(\xi)$ and $\hbar(\xi)$ given by

$$(f * \hbar)(\xi) = \xi + \sum_{m=2}^{\infty} \alpha_m \beta_m \xi^m.$$

We proceed to formulate the linear convolution operator ${}_i\Lambda_j$ as follows:

$$ {}_i\Lambda_j\left[[(\nu_l,2_l)_{1,i};(c_l,2_l)_{1,j};\ \xi\right] = \left[{}_iSupercos_j[(\nu_1,2)\cdots,(\nu_i,2);(c_1,2),\cdots,(c_j,2);\ \xi]\right]^m * \Delta(\xi) = \sum_{m=0}^{\infty}\frac{(\nu_1)_{2m}\cdots(\nu_i)_{2m}}{(c_1)_{2m}\cdots(c_j)_{2m}}\xi^m, \qquad (4)$$

where, $\Delta(\xi):=\sum_{m=0}^{\infty}\frac{(2m)!}{2m(2m-1)\ldots(2m-(m-1))(-1)^m}\xi^m.$

Remark 1. By using the dimidiation formulas

$$(z)_{2m} = 2^{2m}\left(\frac{z}{2}\right)_m\left(\frac{1+z}{2}\right)_m.$$

The operator (4) becomes

$$ {}_i\Lambda_j\left[[(\nu_l,2_l)_{1,i};(c_l,2_l)_{1,j};\ \xi\right] = \sum_{m=0}^{\infty}\frac{(\nu_1)_{2m}\cdots(\nu_i)_{2m}}{(c_1)_{2m}\cdots(c_j)_{2m}}\xi^m$$

$$=\sum_{m=0}^{\infty}\frac{2^{2m}\left(\frac{\nu_1}{2}\right)_m\left(\frac{1+\nu_1}{2}\right)_m\cdots 2^{2m}\left(\frac{\nu_i}{2}\right)_m\left(\frac{1+\nu_i}{2}\right)_m}{2^{2m}\left(\frac{c_1}{2}\right)_m\left(\frac{1+c_1}{2}\right)_m\cdots 2^{2m}\left(\frac{c_j}{2}\right)_m\left(\frac{1+c_j}{2}\right)_m}\xi^m$$

$$=\sum_{m=0}^{\infty}(2^{2m})^{i-j}\left(\frac{\left(\frac{\nu_1}{2}\right)_m\left(\frac{1+\nu_1}{2}\right)_m\cdots\left(\frac{\nu_i}{2}\right)_m\left(\frac{1+\nu_i}{2}\right)_m}{\left(\frac{c_1}{2}\right)_m\left(\frac{1+c_1}{2}\right)_m\cdots\left(\frac{c_j}{2}\right)_m\left(\frac{1+c_j}{2}\right)_m}\right)\xi^m$$

$$=\sum_{m=0}^{\infty}(2^{2m})^{i-j}\left(\frac{(V_1)_m\left(\frac{1}{2}+V_1\right)_m\cdots(V_i)_m\left(\frac{1}{2}+V_i\right)_m}{(C_1)_m\left(\frac{1}{2}+C_1\right)_m\cdots(C_j)_m\left(\frac{1}{2}+C_j\right)_m}\right)\xi^m$$

$$=\left[\sum_{m=0}^{\infty}\left(\frac{(V_1)_m\cdots(V_i)_m}{(C_1)_m\cdots(C_j)_m}\right)\frac{\xi^m}{m!}\right] * \left[\sum_{m=0}^{\infty}\Gamma(m+1)(4^m)^{i-j}\left(\frac{\left(\frac{1}{2}+V_1\right)_m\cdots\left(\frac{1}{2}+V_i\right)_m}{\left(\frac{1}{2}+C_1\right)_m\cdots\left(\frac{1}{2}+C_j\right)_m}\right)\xi^m\right]$$

$:=W(\xi)*\mathrm{E}(\xi),$

where $V_i=\frac{\nu_i}{2}$, $C_j=\frac{c_j}{2}$ $(i,j\in N:=\{1,2,\cdots\})$, and $W(\xi)$ indicates the Fox Write-function. Therefore, our next operator is a generalization to the well-known Fox-Write operator.

Now, a shifted step on (4), we present the following normalized linear operator:

$$\mho_{i,j}[(\nu_l,2_l)_{1,i};(c_l,2_l)_{1,j};\ \xi] \coloneqq \xi\left({}_i\Lambda_j[(\nu_1,\cdots\nu_i);(c_1,\cdots,c_j);\ \xi]\right) = \sum_{m=1}^{\infty}\frac{(\nu_1)_{2(m-1)}\cdots(\nu_i)_{2(m-1)}}{(c_1)_{2(m-1)}\cdots(c_j)_{2(m-1)}}\xi^m = \xi+\sum_{m=2}^{\infty}\Psi_{\nu,c}\ \frac{\prod_{l=1}^{i}\Gamma(\nu_i+2(m-1))}{\prod_{l=1}^{j}\Gamma(c_j+2(m-1))}\xi^m \qquad (5)$$

where

$$\Psi_{\nu,c} = \left(\prod_{l=1}^{i} \Gamma(\nu_l)\right)^{-1} \left(\prod_{l=1}^{j} \Gamma(c_l)\right).$$

Poonam et al. [18] presented a multiplier operator of order k ($k \in \mathbb{Z} \coloneqq \{\dots, -1, 0, 1, \dots\}$) in terms of univalent function by the following form:

$$G_{\gamma,\mathscr{s}}^{k} f(\xi) = \xi + \sum_{m=2}^{\infty} \left[1 + \frac{(m-1)\gamma}{\mathscr{s}+1}\right]^{k} a_m\, \xi^m, \tag{6}$$

where $f(\xi)$ given by (1) and $\mathscr{s} > -1$, and $\gamma > 0$.

Now, let introduce the differential operator $M_{i,j}^{k,\gamma,\mathscr{s}}[(\nu_l, 2)_{1,i}; (c_l, 2)_{1,j};\ \xi]: A \to A$ of supertrigonometric function by:

$M_{i,j}^{k,\gamma,\mathscr{s}}[(\nu_l, 2)_{1,i}; (c_l, 2)_{1,j}]\, f(\xi) =$

$$\xi + \Psi_{\nu,c} \sum_{m=2}^{\infty} \left[1 \frac{(m-1)\gamma}{\mathscr{s}+1}\right]^{k} \frac{\prod_{l=1}^{i} \Gamma(\nu_1 + 2(m-1)) \cdots \Gamma(\nu_i + 2(m-1))}{\prod_{l=1}^{j} \Gamma(c_1 + 2(m-1)) \cdots \Gamma(c_j + 2(m-1))}\, a_m\, \xi^m.$$

(7)

For convenient we denote

$$M_{i,j}^{k,\gamma,\eta}[\nu_l] f(\xi) = M_{i,j}^{k,\gamma,\eta}[(\nu_l, 2)_{1,i}; (c_l, 2)_{1,j}] f(\xi). \tag{8}$$

Remark 2: [19-27] The operator $M_{i,j}^{k,\gamma,\mathscr{s}}[\nu_l]$ is a generalization of the several known operators which are exhibited by:

(i) $M_{j,j}^{k,\gamma,\mathscr{s}}[c_l] f(\xi) = J_{\gamma,\mathscr{s}}^{k} f(\xi)$ $\quad (k \in N_0,\ \textit{Cătas operator})$.

(ii) $M_{j,j}^{k,\gamma,0}[c_l] f(\xi) = D_{\gamma}^{k} f(\xi)$ $\quad (k \in N_0,\ \textit{Al} - \textit{Oboudi operator})$.

(iii) $M_{j,j}^{k,1,0}[c_l] f(\xi) = D_{1}^{k} f(\xi)$ $\quad (k \in N_0,\ \textit{Sălăgean operator})$.

(iv) $M_{j,j}^{k,\gamma,\gamma}[c_l] f(\xi) = M_{\gamma}^{k} f(\xi)$ $\quad (k \in N_0,\ \textit{Swamy operator})$.

(v) $M_{j,j}^{k,\gamma,\nu+\gamma-1}[c_l] f(\xi) = J_{\mathscr{s}}^{k} f(\xi)$ $\quad (k \in N_0,\ \textit{Swamy operator})$.

(vi) $M_{j,j}^{k,1,\mathscr{s}}[c_l] f(\xi) = J_{\mathscr{s}}^{k} f(\xi)$ $\quad (k \in N_0,\ \textit{Cho and Srivastava operator})$.

(vii) $M_{j,j}^{-u,1,\mathscr{s}}[c_l] f(\xi) = J_{u,\mathscr{s}} f(\xi)$ $\quad (u \in Z^{+}, \textit{Srivastava and Attiya})$.

(viii) $M_{j,j}^{-1,1,\mathscr{s}}[c_l] f(\xi) = J_{\mathscr{s}} f(\xi)$ $\quad (\textit{Jung et al.})$.

(ix) $(M_{j,j}^{-1,1,\mathscr{s}}[c_l] f(\xi))' = M_{\mathscr{s}} f(\xi)$ $(\textit{Cho and Kim operator})$, beside that

(x) $M_{i,j}^{0,\gamma,\mathscr{s}}[\nu_l] f(\xi)$ is a linear operator considering a special form of the Dziok and Srivastava operator.

From (8), we have:

Proposition 1: For all $f \in A$, we obtain

$$\frac{\gamma}{\mathscr{s}+1} \xi (M_{i,j}^{k,\gamma,\mathscr{s}}[\nu_l] f(\xi))' + \left(1 - \frac{\gamma}{\mathscr{s}+1}\right) M_{i,j}^{k,\gamma,\mathscr{s}}[\nu_l] f(\xi) = M_{i,j}^{k+1,\gamma,\mathscr{s}}[\nu_1, c_1] f(\xi).$$

Proof. By applying (8), we get

$$\frac{\gamma}{\mathscr{s}+1} \xi (M_{i,j}^{k,\gamma,\mathscr{s}}[\nu_l] f(\xi))' = \xi + \sum_{m=2}^{\infty} \left(1 + \frac{(m-1)\gamma}{\mathscr{s}+1}\right) \left[1 + \frac{(m-1)\gamma}{\mathscr{s}+1}\right]^{k} \Upsilon_{\nu_l, c_l}^{m}\, a_m\, \xi^m$$

$$-\left(1 - \frac{\gamma}{\mathscr{s}+1}\right) \left(\xi + \sum_{m=2}^{\infty} \left[1 + \frac{(m-1)\gamma}{\mathscr{s}+1}\right]^{k} \Upsilon_{\nu_l, c_l}^{m}\, a_m\, \xi^m\right)$$

$$= \frac{\gamma}{\mathscr{s}+1} \left(\xi + \sum_{m=2}^{\infty} m \left[1 + \frac{(m-1)\gamma}{\mathscr{s}+1}\right]^{k} \Upsilon_{\nu_l, c_l}^{m} a_m\, \xi^m\right),$$

where $\Upsilon^{m}_{\nu_l,c_l} = \Psi_{\nu,c}\left(\frac{\prod_{l=1}^{i}\Gamma(\nu_1+2(m-1))\cdots\Gamma(\nu_i+2(m-1))}{\prod_{l=1}^{j}\Gamma(c_1+2(m-1))\cdots\Gamma(c_j+2(m-1))}\right)$ and $\Psi_{\nu,c}$ is given by (5).

Proposition 2: For all $f \in A$ and $l = 1, \ldots, i$, we obtain

$$\xi(M_{i,j}^{k,\gamma,s}[\nu_l]f(\xi))' = \left(\frac{\nu_l}{2}\right)M_{i,j}^{k,\gamma,s}[\nu_l+1]f(\xi) - \left(\frac{\nu_l-2}{2}\right)M_{i,j}^{k,\gamma,s}[\nu_l]f(\xi).$$

The main object here is to make use of a method found by the theory of differential subordination involving the operator $M_{i,j}^{k,\gamma,s}[\nu_l]$. The operator is specified in (8) and obtains adequate requirements for some normalized analytic functions $f(\xi) \neq 0$ and f to meet the condition

$$q_1(\xi) \prec \left(\frac{M_{i,j}^{k,\gamma,s}[\nu_l]f(\xi)}{\xi}\right)^{\omega} \prec q_2(\xi) \quad (\omega \in C;\ \omega \neq 0 \text{ and } \xi \in U), \tag{9}$$

where $q_1(\xi)$ and $q_2(\xi)$ are univalent in U such that $q_1(0) = q_2(0) = 1$. Further, several outcomes and exceptional cases are illustrated in the following.

4. Applications of double differential inequalities

In this section, several findings on the subordination between analytical functions on the open unit disk U are introduced beginning with Theorem 1, which is an application to Lemma 2.

Theorem 1: Define the univalent function $q(\xi)$ with $q(\xi) \neq 0$ and $\frac{\xi q'(\xi)}{q(\xi)}$ is starlike univalent function in U. For $\omega, \zeta, \mu, \sigma \in C$ and $\omega, \sigma \neq 0$, let

$$Re\left\{1 + \frac{\xi q''(\xi)}{q'(\xi)} - \frac{\xi q'(\xi)}{q(\xi)} + \frac{\zeta}{\sigma}q(\xi) + \frac{2\mu}{\sigma}(q(\xi))^2\right\} > 0 \quad (\xi \in U), \tag{10}$$

and

$$\Theta_{\gamma,s}^{k}(\nu_l,\omega,\zeta,\mu,\sigma,f)(\xi) := \tau + \zeta\left(\frac{M_{i,j}^{k+1,\gamma,s}[\nu_l]f(\xi)}{\xi}\right)^{\omega} + \mu\left(\frac{M_{i,j}^{k+1,\gamma,s}[\nu_l]f(\xi)}{\xi}\right)^{2\omega}$$
$$+\sigma\omega\frac{(s+1)}{\gamma}\left[\frac{M_{i,j}^{k+2,\gamma,s}[\nu_l]f(\xi)}{M_{i,j}^{k+1\gamma,s}[\nu_l]f(\xi)} - 1\right]. \tag{11}$$

If the following subordination condition is fulfilled by q:

$$\Theta_{\gamma,s}^{k}(\nu_l,\omega,\zeta,\mu,\sigma,f)(\xi) \prec \tau + \zeta q(\xi) + \mu(q(\xi))^2 + \sigma\frac{\xi q'(\xi)}{q(\xi)}. \tag{12}$$

Then

$$\left(\frac{M_{i,j}^{k+1,\gamma,s}[\nu_l]f(\xi)}{\xi}\right)^{\omega} \prec q(\xi), \tag{13}$$

and $q(\xi)$ is the BDT.

Proof. Assume that $\rho(\xi)$ is formulated by

$$\rho(\xi) := \left(\frac{M_{i,j}^{k+1,\gamma,s}[\nu_l]f(\xi)}{\xi}\right)^{\omega}, \quad (\xi \in U;\ \xi \neq 0; f \in A),$$

such that

$$\xi\rho'(\xi) := \omega\,\xi\left(\frac{M_{i,j}^{k+1,\gamma,s}[\nu_l]f(\xi)}{\xi}\right)^{\omega-1} \times \left\{\frac{\xi(M_{i,j}^{k+1,\gamma,s}[\nu_l]f(\xi))' - M_{i,j}^{k+1,\gamma,s}[\nu_l]f(\xi)}{\xi^2}\right\}.$$

By a straightforward computation, we obtain the following relation:

$$\frac{\xi\rho'(\xi)}{\rho(\xi)} = \omega\left[\frac{\xi(M_{i,j}^{k+1,\gamma,s}[\nu_l]f(\xi))'}{M_{i,j}^{k+1,\gamma,s}[\nu_l]f(\xi)} - 1\right]$$

and by applying the identity equation in Proposition 1, we obtain

$$\frac{\xi\rho'(\xi)}{\rho(\xi)}=\frac{\omega}{\gamma}\left[(s+1)\frac{M_{i,j}^{k+2,\gamma,s}[\nu_l]f(\xi)}{M_{i,j}^{k+1,\gamma,s}[\nu_l]f(\xi)}-(s+1)\right].$$

Depending on the following setting:

$$\theta(w):=\tau+\zeta w+\mu w^2 \quad \text{and} \quad \phi(w):=\frac{\sigma}{w},$$

it is simple to confirm that θ and ϕ are analytic in C and $C\backslash\{0\}$ accordingly and that $\phi(w)\neq 0$ $(w\neq 0)$. Additionally, define

$$Q(\xi):=\xi q'(\xi)\,\phi(q(\xi))=\sigma\frac{\xi q'(\xi)}{q(\xi)}$$

such that

$$\hbar(z):=\theta(q(\xi))+Q(\xi)=\tau+\zeta q(\xi)+\mu(q(\xi))^2+\sigma\frac{\xi q'(\xi)}{q(\xi)}.$$

Obviously, $Q(\xi)$ is starlike in U, and

$$R\left\{\frac{\xi\hbar'(\xi)}{Q(\xi)}\right\}=R\left\{1+\frac{\xi q''(\xi)}{q'(\xi)}-\frac{\xi q'(\xi)}{q(\xi)}+\frac{\zeta}{\sigma}q(\xi)+2\frac{\mu}{\sigma}(q(\xi))^2\right\}>0.$$

By applying Lemma 1, assertion (13) of Theorem 1 is attended.

Now, by taking $q(\xi)=\frac{1+A\xi}{1+B\xi},-1\leq B<A\leq 1$ and $q(\xi)=\frac{1+\xi}{1-\xi}$, in Theorem 1 with an application of Lemma 2, we get:

Corollary 1: For $\omega,\zeta,\mu,\sigma\in$ C such that $\omega,\sigma\neq 0$, if $f\in A$, (10) holds true and

$$\tau+\zeta\left(\frac{M_{i,j}^{k+1,\gamma,s}[\nu_l]f(\xi)}{\xi}\right)^{\omega}+\mu\left(\frac{M_{i,j}^{k+1,\gamma,s}[\nu_l]f(\xi)}{\xi}\right)^{2\omega}$$
$$+\sigma(s+1)\frac{\omega}{\gamma}\left[\frac{M_{i,j}^{k+2,\gamma,s}[\nu_l]f(\xi)}{M_{i,j}^{k+1\gamma,s}[\nu_l]f(\xi)}-1\right]\prec\tau+\zeta\frac{1+A\xi}{1+B\xi}+\mu\left(\frac{1+A\xi}{1+B\xi}\right)^2+\sigma\frac{(A-B)\xi}{(1+A\xi)(1+B\xi)},$$

then

$$\left(\frac{M_{i,j}^{k+1,\gamma,s}[\nu_l]f(\xi)}{\xi}\right)^{\omega}\prec\frac{1+A\xi}{1+B\xi},\quad(\omega\in C;\ \omega\neq 0) \tag{14}$$

and $\frac{1+A\xi}{1+B\xi}$ is the BDT.

Also, by consuming $q(\xi)=\left(\frac{1+\xi}{1-\xi}\right)^d,0<d\leq 1$, in Theorem 1, we have:

Corollary 2: For $\omega,\zeta,\mu,\sigma\in C$ such that $\omega,\sigma\neq 0$, if $f\in A$, (10) holds true and

$$\tau+\zeta\left(\frac{M_{i,j}^{k+1,\gamma,s}[\nu_l]f(\xi)}{\xi}\right)^{\omega}+\mu\left(\frac{M_{i,j}^{k+1,\gamma,s}[\nu_l]f(\xi)}{\xi}\right)^{2\omega}$$
$$+\sigma(s+1)\frac{\omega}{\gamma}\left[\frac{M_{i,j}^{k+2,\gamma,s}[\nu_l]f(\xi)}{M_{i,j}^{k+1\gamma,s}[\nu_l]f(\xi)}-1\right]\prec\tau+\zeta\left(\frac{1+\xi}{1-\xi}\right)^d+\mu\left(\frac{1+\xi}{1-\xi}\right)^{2d}+\sigma\frac{2d\xi}{(1-\xi^2)}.$$

then

$$\left(\frac{M_{i,j}^{k+1,\gamma,s}[\nu_l]f(\xi)}{\xi}\right)^{\omega}\prec\left(\frac{1+\xi}{1-\xi}\right)^d,\quad(\omega\in C;\ \omega\neq 0) \tag{15}$$

and $q(\xi)=\left(\frac{1+\xi}{1-\xi}\right)^d$ is the BDT.

Further, if $(\xi)=\frac{1}{(1-\xi)^{2d}}$ $(d\in C\backslash\{0\}),k=0;$ $c_l=1$ $(l=1,\cdots,j)$, $\nu_l=1$ $(l=1,\cdots,i),\mu=\zeta=0$, $\omega=\tau=1$ and $\sigma=\frac{1}{d}$, in Theorem 1, then the following result is what Srivastava and Lashin [28] get:

Corollary 3: For $\omega,\zeta,\mu,\sigma\in C$ such that $\omega,\sigma\neq 0$ and $0<d\leq 1$, if $f\in A$, (10) holds true and

$$1+\frac{1}{d}\frac{\xi f''(\xi)}{f'(\xi)} \prec \frac{1+\xi}{1-\xi},$$

then

$$f'(\xi) \prec \frac{1}{(1-\xi)^{2d}},$$

and $(1-\xi)^{2\mathrm{d}}$ is the BDT.

In the next outcome, we establish Theorem 2 using justifications similar to those given in Theorem 1 with the equation in Proportion 2.

Theorem 2: Define the univalent function $q(\xi)$ such that $q(\xi) \neq 0$ and $\frac{\xi q'(\xi)}{q(\xi)}$ is starlike in U. For $\omega, \zeta, \mu, \sigma \in C$, $\omega, \sigma \neq 0$ and $l = 1, \dots, i$, let

$$Re\left\{1+\frac{\xi q''(\xi)}{q'(\xi)}-\frac{\xi q'(\xi)}{q(\xi)}+\frac{\zeta}{\sigma}q(\xi)+\frac{2\mu}{\sigma}(q(\xi))^2\right\}>0, \tag{16}$$

and

$$\chi^k_{\gamma,s}(\nu_l,\omega,\zeta,\mu,\sigma,f)(\xi) := \tau+\zeta\left(\frac{M^{k,\gamma,s}_{i,j}[\nu_l+1]f(\xi)}{\xi}\right)^{\omega}+\mu\left(\frac{M^{k,\gamma,s}_{i,j}[\nu_l+1]f(\xi)}{\xi}\right)^{2\omega}$$

$$+\sigma(\nu_l+1)\frac{\omega}{2_l}\left[\frac{M^{k,\gamma,\eta}_{i,j}[\nu_l+2]f(\xi)}{M^{k+1,\gamma,\eta}_{i,j}[\nu_l+1]f(\xi)}-1\right]. \tag{17}$$

If the following subordination condition is fulfilled by q:

$$\chi^k_{\gamma,s}(\nu_l,\omega,\zeta,\mu,\sigma,f)(\xi) \prec \tau+\zeta q(\xi)+\mu(q(\xi))^2+\sigma\frac{\xi q'(\xi)}{q(\xi)}.$$

Then

$$\left(\frac{M^{k,\gamma,s}_{i,j}[\nu_l+1]f(\xi)}{\xi}\right)^{\omega} \prec q(\xi), \quad (\omega \in C;\ \ \omega \neq 0), \tag{18}$$

and $q(\xi)$ is the BDT.

Remark 3: We point out that Theorem 2 can be reformulated for several choices of the function q as are in Theorem 1.

Next, we prove some of the *superordinations* results by applying to Lemma 3:

Theorem 3: Define the univalent function q in U as follows*:* $q(\xi) \neq 0$ and $\frac{q'(\xi)}{q(\xi)}$ to be starlike in U. Assuming that

$$Re\left\{\frac{\zeta}{\sigma}q(\xi)+\frac{2\mu}{\sigma}(q(\xi))^2\right\}>0, \tag{19}$$

for all ζ, σ, μ in C and $\sigma \neq 0$, if $f \in A$,

$$0 \neq \left(\frac{M^{k+1,\gamma,s}_{i,j}[\nu_l]f(\xi)}{\xi}\right)^{\omega} \in Q \cap H[q(0),1] := \nabla,$$

and $\Theta^k_{\gamma,s}(\nu_l,\omega,\zeta,\mu,\sigma,f)$ is univalent in U, then

$$\tau+\zeta q(\xi)+\mu(q(\xi))^2+\sigma\frac{\xi q'(\xi)}{q(\xi)} \prec \Theta^k_{\gamma,s}(\nu_l,\omega,\zeta,\mu,\sigma,f).$$

Implies that,

$$q(\xi) \prec \left(\frac{M^{k+1,\gamma,s}_{i,j}[\nu_l]f(\xi)}{\xi}\right),^{\omega} \quad (\omega \in C; \omega \neq 0) \tag{20}$$

and the q is the best subordinate where $\Theta^k_{\gamma,s}(\nu_l,\omega,\zeta,\mu,\sigma,f)$ is defined by (11).

Proof. By letting

$$\kappa(w) := \tau+\zeta w+\mu w^2 \quad \text{and} \quad \phi(w) := \frac{\sigma}{w},$$

then κ, ϕ are analytic in C and $C\backslash\{0\}$, respectively, with $\phi(w) \neq 0$ $(w \in C\backslash\{0\})$, since q is convex then

$$Re\frac{\kappa'(q(\xi))}{\phi(q(\xi))} = Re\left\{\frac{\zeta}{\sigma}q(\xi) + \frac{2\mu}{\sigma}(q(\xi))^2\right\} > 0,$$

for all ζ, σ, μ in C and $\sigma \neq 0$. The assertion (20) of Theorem 3, follows by an application of Lemma 3.

Theorem 4: Define the univalent function q in U with $q(\xi) \neq 0$ and $\frac{q'(\xi)}{q(\xi)}$ to be starlike univalent in U. Assuming that

$$Re\left\{\frac{\zeta}{\sigma}q(\xi) + \frac{2\mu}{\sigma}(q(\xi))^2\right\} > 0, \tag{21}$$

for all ζ, σ, μ in C and $\sigma \neq 0$, if $f \in A$,

$$0 \neq \left(\frac{M_{i,j}^{k,\gamma,s}[\nu_l + 1]f(\xi)}{\xi}\right)^{\omega} \in \nabla,$$

where $\nabla = [q(0), 1] \cap \phi$ and $\chi_{\gamma,s}^{k}(\nu_l, c_l, \omega, \zeta, \mu, \sigma, f)$ is univalent function in U, then

$$\tau + \zeta q(\xi) + \mu(q(\xi))^2 + \sigma\frac{\xi q'(\xi)}{q(\xi)} \prec \chi_{\gamma,s}^{k}(\nu_l, \omega, \zeta, \mu, \sigma, f)$$

implies that,

$$q(\xi) \prec \left(\frac{M_{i,j}^{k,\gamma,s}[\nu_l+1]f(\xi)}{\xi}\right),^{\omega} \quad (\omega \in C; \omega \neq 0) \tag{22}$$

and the q is the best subordinate where $\chi_{\gamma,s}^{k}(\nu_l, \omega, \zeta, \mu, \sigma, f)$ is defined by (17).

The following differential sandwich theorem is created by merging the above results. By combining Theorem 1 and Theorem 3, we introduce:

Theorem 5: Define two univalent functions q_1 and q_2 in U with $q_1(\xi) \neq 0$ and $q_2(\xi) \neq 0$, $(\xi \in U)$ such that $\frac{\xi q_1'(\xi)}{q_1(\xi)}$ and $\frac{\xi q_2'(\xi)}{q_2(\xi)}$ to be starlike. Suppose q_1 and q_2 achieve (19) and (10) respectively. If $f \in A$,

$$0 \neq \left(\frac{M_{i,j}^{k+1,\gamma,s}[\nu_l]f(\xi)}{\xi}\right)^{\omega} \in \nabla,$$

and $\Theta_{\gamma,s}^{k}(\nu_l, \omega, \zeta, \mu, \sigma, f)$ is univalent in U, then

$$\tau + \zeta q_1(\xi) + \mu(q_1(\xi))^2 + \sigma\frac{\xi q_1'(\xi)}{q_1(\xi)} \prec \Theta_{\gamma,s}^{k}(\nu_l, \omega, \zeta, \mu, \sigma, f)$$
$$\prec \tau + \zeta q_2(\xi) + \mu(q_2(\xi))^2 + \sigma\frac{\xi q_2'(\xi)}{q_2(\xi)},$$

for $\zeta, \mu, \omega, \sigma \in C$, $\omega, \sigma \neq 0$ implies

$$q_1(\xi) \prec \left(\frac{M_{i,j}^{k+1,\gamma,s}[\nu_l]f(\xi)}{\xi}\right)^{\omega} \prec q_2(\xi)$$

and q_1 and q_2 are coordinately the best subordinate and dominant.
Moreover, we have the next double sides result by using Theorem 2 and Theorem 4.

Theorem 6: Define the two univalent functions q_1 and q_2 in U having the properties $q_1(\xi) \neq 0$ and $q_2(\xi) \neq 0$, $(\xi \in U)$ such that $\frac{\xi q_1'(\xi)}{q_1(\xi)}$ and $\frac{\xi q_2'(\xi)}{q_2(\xi)}$ are starlike. Consume q_1 and q_2 admit (21) and (16) correspondingly. If $f \in A$,

$$0 \neq \left(\frac{M_{i,j}^{k,\gamma,s}[\nu_l + 1]f(\xi)}{\xi}\right)^{\omega} \in \nabla,$$

and $\chi_{\gamma,s}^{k}(\nu_l,\omega,\zeta,\mu,\sigma,f)$ is univalent in U, then

$$\tau+\zeta q_1(\xi)+\mu(q_1(\xi))^2+\sigma\frac{\xi q_1'(\xi)}{q_1(\xi)}\prec\chi_{\gamma,s}^{k}(\nu_l,\omega,\zeta,\mu,\sigma,f)$$
$$\prec\tau+\zeta q_2(\xi)+\mu(q_2(\xi))^2+\sigma\frac{\xi q_2'(\xi)}{q_2(\xi)},$$

for $\zeta,\mu,\omega,\sigma\in C$, $\omega,\sigma\neq 0$ implies

$$q_1(\xi)\prec\left(\frac{M_{i,j}^{k,\gamma,s}[\nu_l+1]f(\xi)}{\xi}\right)^{\omega}\prec q_2(\xi)$$

and q_1 and q_2 are correspondingly the best subordinate and dominant.

5. Conclusions

We defined and studied aspects of subordination applications for the convoluted operator presented in (8) as the differential of a super-trigonometric function using notions of differential subordination and superordination. The original theorems established inequalities, which give the best subordinate and dominant and intriguing corollaries, correspondingly. As a consequence, sandwich results are presented. In investigations using geometric theories, the convoluted operator of differential of a super trigonometric function produces good results; it may be used to propose new classes of analytic functions. Theorems relating to neighborhood, the radii starlikeness or convexity, closure theorems, distortion theorems, and coefficient estimations can all be examined.

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