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Boundary Exponential Gradient Reduced Order Detectability in Neumann Conditions

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Abstract

This work, aims to study and examine the description of the gradient reduced order-strategic sensors of type boundary exponential ($\partial\Theta^*{}_{EGRO}$-strategic sensors) for completion gradient reduced order-detectability of type boundary exponential ($\partial\Theta^*{}_{EGRO}$-detectability). Thus, this concept is linked to an estimator in distributed parameter systems (DPSs) in Neumann problem. So, we present numerous consequences regarding to diverse kinds of information, region Θ^* and conditions of boundary region to allow existence of $\partial\Theta^*{}_{EGRO}$-detectable systems. In addition, we have estimated at the junction interface that the interior solution is harmonized with the exterior solution for $\partial\Theta^*{}_{EGRO}$-detectable and, we give the relationship between this concept and sensors structures. Finally, we demonstrate some applications with many circumstances of sensor positions.

Keywords: Strategic sensors, $\partial\Theta^*{}_{EGRO}$-detectability, estimator, Neumann conditions.

انحدار القابلية على الاكتشاف ذات الرتبة المخفضة الاسية الحدودية في شروط نيومان

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الخلاصة

هذا العمل يهدف لدراسة واثبات وصف المجسات الاستراتيجية ذات الرتبة المخفضة للانحدار من النوع الاسي الحدودي (المجسات الاستراتيجية$-\partial\Theta^*{}_{EGRO}$) لأجل انجاز انحدار القابلية على الاكتشاف ذات الرتبة المخفضة من النوع الأسية الحدودية (القابلية على الاكتشاف$\partial\Theta^*{}_{EGRO}-$). وعلية هذا المفهوم يرتبط بالمخمن في الانظمة ذات البارامترات التوزيعية (DPSs) في مسالة نيومان. ثم قدمنا العديد من النتائج المهمة والتي تتعلق بأصناف متنوعة من المعلومات والمناطق Θ^* والشروط لمنطقة حدودية والتي تسمح بوجود انظمة القابلة للاكتشاف من النوع $\partial\Theta^*{}_{EGRO}$. بالإضافة الى ذلك, تم تخمين انحدار القابلية على الاكتشاف من النوع $\partial\Theta^*{}_{EGRO}$ بالاعتماد على الحل الداخلي المتوافق مع الحل الخارجي لتلك المنطقة وكذلك برهنا العلاقة بين

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القابلية على الاكتشاف وتركيبات المجسات. واخيرا اثبتنا بعض التطبيقات المعتمدة على العديد من حالات مواقع المجسات.

1. Introduction

The important issue in connection with detecting notion for usual case in linear or nonlinear DPS$_S$ has received a lot of consideration, see [1-2] and references therein. Therefore, the regional case of detectability notion in this systems has been developed through an extension to previous works as in [3-5]. In addition, boundary detectability [6-7] and boundary strategic sensors [8-10] have been deliberated and analyzed. On a time finite horizon, the outcomes of a boundary gradient analysis were obtained [11-12]. But we still need to expand these outcomes to problem for infinite interval [13-14]. Thus, $\partial\Theta^*_{EG}$-detectability conception which is extension of internal detectability notion that has been introduced and developed by [15], which is very important in practical application [4, 12].

This article aims to discover the notion of $\partial\Theta^*_{EGRO}$-detectability by means of the sensors selection [13-15]. The motive of studying conception, there are existent several problems in the real world need to be studied as in [16-17] and for another system types (see [18-20]). This means, one maybe devoted to the boundary gradient exponential problem of type detection in $\partial\Theta^*$ (see Figure 1) [21].

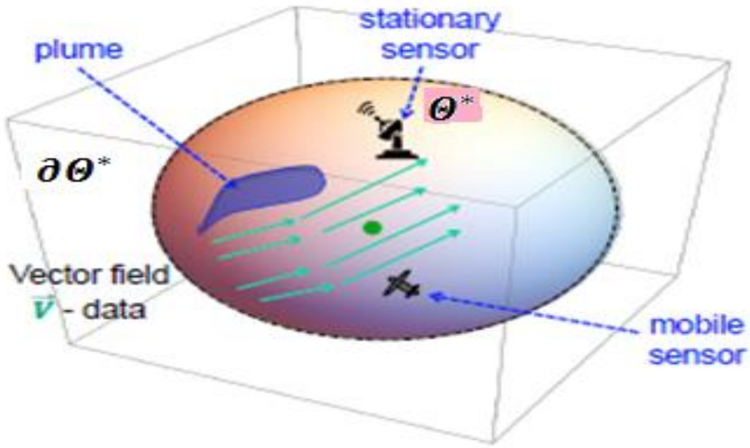

Figure 1: Static and mobile sensor system region.

The objective of this article is to explore $\partial\Theta^*_{EGRO}$-detectability conception depending on the outcomes in [22-24]. This work is planned via the following:

Firstly, we recall the class of considered system and preliminaries, with some definitions with descriptions of $\partial\Theta^*_{EG}$-observability in linking to the detectability. Section three introduces the matching of inside to the outside solution at a junction interface has been studied in the sense of Al-Saphory *et al.* [24]. Also, we discuss the connection between $\partial\Theta^*_{ERO}$-detectability of state gradient and $\partial\Theta^*_{ERO}$-estimator and, it will explain the relation of $\partial\Theta^*_{EGRO}$-detectability with structures sensors. Finally, the last section will offer some application to different circumstances of sensor locations on $\partial\Theta^*_{EGRO}$-detectability and in Neumann boundary conditions.

2. $\partial\Theta^*_{EG}$-Observability and $\partial\Theta^*_{EG}$-Detectability

This division deals with a class of parabolic system in DPS$_s$ and some preliminaries, also we come across to some definitions and descriptions which is related to elucidate the concept of $\partial\Theta^*_{EG}$-detectability and $\partial\Theta^*_{EG}$-observability in the Neumann problem in it's an extension from [2, 14].

2.1 Considered System and Preliminaries

We consider the following system

$$\begin{cases} \frac{\partial x}{\partial t}(\zeta,t) = A\,x(\zeta,t) + B\,u(t) \ in\ \partial\Theta^* \times]0,T[\\ x(\zeta,0) = x_0(\zeta) \qquad in\ \overline{\Theta^*} \\ \frac{\partial x}{\partial v}(\eta,t) = \qquad on\ \partial\Theta^* \times]0,T[\end{cases} \tag{1}$$

wherever

$$\zeta \in \Theta^*, \eta \in \partial\Theta^*;\ t\epsilon[0\,,T];$$

and

$$(\zeta,t) \in \Theta^* \times]0\,,T[, (\eta\,,t)\epsilon\,\partial\Theta^* \times]0,T[, (\xi,0) \in \overline{\Theta^*}.$$

The system (1) is increased with output function

$$y\,(.,t) = C\,x(.,t) \tag{2}$$

and provided with some details as follows:

- Θ^* stay bounded and open in $\mathcal{R}^n$, with smooth boundary $\partial\Theta^*$.
- The Hilbert spaces $X = \mathcal{H}^1(\partial\Theta^*)$ is the state space of x, $\mathbb{U} = \mathcal{L}^2(0, T, \mathcal{R}^p)$ is the control space and $\mathcal{O} = \mathcal{L}^2(0, T, \mathcal{R}^q)$ is the observation spaces are designed in this paper of separable type with p and q are the sensors; actuator numbers [1-6].
- $[0,T], T > 0$ stand to a space-time interval cylinder.
- Mapping A stratify linearity and differentiability property of order two produced a strongly continuous semigroup (SCS-group) $\left(S_A\,(t)\right), t \geq 0$ on $X = \mathcal{H}^1\,(\overline{\Theta^*})$[25].
- Mappings $B \in \mathcal{L}(\mathbb{U}, X)$ and $C \in \mathcal{L}(X, \mathcal{O})$ be determined by the construction of actuator and sensor [10-12].
- System (1) acquire only one solution [24-25] exemplified via formulae

$$x\,(\zeta\,,t) = S_A\,(t)\,x_0(\zeta) + \int_0^t S_A\,(t-\tau)\,B\,u\,(\tau)\,d\tau \tag{3}$$

- Now, we consider the bounded linear operator K is given by

$$K: x \in X \to Kx = C\,S_A(.)x\,,\ x \in \mathcal{O}$$

where $S_A(.)$ is a SCS-group product an operator A such that

$$y(t) = K(t)$$

Besides, the adjoint mapping of K designates by $K^*: \mathcal{O} \to X$, recognized by

$$y \to \int_0^T S_A^*(s)\,C^* y(s)\,ds$$

where S^* and C^* are the adjoint of S and C respectively.

- The mapping ∇ is define by

$$\begin{cases} \nabla: \mathcal{H}^1(\Theta^*) \to (\mathcal{H}^1(\Theta^*))^n \\ x \to \nabla x = \left(\frac{\partial x}{\partial \zeta_1}, \dots, \frac{\partial x}{\partial \zeta_n}\right) \end{cases}$$

with ∇^* the adjoints of ∇ assumed by:

$$\begin{cases} \nabla^*: (\mathcal{H}^1(\Theta^*))^n \to \mathcal{H}^1(\Theta^*) \\ x \to \nabla^* x = v \end{cases}$$

Whereas u is a solution of the Neumann problems

$$\begin{cases} \Delta u = -f(x) \ ; \ \Theta^* \\ \partial u/\partial v = 0 \ ; \ \Theta^* \end{cases}$$

- Thus, mapping of trace transformation of order zero class is defined in [25] via

$$\gamma_0: \mathcal{H}^1(\partial\Theta^*) \to \mathcal{H}^{1/2}(\partial\Theta^*)$$

realize continuity, surjectivity and linearity. Hence, the extending of a trace mapping wherever [24]:

$\gamma: (\mathcal{H}^{1}(\partial\Theta^*))^n \to \left(\mathcal{H}^{1/2}(\partial\Theta^*)\right)^n$

provided through γ_0^* and γ^* represent the adjoint mappings of γ_0, γ(respectively).

• Finally, we introduced the operator $H_{\partial\Theta^*_{EG}} = \chi_{\partial\Theta^*}\gamma\nabla K^*$: $\mathcal{O} \to (H^{1/2}(\partial\Theta^*))^n$ and the adjoint of this operator given by: $H^*_{\partial\Theta^*_{EG}} = K\nabla^*\gamma^*\chi^*_{\partial\Theta^*}$.

• Then, allowing to the select of the variables $\mathcal{D}_i$ and f_i, so, we diverse categories of sensors

▪ ***Case of zone:*** If $\mathcal{D}_i \subset \overline{\Theta^*}$ and $f_i \epsilon\, \mathcal{L}^2(\mathcal{D}_i)$, then, the information output (2) may be given by

$$y(t) = \int_{\mathcal{D}_i} x(\zeta, t)\, f_i(\zeta)\, d\zeta$$

▪ ***Case of pointwise:*** If $\mathcal{D}_i = \{b_i\}$ and $b_i \epsilon\, \overline{\Theta^*}$ and $f = \delta(.-b_i)$, whereas δ be Dirac mass concentrated in b [8-10]. So, equation (2) certainly defined by

$$y(t) = \int_{\Theta^*} x(\zeta,t)\, \delta_{b_i}(\zeta - b_i)\, d\zeta$$

• For linear systems, the $\partial\Theta^*_{EGRO}$-observability which is extension of [14-15]. One can reflected the devoted system to (1) and characterized by (4) by the formula.

$$\begin{cases} \frac{\partial x}{\partial t}(\zeta, t) = A\, x(\zeta, t) & \Theta^* \times]0, T[\\ x(\zeta, 0) = x_0(\zeta) & \overline{\Theta^*} \\ \frac{\partial x}{\partial v}(\eta, t) = 0 & \partial\Theta^* \times]0, T[\end{cases} \quad (4)$$

The problem solution of system (4) is described by

$$x(\zeta, t) = S_A(t) x_0(\zeta) \quad (5)$$

2.2 Descriptions and Definitions

In this subdivision, we offer several explanations and concept definitions and for observability of class exact boundary gradient ($E\,\partial\Theta^*G$-observability), and strategic sensor on $\partial\Theta^*$, which is derived from [11-12].

Definition2.1: System (4) is increased with the measurement function (2) is called observable of class exactly boundary gradient ($E\,\partial\Theta^*G$-observable), if

$$Im\, \chi_{\partial\Theta^*}\gamma\nabla K^* = \left(H^{1/2}(\partial\Theta^*)\right)^n$$

Definition2.2: System (4) is increased with the measurement function (2) is said to be weakly boundary gradient observable ($W\,\partial\Theta^*G$-obsevable)

$$\overline{Im\, \chi_{\partial\Theta^*}\gamma\nabla K^*} = \left(H^{1/2}(\partial\Theta^*)\right)^n$$

Remark2.3: So, we realize that,

$$\overline{Im\, \chi_{\partial\Theta^*}\gamma\nabla K^*} = \left(H^{1/2}(\partial\Theta^*)\right)^n \Leftrightarrow ker\, K\nabla^*\gamma^*\chi^*_{\partial\Theta^*} = \{0\}.$$

Definition2.4: Couple $(\mathcal{D}, f)$ is $\partial\Theta^*G$-strategic sensor, if a linked system is $W\partial\Theta^*G$-obsevable.

Definition2.5: Semigroup $(S_A(t))_{t\geq 0}$ is $\partial\Theta^*EG$-stable on $\left(\mathcal{H}^{1/2}(\partial\Theta^*)\right)^n$, if $\forall\, x_0 \epsilon\, \mathcal{H}^1(\overline{\Theta^*})$, then, the system solution linked with system (1) has an exponentially convergence to zero when $t \to \infty$.

• If a semigroup $(S_A(t))_{t\geq 0}$ is $\partial\Theta^*EG$-stable, then, $\forall\, x_0 \in \mathcal{H}^1(\overline{\Theta^*})$, with $x(.,t)$ linked with (1) satisfied

$$\lim_{t\to\infty} \|\gamma\nabla x(.,t)\|_{(\mathcal{H}^{1/2}(\partial\Theta^*))^n} = \lim_{t\to\infty} \|\gamma\nabla S_A(.)x_0\|_{(\mathcal{H}^{1/2}(\partial\Theta^*))^n} = 0 \quad (6)$$

Definition2.6: System (1) is $\partial\Theta^*EG$-stable, if the operator A creates a semigroup realize $\partial\Theta^*EG$-stable on $\left(\mathcal{H}^{1/2}(\partial\Theta^*)\right)^n$.

Definition2.7: System (1) is increased with the output function (2) is said to be $\partial\Theta^*EG$-detectable, if there exists an operator $H_{\partial\Theta^*EG}: R^q \to \left(\mathcal{H}^{1/2}(\partial\Theta^*)\right)^n$ such that $(A\text{-}H_{\partial\Theta^*EG}C)$ products SCS-group $(S_{H_{\partial\Theta^*EG}}(t))_{t\geq 0}$, realize $\partial\Theta^*EG$-stable.

Proposition2.8: System (1) is increased with the output function (2) is an $E\partial\Theta^*G$-observable, then it is $\partial\Theta^*EG$-detectable.

This outcomes provides the subsequent important inequality

$$\|\gamma\nabla S_A(.)x\|_{(\mathcal{H}^{1/2}(\partial\Theta^*))^n} \leq \upsilon\, \|CS_A(.)x\|_{L^2(0,\infty,\mathcal{O})}, \text{ for all } x\,\epsilon\left(\mathcal{H}^{1/2}(\partial\Theta^*)\right)^n \tag{7}$$

where $\upsilon > 0$.

Proof: We conjecture the proof of this outcome is accomplish from preceding of general observability operator $\chi_{\partial\Theta^*}\gamma\nabla K^*$. Then,

1. $Imf \subset Img.$
2. $\exists\, \upsilon > 0, \ni$

$$\|f^*x^*\|_{E^*} \leq \upsilon\, \|g^*x^*\|_{F^*},\ \forall\, x^*\epsilon G^*. \tag{8}$$

As of inequality (8):

$\exists\, M_{\partial\Theta^*EG}, \alpha_{\partial\Theta^*EG} > 0$;with $\upsilon < M_{\partial\Theta^*EG}$, such that

$$\upsilon\, \|g^*x^*\|_{F^*} \leq M_{\partial\Theta^*EG}\, e^{-\alpha_{\partial\Theta^*EG}t}\, \|x^*\|_{F^*}$$

where F^* and G^* be a reflexives spaces of type Banach and $f \in \mathcal{L}(E, G)$, $g \in L(F, G)$. If $E = G = (\mathcal{H}^{1/2}(\partial\Theta^*))^n$, $F = \mathcal{O}, f = Id_{(\mathcal{H}^{1/2}(\partial\Theta^*))^n}$ and $g = S_A^*(.)\gamma^*\nabla^*C^*$, where $S_A(.)$ is a SCS-group generates by A, which is $\partial\Theta^*EG$-stable, then it is $\partial\Theta^*EG$-detectable.

3. Boundary Exponential Gradient Reduced Order Detectability

Let us consider $X = X_1 \oplus X_2$ where X_1 and X_2 are sub-space of X. Under the assumption in [3, 6,13-14] , system (1) is decomposed via two subsystems

$$\begin{cases} \dfrac{\partial x_2}{\partial t}(\zeta,t) = A_{21}x_1(\zeta,t) + A_{22}x_2(\zeta,t) + B_2\, u(t) & \Theta^* \times]0,T[\\ x_2(\zeta,0) = x_{2_0}(\zeta) & \overline{\Theta^*} \\ \dfrac{\partial x_2}{\partial v}(\eta,t) = 0 & \partial\Theta^* \times]0,T[\end{cases} \tag{9}$$

Increased with the output function

$$y(.,t) = C\, x_1(\zeta,t). \tag{10}$$

The problem consists in constructing boundary exponential gradient reduced order estimator ($\partial\Theta^*EGRO$-estimator), that permits to calculate the anonymous component $x_2(\zeta\,,t)$. Consequently, by changing of variable in system (9) implies

$$\begin{cases} \dfrac{\partial a}{\partial t}(\zeta,t) = A_{22}\, a(\zeta,t) + [B_2u(t) + A_{21}y(\zeta,t)] & \Theta^* \times]0,T[\\ a(\zeta,0) = a_0(\zeta) & \overline{\Theta^*} \\ \dfrac{\partial a}{\partial v}(\eta,t) = 0 & \partial\Theta^* \times]0,T[\end{cases} \tag{11}$$

and with information output

$$\tilde{y}(.,t) = A_{12}\, a(\zeta,t). \tag{12}$$

Whereas a be a system state of (11) representing x_2 system state of (9).

3.1 $\partial\Theta^*{}_{EGRO}$-Reconstruction State via Internal Region

This sub-section is related with the rebuilding the flux state on $\partial\Theta^*$ for the considered problematic, so it is an extension of ref. [10-12], and then, if we consider $\bar{F}_2 \subset \overline{\Theta^*}$ [9].

- Let $R: \left(\mathcal{H}^{1/2}(\partial\Theta^*)\right)^n \longrightarrow (\mathcal{H}^1(\Theta^*))^n$, an operator satisfy continuity and linearity property [25].

$$\gamma\nabla Rh(\zeta,t) = h(\zeta\,,t), \text{ for all } h\,\epsilon\left(\mathcal{H}^{1/2}(\partial\Theta^*)\right)^n \tag{13}$$

then for all $x_0 \epsilon\ \partial\Theta^*$there exists $\mathcal{r} > 0$, with following collections:

$$F = \cup_{z_0 \in\ \partial\Theta^*} B\ (x_0, \mathcal{r}) = \{x\epsilon\ \Theta^* \text{ or } x \in F_1 : \|x - x_0\| < \mathcal{r},\ x_0 \epsilon\ \partial\Theta^*\}$$

where

$$F_1 = \cup_{x_0 \in\ \partial\Theta^*} B(x_0, \mathcal{r}) = \{x\epsilon\ F \text{ and } x \notin\ \Theta^* : \|x - x_0\| < \mathcal{r},\ x_0 \epsilon\ \partial\Theta^*\}$$

and

$$F_2 = \cup_{x_0 \in \partial\Theta^*} B(x_0, \mathcal{r}) = \{x\epsilon\ F \text{ and } x \notin F_1 : \|x - x_0\| < \mathcal{r},\ x_0 \epsilon\ \partial\Theta^*\} \subset\ \Theta^*$$

and then, we have

$$\boldsymbol{F} = \boldsymbol{F_1} \cup \boldsymbol{F_2},\ \ \boldsymbol{\partial\Theta}^* = F_1 \cap \bar{F}_2 \text{ and } F_2 = F \cap\ \Theta^*$$

wherever $B\ (x_0, \mathcal{r})$ represents open sets collection cover $\partial\Theta^*$ region **(**see Figure 2**)**.

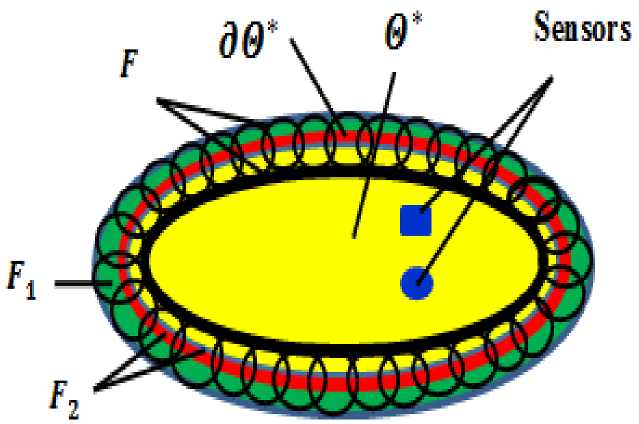

Figure 2: $\boldsymbol{\Theta}^*$**;** $\boldsymbol{\partial\Theta}^*$ with Junction Interface Conditions.

Definition3.1: System (11) is said to be an $EGRO$-stable in F_2, if a mapping A produces a semigroup which is an $EGRO$-stable in F_2.

Definition3.2: System (11) is increased with the output function (12) is said to be an $EGRO$-detectable in F_2, if $\exists\ H_{F_2} : \mathcal{O} \longrightarrow (\mathcal{H}^1(F_2))^n$, such that the mapping (A-$H_{F_2}C$) create a SCS-group $\left(S_{H_{F_2}}(t)\right)_{t \geq 0}$, realize an $EGRO$-stable, in F_2.

By the same technique in [23], we can extend these results to the junction interface method for accomplishing $\partial\Theta^*{}_{EGRO}$-detectable system, will be given in the following theorem.

Proposition3.3: If system (11) increased with the output function (12) is an $EGRO$-detactable on $\bar{F}_2$, then, it is an $\partial\Theta^*{}_{EGRO}$-detectable.

Proof: Let $x(\zeta, t)\epsilon\left(\mathcal{H}^{1/2}(\partial\Theta^*)\right)^n$ in addition $\bar{x}(\zeta, t)$ be an prolongation to $\left(\mathcal{H}^{1/2}(\partial\Theta^*)\right)^n$. Through equation (13) $\exists R\bar{x}(\zeta, t)\ \epsilon\ \left(\mathcal{H}^1(\Theta^*)\right)^n$ satisfying

$$\gamma\left(R\bar{x}(\zeta, t)\right) = \bar{x}(\zeta, t)$$

Subsequently, the system (11) provided by (12) an $EGRO$-detectable on $\bar{F}_2$, then it is an $EGRO$-detectable in F_2.

Thus, $\exists\ \gamma\nabla K^* : \mathcal{O} \to (\mathcal{H}^1(F_2))^n$ defined by

$$H_{F_2}\ y(.,t) = \nabla K^* y(\xi, t)$$

such that the operator (A-$H_{F_2}C$) generates SCS-group $\left(S_{H_{F_2}}(t)\right)_{t \geq 0}$**,** realize an $EGRO$-stable in F_2. Hence, $\forall\ y \in \mathcal{O}$; then we acquire

$$\nabla K^* y\ (\zeta, t) = \nabla R\ \bar{x}(\zeta, t)$$

therefore,

$(\gamma \nabla K^*)\, y(.,t) = x(\zeta,t)$.

Consequently, there exists an operator

$\mathbb{H}_{\partial\Theta^*_{EGRO}} = (\gamma\nabla K^* y) : \mathcal{O} \to \left(\mathcal{H}^{1/2}(\,\partial\Theta^*)\,\right)^n$,

such that (A -$\mathbb{H}_{\partial\Theta^*_{EGRO}} C$) generates a SCS-group $\left(S_{\mathbb{H}_{\partial\Theta^*_{EGRO}}}(t)\,\right)_{t\,\geq\,0}$, which is an $\partial\Theta^*_{EGRO}$-stable. Finally, system (11) is increased with (12) is an $\partial\Theta^*_{EGRO}$-detectable.

3.2 $\partial\Theta^*_{EGRO}$-Detectability and Related Estimator

Such as in [8] we can prolong the outcomes to the case $\partial\Theta^*_{EG}$-detectability and liked dynamical estimator. So, this case, system (11) increased with (12) it can be given by:

$K : x_2 \to Kx_2 = A_{12}\, S_{A_{22}}(t)\, x_2 \epsilon\, \mathcal{O}$,

then

$y(.,t) = K\, x_{2_0}(.)$, with the adjoint $K^* : \mathcal{O} \to x_2$

such that

$K^* y^*(.,t) = \int_0^t S_{A_{22}}(\tau)\, A^*_{12}\, y^*(.,\tau)\, d\tau$.

And $\chi_{\partial\Omega^*} : (\mathcal{H}^{1/2}(\Theta^*))^n \to (\mathcal{H}^{1/2}(\partial\Theta^*))^n = x_2$,

$x_2 \to x_{\partial\Omega^*}\, x_2 = x_2|_{\,\partial\Theta^*}$

wherever $x_2|_{\,\partial\Theta^*}$ is the restricting of the state x_2 on $\partial\Theta^*$.

Now, choose the following decomposition:

$$\widehat{w} = \begin{bmatrix} \widehat{w}_1 \\ \widehat{w}_2 \end{bmatrix} = \begin{bmatrix} y \\ \varphi + \mathbb{H}_{\partial\Theta^*_{EGRO}} y \end{bmatrix}$$

Which estimates

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Definition3.4: Suppose there exists a system is specified by the following system:

$$\begin{cases} \frac{\partial\varphi}{\partial t}(\zeta,t) = \left(A_{22} - \mathbb{H}_{\partial\Theta^*_{EGRO}} A_{12}\right)(\zeta,t) + (A_{22}\, A_{12}\mathbb{H}_{\partial\Theta^*_{EGRO}} & \\ -\mathbb{H}_{\partial\Theta^*_{EGRO}} A_{11} + A_{21})\, y(\zeta,t) + (B_2 - \mathbb{H}_{\partial\Theta^*_{EGRO}} B_1)\, u(t) & \Theta^* \times\,]0,T[\\ \varphi\,(\zeta,0) = \varphi_0(\zeta) & \overline{\Theta^*} \\ \frac{\partial\varphi}{\partial v}\,(\eta,t) = 0 & \partial\Theta^* \times\,]0,T[\end{cases} \quad (14)$$

which describes an $\partial\Theta^*_{EGRO}$-estimator for $T_{\partial\Theta^*_{EGRO}} x_2(\zeta,t)$, if

I. $\lim_{t\to\infty} \left\|\varphi\,(\zeta,t) - T_{\partial\Theta^*_{EGRO}} x_2(\zeta,t)\right\|_{(\mathcal{H}^{1/2}(\,\partial\Theta^*)\,)^n} = 0$

II. $T_{\partial\Theta^*_{EGRO}} : \mathfrak{D}(A_{22}) \to \mathfrak{D}\left(A_{22} - \mathbb{H}_{\partial\Theta^*_{EGRO}} A_{12}\right)$ where $T_{\partial\Theta^*_{EGRO}} = x_{\,\partial\Theta^*_{EGRO}} T$, in addition to $\varphi\,(\,\xi,t)$ is the system solution of (14).

1xs2za

Remark3.5: System is $\partial\Theta^*_{EGRO}$-detectable, then, it is conceivable to rebuild a $\partial\Theta^*_{EGRO}$-estimator

.

Definition3.6: The system (11) is so called $\partial\Theta^*_{EGRO}$-stable, if the mapping A_{22} produces a semigroup which is $\partial\Theta^*_{EGRO}$-stable.

Definition3.7: System (11) increased with information output (12) is so called $\partial\Theta^*_{EGRO}$-detectable, if $\exists\mathbb{H}_{\partial\Theta^*_{EGRO}} : \mathcal{R}^q \longrightarrow (H^{1/2}(\,\partial\Theta^*))^n$ in the case the mapping (A_{22}-$\mathbb{H}_{\partial\Theta^*_{EGRO}} A_{12}$) produces a SCS-group $(S_{A_{22}}(t))t \geq 0$, realize $\partial\Theta^*_{EGRO}$-stable.

Therefore, we have the dynamical system $\partial\Theta^*_{EGRO}$-estimator for system (11) increased with (12) may be given by

$$\begin{cases} \frac{\partial \hat{w}}{\partial t}(\zeta,t) = A_{22}\hat{w}(\zeta,t) + [B_2u(t) + A_{21}y(\zeta,t)] & \\ + \mathbb{H}_{\partial\Theta^*_{EGRO}}(\tilde{y}(.,t) - A_{12}\hat{w}(\zeta,t)) & \Theta^* \times]0,T[\\ \hat{w}(\zeta,0) = \hat{w}_0(\zeta) & \overline{\Theta^*} \\ \frac{\partial \hat{w}}{\partial v}(\eta,t) = 0 & \partial\Theta^* \times]0,T[\end{cases} \tag{15}$$

Where $(A_{22} - \mathbb{H}_{\partial\Theta^*_{EGRO}}A_{12})$ generates a SCS-group $(S_{A_{22}}(t))t\geq 0$ which is $\partial\Theta^*_{EGRO}$-stable on $X_2 \subset X = (H^{1/2}(\partial\Theta^*))^n$, $(B_2 - \mathbb{H}_{\partial\Theta^*_{EGRO}}B_1) \in L(R^p, X_2)$
and

$$(A_{22}\mathbb{H}_{\partial\Theta^*_{EGRO}} - \mathbb{H}_{\partial\Theta^*_{EGRO}}A_{12}\,\mathbb{H}_{\partial\Theta^*_{EGRO}} - \mathbb{H}_{\partial\Theta^*_{EGRO}}A_{11} + A_{21}) \in L(R^p, X_2)$$ as in [6].

Proposition3.8: If the system (11) with (12) are an $\partial\Theta^*_{EGRO}$-detectable, then the system (15) is a $\partial\Theta^*_{EGRO}$-estimator of the systems (11)-(12), if

$$\lim_{t\to\infty}\|\hat{w}(\zeta,t) - x(\zeta,t)\|_{(\mathcal{H}^{1/2}(\partial\Theta^*))^n} = 0\,,\ \ \zeta\epsilon\,\partial\Theta^*.$$

Proof: Let $e(\zeta,t) = \hat{w}(\zeta,t) - x(\zeta,t)$, where $\hat{w}(\zeta,t)$ realize the dynamical system solution of (15). Via the use of derivation property of (ζ,t)**,** we obtain

$$\begin{aligned}\frac{\partial e}{\partial t}(\zeta,t) &= \frac{\partial x}{\partial t}(\zeta,t) - \frac{\partial \hat{w}}{\partial t}(\zeta,t)\\ &= A_{22}x(\zeta,t) + A_{21}y(\zeta,t) + B_2u(t) - A_{22}\hat{w}(\zeta,t)\\ &\quad -B_2u(t) - A_{21}y(\zeta,t) - \mathbb{H}_{\partial\Theta^*_{EGRO}}((\tilde{y}(\zeta,t) - A_{12}\hat{w}(,t))\\ &= A_{22}\big(x(\zeta,t) - \hat{w}(\zeta,t)\big) - \mathbb{H}_{\partial\Theta^*_{EGRO}}((A_{12}\,x(\zeta,t) - A_{12}\hat{w}(\zeta,t))\\ &= \big(A_{22} - \mathbb{H}_{\partial\Theta^*_{EGRO}}A_{12}\big)\big(x(\zeta,t) - \hat{w}(\zeta,t)\big)\\ &= \big(A_{22} - \mathbb{H}_{\partial\Theta^*_{EGRO}}A_{12}\big)\ e(\zeta,t)\end{aligned}$$

Then the system (11) is a $\partial\Theta^*_{EGRO}$-detectable, thus, $\exists\mathbb{H}_{\partial\Theta^*_{EGRO}} \in L(\mathcal{O}, (\mathcal{H}^{1/2}(\partial\Theta^*))^n$, in view the mapping $(A_{22} - \mathbb{H}_{\partial\Theta^*_{EGRO}}A_{12})$ produces a SCS-group $\left(S_{H_{\partial\Theta^*_{EGRO}}}(t)\right)_{t\geq 0}$, which is a $\partial\Theta^*_{EGRO}$-stable on $(H^{1/2}(\partial\Theta^*))^n$, $\exists\, M_{\partial\Theta^*_{EGRO}}, \alpha_{\partial\Theta^*_{EGRO}} > 0$, so that

$$\left\|\gamma\nabla S_{\mathbb{H}_{\partial\Theta^*_{EGRO}}}(t)\right\|_{(\mathcal{H}^{1/2}(\partial\Theta^*))^n} \leq M_{\partial\Theta^*_{EGRO}}\, e^{-\alpha_{\partial\Theta^*_{EGRO}}t},\ \text{for all}\ \ t \geq 0$$

Finally, we have

$$\|e\|_{(\mathcal{H}^{1/2}(\partial\Theta^*))^n} \leq \left\|\gamma\nabla S_{H_{\partial\Theta^*_{EGRO}}}(t)\right\|_{(\mathcal{H}^{1/2}(\partial\Omega^*))^n}\|e_0\| \leq M_{\partial\Theta^*_{EGRO}}e^{-\alpha_{\partial\Theta^*_{EGRO}}t}\,\|e_0\|$$

With $e_0(\zeta) = x_0(\zeta) - \hat{w}_0(\zeta)$, and the above inequality permits the consequential outcome

$$\lim_{t\to\infty}\|\hat{w}(\zeta,t) - x(\zeta,t)\|_{(\mathcal{H}^{1/2}(\partial\Theta^*))^n} = 0\,,\ \ \zeta\epsilon\,\partial\Theta^*.$$

Here and now, for determining an specific estimator, assume there are $\mathcal{J}$ parameters of unstable classes linked to sensor descriptions[14].

Definition3.9: The couple $(\mathcal{D}, f)$ is said to be $\partial\Theta^*_{EGRO}$-strategic sensor, if the dynamical system (15) is $\partial\Theta^*_{EGRO}$-estimator for system (11) increased with output information (12).

Theorem3.10: If there are q sensors $(\mathcal{D}_i, f_i)_{1\leq i\leq q}$ and the spectrum of A_{22} have $\mathcal{J}$ eigenvalues with non-negative real parts. The system (11) increased with output information (12) are $\partial\Theta^*_{EGRO}$-detectable if and only if

I. $q \geq m_2$

II. $rank\ \mathrm{G}_{2_i} = m_{2_i}, \forall\ i,\ i = 1, \ldots, \mathcal{J}$ realize

$$\mathrm{G}_2 = \mathrm{G}_{2_{ij}} = \begin{cases} < \varphi_j(.), f_i(\,.\,) >_{\mathcal{L}^2(\mathcal{D}_i)} \\ \varphi_j(b_i) \end{cases}$$

such that,

$Sup(m_{2_i}) = m_2 < \infty$, and $j = 1, \dots, \infty$.

Proof:

Firstly: System (11) decomposing by the projecting mapping $\mathcal{P}$ and $I - \mathcal{P}$, on stable and instable portions of (11) [14]. Then,

$$x_2(\zeta, t) = [x_{2_1}(\zeta, t)\; x_{2_2}(\zeta, t)]^{tr},$$

wherever $x_{2_1}(\zeta, t)$ is the portion of unstable class for (11),given form by

$$\begin{cases} \frac{\partial x_{2_1}}{\partial t}(\zeta, t) = A_{22_1} x_{2_1}(\zeta, t) + \mathcal{P}[A_{21_1} x_{1_1}(\zeta, t) + B_2 u(t)] & \Theta^* \times]0, T[\\ x_{2_1}(\zeta, 0) = x_{2_{1_0}}(\zeta) & \overline{\Theta^*} \\ \frac{\partial x_{2_1}}{\partial v}(\eta, t) = 0 & \partial\Theta^* \times]0, T[\end{cases} \quad (16)$$

and $x_{2_2}(\zeta, t)$ is portion stable of (11), specified by

$$\begin{cases} \frac{\partial x_{2_2}}{\partial t}(\,\zeta, t) = A_{22_2} x_{2_2}(\zeta, t) + (I - \mathcal{P})\big[A_{21_2} x_{1_2}(\zeta, t) + B_2 u(t)\big] & \Theta^* \times]0, T[\\ x_{2_2}(\zeta, 0) = x_{2_{2_0}}(\zeta) & \overline{\Theta^*} \\ \frac{\partial x_{2_2}}{\partial v}(\eta, t) = 0 & \partial\Theta^* \times]0, T[\end{cases} \quad (17)$$

and A_{22_1} is a mapping characterized by a transformation of type $\left(\sum_{i=1}^{\mathcal{J}} m_{2_i}, \sum_{i=1}^{\mathcal{J}} m_{2_i}\right)$ described by the following formula

$$A_{22_1} = diag[\lambda_{2_1}, \dots, \lambda_{2_1}, \dots, \lambda_{2_{\mathcal{J}}}, \dots, \lambda_{2_{\mathcal{J}}}] \; and \; \mathcal{P}B_2 = [G_{2_1}^{tr}, G_{2_2}^{tr}, \dots, G_{2_J}^{tr}]$$

Secondly: System (16) is $\partial\Theta^*_{EGRO}$-detectable, then, the couple $(D_i, f_i)_{1 \le i \le q}$ are $\partial\Theta^*_{EGRO}$-strategic for subsystem unstable of (11). Hence, and then, (16) is $W\,\partial\Theta^*_{EGRO}$-observable system. As it is bounded dimension, then it is $E\partial\Theta^*_{EGRO}$-observable.

Consequently, realize $\partial\Theta^*_{EGRO}$-detectable, and so, $\exists\, \mathbb{H}^1_{\partial\Theta^*_{EGRO}}$, such that $(A_{22_1} - \mathbb{H}^1_{\partial\Theta^*_{EGRO}} A_{12_1})$ which satisfies the following

$\exists\, M^1_{\partial\Theta^*_{EGRO}}, \alpha^1_{\partial\Theta^*_{EGRO}} > 0$, such that

$$\| e^{\left(A_{22_1} - \mathbb{H}^1_{\partial\Theta^*_{EGRO}} A_{12_1}\right)t} \|_{(\mathcal{H}^{1/2}(\,\partial\Theta^*))^n} \le M^1_{\partial\Theta^*_{EGRO}} e^{-\alpha^1_{\partial\Theta^*_{EGRO}}(t)}$$

and we have

$$\| x_{2_1}(\zeta, t) \|_{(\mathcal{H}^{1/2}(\,\partial\Theta^*))^n} \le M^1_{\partial\Theta^*_{EGRO}} e^{-\alpha^1_{\partial\Theta^*_{EGRO}}(t)} \| \mathcal{P} x_{2_0}(.) \|_{(\mathcal{H}^{1/2}(\,\partial\Theta^*))^n}$$

Since the semigroup produced by the mapping A_{22_2} is $\partial\Omega^*_{EGRO}$-stable, $\exists\, M^2_{\partial\Theta^*_{EGRO}}, \alpha^2_{\partial\Theta^{**}_{EGRO}} > 0$, satisfy

$$\| x_{2_2}(\xi, t) \|_{(\mathcal{H}^{1/2}(\,\partial\Theta^*))^n} \le M^2_{\partial\Theta^*_{EGRO}} e^{-\alpha^2_{\partial\Theta^*_{EGRO}}(t)} \| (I - \mathcal{P}) x_{2_0}(.) \|_{(\mathcal{H}^{1/2}(\,\partial\Theta^*))^n}$$
$$+ \int_0^t M^2_{\partial\Theta^*_{EGRO}} e^{-\alpha^2_{\partial\Theta^*_{EGRO}}(t-\tau)} \| (I - \mathcal{P}) x_{2_0}(.) \|_{(\mathcal{H}^{1/2}(\,\partial\Theta^*))^n} \| u(\tau) \| d\tau$$

therefore, $x_2\,(\zeta, t) \to 0$ when $t \to \infty$. Thus, the system (11) increased with output information (12) are $\partial\Theta^*_{EGRO}$-detectable.

Now, If the system (11) increased with output information (12) are $\partial\Theta^*_{EGRO}$-detectable, then, $\exists\, H_{\partial\Theta^*_{EGRO}} \in \mathcal{L}(\mathcal{L}^2(0, \infty, \mathbb{R}^q), \left(\mathcal{H}^{1/2}(\,\partial\Theta^*)\right)^n\big)$, so that $(A_{22} - H_{\partial\Theta^*_{EGRO}} A_{12})$ produces a SCS-group $(S_{A_{22}}(t))t \ge 0$, realize $\partial\Theta^*_{EGRO}$-stable on $\left(\mathcal{H}^{1/2}(\,\partial\Theta^*)\right)^n\big)$, and then, $\exists\, M_{\partial\Theta^*_{EGRO}}, \alpha_{\partial\Theta^*_{EGRO}} > 0$, with

$$\| \chi_{\partial\Theta^*_{EGRO}} S_{A_{22}}(t) \|_{(\mathcal{H}^{1/2}(\,\partial\Theta^*))^n} \le M_{\partial\Theta^*_{EGRO}} e^{-\alpha_{\partial\Theta^*_{EGRO}}(t)}$$

Thus, the unstable sub-system (16) is $\partial\Theta^*{}_{EGRO}$-detectable.
Consequently, (16) realize $W\ \partial\Theta^*{}_{EGRO}$-observable, so it is $\partial\Theta^*{}_{EGRO}$-strategic, This means

$[K\chi^*_{\partial\Theta^*} x_2^*(.,t) = 0 \Longrightarrow x_2^*(.,t) = 0]$, used for $x_2^*(.,t) \epsilon\ \mathcal{H}^{1/2}(\partial\Theta^*)$.

We have, $[K\chi^*_{\partial\Theta^*{}_{EGRO}} x_2^*(.,t) = (\sum_{j=1}^{J} e^{\lambda_j t} \langle\varphi_j(.), x_2^*(.,t)\rangle_{(\mathcal{H}^{1/2}(\partial\Theta^*))^n} \langle\varphi_j(.), f_i(.)\rangle_{(\mathcal{H}^{1/2}(\partial\Theta^*))^n})_{1\le i\le q}$

Here and now in the case, if the couples $(\mathcal{D}_i, f_i)_{1\le i\le q}$ is not $\partial\Theta^*{}_{EGRO}$-strategic sensors intended for the system part one of (11), $\exists\ x_2^*(.,t) \epsilon\ (\mathcal{H}^{1/2}(\partial\Theta^*))^n$ such that

$$K\chi^*_{\partial\Theta^*{}_{EGRO}}\ x_2^*(.,t) = 0,$$

this leads to

$$\sum_{j=1}^{J} \langle\varphi_j(.), x_2^*(.,t)\rangle_{(H^{1/2}(\partial\Theta^*))^n} \langle\varphi_j(.), f_i(.)\rangle_{(H^{1/2}(\partial\Theta^*))^n} = 0.$$

the estimated component x_{2_i} might be known

$$x_{2_i}(.,t) = [\langle\varphi_j(.), x_2^*(.,t)\rangle_{(H^{1/2}(\partial\Theta^*))^n} \langle\varphi_j(.), x_2^*(.,t)\rangle_{(H^{1/2}(\partial\Theta^*))^n}]^{tr} \neq 0.$$

Finally, $G_{2_i} x_{2_i} = 0$ for all $i = 1, \dots, \mathcal{J}$ and then, $rank\ \mathrm{G}_{2_i} \neq m_{2_i}$.

4. Application to Sensor Location of $\partial\Theta^*{}_{EGRO}$-Detectability

Reflect the systems of two phase exchange represented via the next pair of equations

$$\begin{cases} \frac{\partial x_1}{\partial t}(\zeta_1,\zeta_2,t) = \alpha\frac{\partial^2 x_1}{\partial \xi_1^2}(\zeta_1,\zeta,t) + \beta\big(x_1(\zeta_1,\zeta_2,t) - x_2(\zeta_1,\zeta,t)\big) & \partial\Theta^* \times]0,T[\\ \frac{\partial x_2}{\partial t}(\zeta_1,\zeta_2,t) = \alpha\frac{\partial^2 x_2}{\partial \xi_2^2}(\zeta_1,\zeta_2,t) + \beta\big(x_2(\zeta_1,\zeta_2,t) - x_1(\zeta_1,\zeta_2,t)\big) & \Theta^* \times]0,T[\\ x_1(\zeta_1,\zeta_2,0) = x_{1_0}(\zeta_1,\zeta_2),\ x_2(\zeta_1,\zeta_2,0) = x_{2_0}(\zeta_1,\zeta_2) & \overline{\Theta^*} \\ \frac{\partial x_1}{\partial v}(\eta_1,\eta_2,t) = 0,\ \ \frac{\partial x_2}{\partial v}(\eta_1,\eta_2,t) = 0 & \partial\Theta^* \times]0,T[\end{cases} \tag{18}$$

For two-dimensional system, deliberate $\Theta^* =]0,1[\times]\ 0,1[$ with the boundary is given by the following form

$$\partial\Theta^* = [0,1] \times \{1\} \cup [0,1] \times \{0\} \cup \{0\} \times [0,1] \cup \{1\} \times [0,1]$$

is a region of $\overline{\Theta^*}$, and assume that it is probable to observe the state $x_1(.,t)$.
So, via utilizing $(\mathcal{D}_i, f_i)_{i\le 1\le q}$ sensors of type zone. Then, measurement information (2) is specified by

$$y(.,t) = C\ x_1(.,t) \tag{19}$$

Here and now, the problem is to exponential estimate $x_2(\xi_1,\xi_2,t)$. Contemplate now

$$\frac{\partial x}{\partial t} = \begin{bmatrix} \frac{\partial x_1}{\partial t} \\ \frac{\partial x_2}{\partial t} \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

where

$$A_{11} = \alpha\frac{\partial^2 x_1}{\partial \xi^2}(\zeta_1,\zeta_2,t) + \beta,\ A_{22} = \gamma\frac{\partial^2 x_2}{\partial \xi^2}(\zeta_1,\zeta_2,t) + \beta$$

and

$$A_{12} = A_{21} = -\beta I.$$

Reflect the system of two-dimensional

$$\begin{cases} \frac{\partial x_2}{\partial t}(\zeta_1,\zeta_2,t) = \gamma\frac{\partial^2 x_2}{\partial \xi^2}(\zeta_1,\zeta_2,t) + \beta(x_2(\zeta_1,\zeta_2,t) - x_1(\xi_1,\xi_2,t)) & \Theta^* \times]0,T[\\ x_2(\xi_1,\xi_2,0) = x_{2_0}(\xi_1,\xi_2) & \overline{\Theta^*} \\ \frac{\partial x_2}{\partial v}(\eta_1,\eta_2,t) = 0 & \partial\Theta^* \times]0,T[\end{cases} \tag{20}$$

together with output function by (2). The eigenfunctions of the system (20) denoted by

$$\varphi_{nm}(\zeta_1,\ \zeta) = 2\ \cos n\pi(\zeta_1)\cos m\pi(\zeta_2) \tag{21}$$

associated with eigenvalues

$$\lambda_{nm} = -(n^2 + m^2)\pi^2 , \; n, m \geq 1 \tag{22}$$

The subsequent consequences in the next sections give information maybe with one internal sensors ensuring an $\partial\Omega^*_{EGRO}$-detectability [15].

4.1.Domain of Rectangular Case

Numerous positions of sensor types is augmented in this section.

4.1.1.Sensor of internal zone case

Reflect the following system that is distinct by parabolic equations

$$\begin{cases} \frac{\partial x_2}{\partial t}(\zeta_1, \zeta_2, t) = \gamma \frac{\partial^2 x_2}{\partial \xi^2}(\zeta_1, \zeta_2, t) + \beta x_2(\zeta_1, \zeta_2, t) - \beta x_1(\zeta_1, \zeta_2, t)\,) & \Theta^* \times]0, T[\\ x_2(\zeta_1, \zeta_2, 0) = x_{2_0}(\zeta_1, \zeta_2) & \overline{\Theta^*} \\ \frac{\partial x_2}{\partial v}(\eta_1, \eta_2, t) = 0 & \partial\Theta^* \times]0, T[\end{cases} \tag{23}$$

increased with the output information, is represented via internal zone sensor

$$y(.,t) = [\int_{\mathcal{D}} x_1(\zeta_1, \zeta_2, t) f_1(\zeta_1, \zeta_2) d\zeta_1 d\zeta_2]^{tr} \tag{24}$$

where $\mathcal{D}$ specify by

$\boldsymbol{\mathcal{D}} =]\zeta_{0_1} - l_1, \zeta_{0_1} + l_1[\times]\zeta_{0_2} - l_2, \zeta_{0_2} + l_2[\subset \Theta^*$ where $f \epsilon \mathcal{L}^2(\mathcal{D})$(see Figure 3)

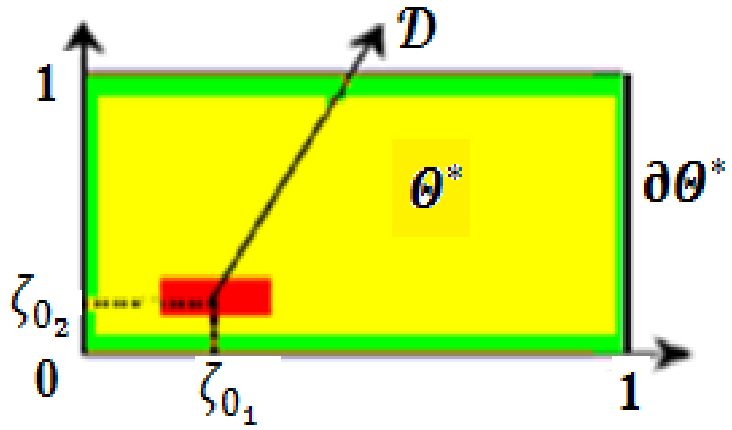

Figure 3: $\boldsymbol{\Theta^*}$; $\boldsymbol{\partial\Theta^*}$ with sensor position $\boldsymbol{\mathcal{D}}$ of internal zone type.

Hence, the subsequent essential outcome is obtained by.

Proposition4.1: Let f_i satisfy symmetry property around $\zeta = \zeta_{0_i}, i = 1,2,$ then the system (23) increased with output information (24) is $\partial\Theta^*_{EGRO}$-detectable, if $\exists n, m = \{1, \ldots, J\}$, realize $n\zeta_{0_1}$ and $m\zeta_{0_2} \notin N$.

4.1.2.Sensors of internal pointwise Case

Assume the system (23) together with information (24), then, the output information can be formulated as

$$y(t) = \int_{\partial\Theta^*} x_2\,(\zeta_1, \zeta_2, t)\, \delta(\zeta_1 - b_1, \zeta_2 - b_2)\, d\zeta_1\, d\zeta_2 \tag{25}$$

anywhere $\boldsymbol{b} = (\boldsymbol{b_1}, \boldsymbol{b_2})$ is the located sensor inside region Θ^* as shown in (Figure 4).

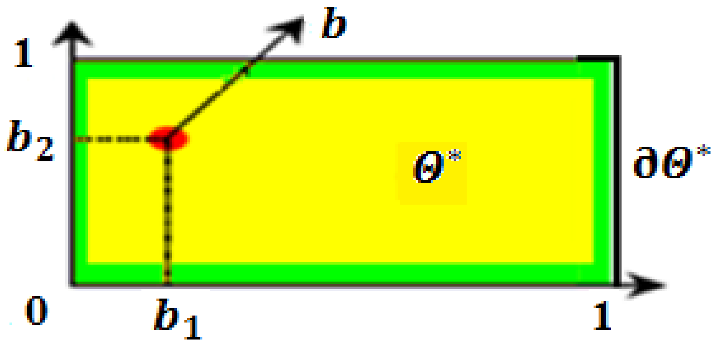

Figure 4: $\boldsymbol{\Theta}^*$; $\boldsymbol{\partial\Theta}^*$ with sensor position $\boldsymbol{b}$ of pointwise type.

So, the following result is proposed.

Proposition 4.2: Let $b = (b_1, b_2)$ is the sensor positioned in Θ^*, then the system (23) increased with the output information (25) is $\partial\Theta^*{}_{EGRO}$-detectable, if nb_1 and $mb_2 \notin N$, for every $n, m = \{1, \dots, \mathcal{J}\}$.

4.1.3. Internal filament pointwise sensors case

Assume that the filament sensor positioned in $\partial\Theta^*$ where **σ** = $Im(\gamma) \subset \Theta^*$ is symmetric with respect to the line $\boldsymbol{b} = (\boldsymbol{b_1}, \boldsymbol{b_2})$ (Figure 5). More precisely, the sensor is line of pointwise positioned in Θ^*, then the output function still given by equation (24).

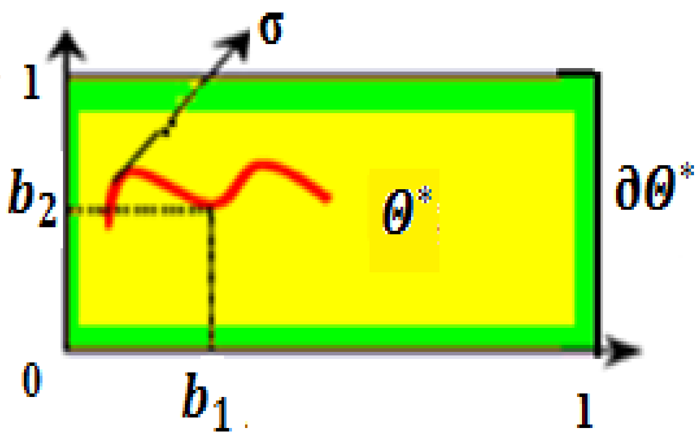

Figure 5: $\boldsymbol{\Theta}^*$; $\boldsymbol{\partial\Theta}^*$ with sensor position σ of filament zone type.

Proposition 4.3: Let the sensor is located in $b = (b_1, b_2)$, then, the system (23) increased with output information (24) is $\partial\Theta^*{}_{EGRO}$-detectable,, if nb_1 and $mb_2 \notin N$, for every $n, m = \{1, \dots, \mathcal{J}\}$.

4.2. Domain of Circular Case

Circular problem discusses via the following outcome.

Remark4.4:

The consequences in section 4.1 is prolonged to the domain of circular type as in [3-7] in parabolic system or hyperbolic as in [26].

5. Conclusion

In this paper, we established the original notion related to $\partial\Theta^*{}_{EGRO}$-detection a desired region in connection with the sensors structure and we have presented the existence of the sufficient condition of $\partial\Theta^*{}_{EGRO}$-detectability. The crossing problem from interior to exterior and have been explored and achieved in rigorous results, Besides, these results are extended to the problem of Neumann boundary type of the system domain which is under consideration. Numerous problem sill opened, maybe interested to study in the next time the likelihood the problem of $\partial\Theta^*{}_{EGRO}$-detectable in association with the diffusion systems.

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