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A robust Study of the Treatment Delay Effect on the Dynamics of Epidemic Disease

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Abstract
A general treatment function with an epidemic model that involves the delay in the treatment period has been proposed and studied in this work. This model contains two compartments, namely susceptible denoted by $S(t)$ and infected denoted by $I(t)$. The existence of all the fixed points has been determined. The system has two equilibrium points, namely the uninfected equilibrium point (UIEP) and the endemic equilibrium point (EEP). The conditions for local stability and Hopf bifurcation have been discussed. The stability of the periodic solutions and the direction of the Hopf bifurcation properties have been studied analytically and numerically.

Keyword.: Infection diseases, Stability, Time delay, Hopf bifurcation, Treatment function.

دراسة قوية لتأثير تأخير العلاج على ديناميكات المرض الوبائي

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الخلاصة
تم دراسة دالة علاج عامة لنموذج وبائي يتضمن التأخير في فترة العلاج. يحتوي هذا النموذج على جزأين الافراد الاصحاء $S(t)$ و الافراد المصابين$I(t)$. تم تحديد وجود جميع النقاط التوازن ، حيث يحتوي النظام على نقطتي توازن ، الأولى هي نقطة التوازن غير المصابة (UIEP) والثانية وهي نقطة التوازن المصابه (EEP) ، وقد تمت مناقشة شروط الاستقرار المحلي وتشعب هوبف المناسبة عند كل نقطة توازن. وأخيرا ,تم إجراء المحاكاة العددية لدراسة تأثير المعلمات على ديناميكات النموذج المقترح ودعم النتائج التحليلية . تمت دراسة استقرار الحلول الدورية واتجاه خصائص تشعب هوبف تحليليًا وكذلك عدديًا.

1. Introduction
Mathematical models are powerful tools for describing the population dynamics of infectious diseases. Mathematical models have played an essential role in the studying of disease dynamics and control of the disease. The appearance of different infectious diseases represents a main challenge in the modern world. The incident of infectious disease causes a

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large loss of lives and other resources. Although, there are advances in medical science and an increased understanding of infectious disease mechanisms. Infectious diseases have caused millions of deaths and disabilities Worldwide. As a result, researchers from various fields of science and medicine are working to develop an effective solution to stop the transmission of infectious diseases see [1-3]. A lack of safe drinking water causes waterborne diseases such as cholera, typhoid, and hepatitis. The prospect of various modes of disease transmission makes the study of waterborne disease more significant. Cholera is considered one of these diseases that belongs to this class. Many researchers have studied the dynamic behavior of Cholera disease. Joh et al. [4] suggested a mathematical model of Cholera disease so that the primary mode of transmission is indirect and occurs by contact with a contaminated reservoir. In [5] Cui et al. formulated an SVR-B Cholera model with imperfect vaccination. Wang et al. [6] discussed and formulated a mathematical model including the of human behavior on cholera dynamics. Ayoade et al. studied the dynamics of cholera disease with vaccination and treatment control for Cholera outbreaks [7]. Olaniyi S .at el. presented a mathematical model for controlling cholera outbreaks without natural recovery [8]. The hepatitis B virus causes infection in liver cells a potentially life-threatening infection. It is a global health problem, with HBV being the most dangerous type of viral hepatitis. Infection with hepatitis can result in both acute and chronic symptoms (transient i.e lasting less than six months) [9]. The ability of the infected individual's immune system to contain and eliminate the virus determines whether the sickness is chronic (lifelong) [10]. Theoretical biologists can improve our understanding of contributing factors in disease by applying mathematical modeling hepatitis virus dynamics. As a result, a number of research studies have been conducted that deal with modeling and simulating the dynamic behavior of hepatitis. Zahura et al. [11] analyzed and studied the hepatitis B virus mathematical model. John et al. [12] developed a mathematical model including Cytotoxic T-Lymphocytes (CTLs) immune response and focusing on the hepatitis B and C viruses (HBV and HCV) infection in the liver and blood cells. They also developed a mathematical model including the Cytotoxic T-Lymphocytes (CTLs) immune response.

Time delay plays a significant role in population dynamics. The dynamics of the state variables in many real-world processes, specifically in many biological phenomena, depend not only on the processes' current state but also on the phenomenon's history, that is, on the state variables' previous values. The dynamics of infectious diseases may be affected by the time delay as shown in Zuo et al. [13]. Their formulated and studied the effect of media on recruitment and delaying the epidemic's spread. Aekabut at el. [14] studied a delayed of SEIR epidemic model in which the latent and infected states are infective. Naji and Majeed [15] investigated the impact of delay on a stage-structure prey-predator model. Zhe Yin at el. [16] studied the effect of time delay on an age-structured SEIRS Model. Mohsen and Naji [17] studied the stability delay cancer model in a polluted environment. Zizhen et al. [18] The proposed SVIRS epidemic model contains multiple delays with Holling type II incidence rate and treatment rate. Naji and Mohsen [19] studied the SVIR epidemic model including immigrants. Hassan [20] studied the affected vaccine in the epidemic model of stage structure. Mohsen et al. [21] suggested a mathematical model for the dynamics of the COVID-19 epidemic including infected immigrants. Hassan and Ahmed [22] proposed the following model

$$\begin{aligned} \frac{dS}{dt} &= (1-P)A - \frac{\beta SI}{K+I} - \mu S, \\ \frac{dI}{dt} &= PA + \frac{\beta SI}{K+I} - (\mu+\alpha)I - \frac{rI}{n+I} \ . \end{aligned} \tag{1}$$

Where $S(t)$ and $I(t)$ represent the number of susceptible and infected at time t, respectively. $A > 0$ is the birth rate in $S(t)$, P is the external sources of disease such that $0 \leq P < 1$; β is the infection rate; μ is the natural death rate from both $S(t)$ and $I(t)$; α is the disease related death from $I(t)$; K refers to the carrying capacity of disease; r represents the maximal medical resource per unit of time ; n is the infected size. Clearly, the above model is without delay. We will focus on the time delay effect. So, in the first step, system (1) is modified in this paper. In section 2, The positivity and bounded of solutions to the modified system are discussed. In section 3, we mainly study the stability and the existence of the Hopf bifurcation. In section 4, we study the properties of the Hopf bifurcation by using the normal theory and the center manifold theorem. In section 5, some numerical simulations are performed to illustrate the main results. In section 6, conclusions are given. Now, the modified system (1) can be expressed as follows:

$$\begin{aligned} \frac{dS}{dt} &= (1-P)A - \frac{\beta SI}{K+I} - \mu S \ , \\ \frac{dI}{dt} &= PA + \frac{\beta SI}{K+I} - (\mu+\alpha)I - \frac{rI(t-\tau)}{n+I(t-\tau)} \ . \end{aligned} \tag{2}$$

Where τ is the time delay due to the latent period of treatment. All parameters in system (2) have the same biological meaning as those in system (1).

2. Postive and Boundedness.

In this part, we will study the positive and boundedness of the solutions to the system (2).

Theorem 1. The solutions to the system (2) are positive and bounded for $t \geq 0$.
Proof. First, we prove that the solutions of the system (2) are positive
From the first equation of the system (2) for $t \geq 0$, we have

$$\frac{dS}{dt} \geq -S(\frac{\beta I}{K+I} + \mu).$$

As a result and through computation, we obtained

$$S(t) \geq S(0)exp - \left\{\int_0^t \left(\frac{\beta I(\zeta)}{K+I(\zeta)} + \mu\right) d(\zeta)\right\}$$

Since $(0) > 0$, we get $S(t) > 0$ for all $t \geq 0$.
Now, we show that $I(t)$ is positive
From the second equation of the system (2), it is noted that :

$$\frac{dI}{dt} \geq I \left[\frac{\beta S(t)}{K+I(t)} - (\mu+\alpha) - \frac{rI(t-\tau)}{[n+I(t-\tau)]I(t)}\right]$$

Thus, we obtain:

$$I(t) \geq I(0)exp\left\{\int_0^t \left(\frac{\beta s(\zeta)}{K+I(\zeta)} - (\mu+\alpha) - \frac{\mathrm{rI}(\zeta-\tau)}{\mathrm{n+I}(\zeta-\tau)\mathrm{I}(\zeta)}\right) d(\zeta)\right\}.$$

Since $(0) > 0$, we have $I(t) > 0$ for all $t \geq 0$.

Following that, the proof that the system's solutions are bounded for all $t \geq 0$.
Define $\mathcal{H}(t) = S(t) + I(t)$
Therefore, it is obtained that $\frac{d\mathcal{H}}{dt} \leq A - \mu\mathcal{H}$.
By using Gronwell's lemma [23], we have:

$$\mathcal{H}(t) \leq \mathcal{H}(0)e^{-\mu t} + \frac{A}{\mu}(1 - e^{-\mu t}),$$

which gives $\lim_{n\to\infty} \mathcal{H}(t) \leq \frac{A}{\mu}$, that is independent of the initial condition. Therefore, the solutions are bounded.

3. The Local stability and Hopf Bifurcation .

The local stability and Hopf bifurcation of the system (2) will be discussed in this section. It is known that the location and number of equilibrium points do not change with time delay. It is clear that system (2) has two equilibrium points, see [22].

- The first equilibrium point, namely the uninfected equilibrium point(UIEP) denoted $E_0 = (S_0, 0)$, where

$$S_0 = {}^{A}\!/_{\mu} \tag{3}$$

Clearly, E_0 exists if $I = 0$, $p = 0$ and the basic reproduction number $R_0 < 1$. With

$$R_0 = \frac{n\beta A}{\mu K[n(\mu+\alpha)+r]}. \tag{4}$$

- The second equilibrium point, namely the endemic equilibrium point (EEP) denoted $E_1 = (S_1, I_1)$, where

$$S_1 = \frac{(1-P)(K+I)A}{\beta I+\mu(K+I)} \tag{5}$$

While I_1 is the positive root of the following fourth order polynomial equation

$$\Omega_1 I_1^4 + \Omega_2 I_1^3 + \Omega_3 I_1^2 + \Omega_4 I_1 + \Omega_5 = 0. \tag{6}$$

Here

$$\Omega_1 = -[\mu^2 + \alpha\mu + \beta] < 0$$
$$\Omega_2 = (\beta+\mu)(PA - r) + \beta A(1-P) - (\mu+\alpha)[\beta(k+n) + \mu(2+n)]$$
$$\Omega_3 = \beta A(K+n) - (\beta+2\mu)(rk+\alpha n) + \mu k(2PA - \alpha) + PAn\,\mu$$
$$\Omega_4 = PAK\big(n[\beta + \mu(2 - rK) + (1-P)] + K\mu(1-K)\big)$$
$$\Omega_5 = K^2 A\mu n P > 0$$

Clearly, E_1 exists if $I \neq 0, p \neq 0$ and the basic reproduction number $R_0 > 1$.

Now, the local stability analysis of the equilibrium points UIEP and EEP are discussed by using the linearization method. This method depends on computing the Jacobian matrix that is evaluated at each equilibrium point.

The general Jacobian matrix (JM) of the system (2) at any equilibrium point $E = (S, I)$ is given by

$$J(E) = \begin{bmatrix} -(\frac{\beta I}{K+I} + \mu) & -\frac{\beta KS}{(K+I)^2} \\ \frac{\beta I}{K+I} & \frac{\beta KS}{(K+I)^2} - (\mu+\alpha) - \frac{rne^{-\lambda\tau}}{(n+I)^2} \end{bmatrix}. \tag{7}$$

Then, the characteristic equation of the above matrix is

$$P_1(\lambda) + P_2(\lambda)e^{-\lambda\tau} = 0\,. \tag{8}$$

Here $P_1(\lambda)$ and $P_2(\lambda)$ are polynomials of λ , Accordingly, the (JM) of the system (2) at the UIEP is

$$J(E_0) = \begin{bmatrix} -\mu & -\frac{\beta S_0}{K} \\ 0 & \frac{\beta S_0}{K} - (\mu+\alpha) - \frac{re^{-\lambda\tau}}{n} \end{bmatrix} . \quad (9)$$

Then the characteristic equation of $J(E_0)$ is given by

$$\lambda^2 + C_1\lambda + C_2 + (D_1\lambda + D_2)e^{-\lambda\tau} = 0. \quad (10)$$

where

$$C_1 = -\left[\frac{\beta S_0}{K} - (2\mu+\alpha) \right].$$

$$C_2 = -\mu\left[\frac{\beta S_0}{K} - (\mu+\alpha) \right].$$

$$D_1 = \frac{r}{n}$$

$$D_2 = \frac{r\mu}{n}$$

Now, if $\tau = 0$, then equation (10) becomes

$$\lambda^2 + (C_1 + D_1)\lambda + (C_2 + D_2) = 0 . \quad (11)$$

Clearly, equation (11) has two roots which are negative if the following condition holds

$$R_0 < 1 \quad (12)$$

Hence, the UIEP is locally asymptotically stable under the condition (12) holds for $\tau = 0$
On the other hand, for $\tau > 0$, suppose that equation (10) has a pair of purely imaginary roots, namely $\lambda = \pm i\omega_0$ ($\omega_0 > 0$) if ,in addition to condition (12), the following condition holds

$$\mathrm{D}_2 > C_2 . \quad (13)$$

By substituting $\lambda = \pm i\omega_0$ in equation (10) we get :

$$D_1\omega_0 \sin\omega_0\tau + D_2 \cos\omega_0\tau + i[D_1\omega_0 \cos\omega_0\tau - D_2 \sin\omega_0\tau] = \omega_0^2 - C_2 - C_1\omega_0 i$$

Consequently, we obtain by separating the real and imaginary components

$$\left.\begin{aligned} D_1\omega_0 \sin\omega_0\tau + D_2 \cos\omega_0\tau &= \omega_0^2 - C_2 \\ D_1\,\omega_0 \cos\omega_0\tau - D_2 \sin\omega_0\tau &= -C_1\omega_0 \end{aligned}\right\} \quad (14)$$

Squaring the equation (14) and adding them, we obtain

$$\omega_0^4 + b_1\omega_0^2 + b_2 = 0 . \quad (15)$$

where

$b_1 = C_1^2 - D_1^2 - 2C_2$

$b_2 = C_2^2 - D_2^2 = (C_2 - D_2)(C_2 + D_2)$.

Let $h = \omega_0^2$, then equation (15) becomes

$$h^2 + b_1 h + b_2 = 0 . \quad (16)$$

Obviously, due to conditions (12) and (13) , we have $b_2 < 0$**.** According to Descartes rule of sign, there is a unique non-negative root say $\omega_0^{\sim}$ that satisfies equation (16). Thus $\omega_0^{\sim}$ is the non-negative root of equation (15) too.

Therefore, there are roots represented by $\pm i\omega_0^{\sim}$ that satisfy equation (10)
Corresponding to the time delay, Moreover, by substituting $\omega_0^{\sim}$ in equation.(14) and solving the resulting system for τ, we have

$$\tau_n = \frac{1}{\omega_0^{\sim}} \cos^{-1} \frac{(D_2 - C_1 D_1)\left(\omega_0^{\sim}\right)^2 - C_2 D_2}{D_1^2 \left(\omega_0^{\sim}\right)^2 + D_2^2} + \frac{2\pi\iota}{\omega_0^{\sim}} \; ; n = 0,1,2, \ldots. \tag{17}$$

Thus the system (2) has no periodic when $\tau \geq 0$, and UIEP is absolutely stable for all $\tau \geq 0$ [24]
While ,The (JM) of the system (2) at the EEP is:

$$J(E_1) = \begin{bmatrix} -(a_1 + \mu) & -a_2 \\ a_1 & a_2 - (\mu + \alpha) - a_3 e^{-\lambda\tau} \end{bmatrix} \tag{18}$$

where
$a_1 = \frac{\beta I_1}{K + I_1}; a_2 = \frac{\beta K S_1}{(K + I_1)^2}; a_3 = \frac{rn}{(n + I_1)^2}.$
Then, the characteristic equation of $J(E_1)$ can be written as follows:

$$\lambda^2 + \Psi_1 \lambda + \Psi_2 + (\Upsilon_1 \lambda + \Upsilon_2) e^{-\lambda\tau} = 0 \ . \tag{19}$$

Where

$$\Psi_1 = -\left[\frac{\beta K S_1}{(K + I_1)^2} - \frac{\beta I_1}{K + I_1} - \mu - (\mu + \alpha)\right]$$

$$\Psi_2 = -\left[\frac{\beta I_1}{K + I_1} + \mu\right]\left[\frac{\beta K S_1}{(K + I_1)^2} - (\mu + \alpha)\right] + \left[\frac{\beta^2 K S_1 I_1}{(K + I_1)^3}\right]$$

$$\Upsilon_1 = \frac{rn}{(n + I_1)^2} > 0$$

$$\Upsilon_2 = \left[\frac{rn}{(n + I_1)^2}\right]\left[\frac{\beta I_1}{K + I_1} + \mu\right] > 0$$

So for $\tau = 0$, then equation (19) becomes

$$\lambda^2 + (\Psi_1 + \Upsilon_1)\lambda + \Psi_2 + \Upsilon_2 = 0 \ . \tag{20}$$

Clearly, the above equation has two roots, these roots have a negative real part if the following conditions are satisfied

$$R_0 > 1 \tag{21}$$

$$(K + I_1)\left[\left((n + I_1)^2(\beta I_1 + (2\mu + \alpha)(K + I_1) + rn(K + I_1)\right) > \beta K S_1 (n + I_1)^2 \tag{22}$$

$$(K + I_1)(n + I_1)^2[(\mu + \alpha)(\beta I_1 + \mu)] + rn\left[\mu (n + I_1)^2 + \beta I_1 (K + I_1)\right) > \beta \mu K S_1 (n + I_1)^2. \tag{23}$$

Hence, for $= 0$, the equilibrium point E_1 is locally asymptotically stable if conditions (21) – (23) are satisfied.

On the other hand, for $\tau > 0$, suppose that equation(19) has a pair of purely imaginary roots, namely $\lambda = \pm i\omega_1$ ($\omega_1 > 0$) if in addition to condition (23) the following condition holds

$$\Upsilon_2 > \Psi_2. \tag{24}$$

By substituting $\lambda = \pm i\omega_1$ in equation(19), we obtain

$$\Upsilon_1\omega_1 \sin\omega_1\tau + \Upsilon_2 \cos\omega_1\tau + i[\Upsilon_1\omega_1 \cos\omega_1\tau - \Upsilon_2 \sin\omega_1\tau] = \omega_1^2 - \psi_2 - \psi_1\omega_1 i$$

Which implies

$$\left.\begin{array}{l} \Upsilon_1\omega_1 \sin\omega_1\tau + \Upsilon_2 \cos\omega_1\tau = \omega_1^2 - \psi_2 \\ \Upsilon_1\omega_1 \cos\omega_1\tau - \Upsilon_2 \sin\omega_1\tau = -\psi_1\omega_1 \end{array}\right\} . \tag{25}$$

Squaring the above equations and adding them, we have

$$\omega_1^4 + e_1\omega_1^2 + e_2 = 0 , \tag{26}$$

where

$e_1 = {\psi_1}^2 - {\Upsilon_1}^2 - 2\psi_2,$

$e_2 = {\psi_2}^2 - {\Upsilon_2}^2 = (\psi_2 - \Upsilon_2)(\psi_2 + \Upsilon_2).$

Let $k = \omega_1^2$, then equation (26) becomes

$$k^2 + e_1k + e_2 = 0 . \tag{27}$$

Obviously, due to conditions (23) and (24) , we have $e_2 < 0$.

According to Descartes rule of the sign there is a unique non-negative root say $\omega_1^{\sim}$ that satisfies equation (27). Thus $\omega_1^{\sim}$ is the non-negative root of equation (26) too.

Therefore, there are roots represented by $\pm i\omega_1^{\sim}$ that satisfy equation (19)

Corresponding to the time delay, Moreover, by substituting $\omega_1^{\sim}$ in equation (25) and solving the resulting system for τ, then we have

$$\tau_\iota = \frac{1}{\omega_1^{\sim}}\cos^{-1}\frac{(\gamma_2-\gamma_1\psi_1)(\omega_1^{\sim})^2-\psi_2\gamma_2}{\gamma_1^2\ (\omega_1^{\sim})^2+\gamma_2^2} + \frac{2\pi\iota}{\omega_1^{\sim}}\ ;\iota = 0,1,2, ... \tag{28}$$

Now define that $\tau_0 = \min_{\iota\geq 0}\tau_\iota$, then $\lambda(\tau) = \eta(\tau) + i\omega_1(\tau)$ is a root of equation (19) that satisfies $\eta(\tau_0) = 0$ and $\omega_1(\tau_0) = \omega_1^{\sim} > 0$. Then, we obtain the following theorem.

Theorem 2 . If the following condition is satisfied

$$\psi_1^2 - 2\psi_1 < \gamma_1^2 - 2\omega_1^{\sim} \tag{29}$$

Then E_1 is conditionally stable.

Proof. We will show that E_1 is conditionally stable. Firstly, we show that E_1 is asymptotically stable for $\tau \in [0, \tau_0)$. Moreover, as shown in conditions (21) - (23).

Secondly, we show the transcendental characteristic equation(19) has roots which are represented by $\pm i\omega_1^{\sim}$ at $\tau = \tau_0$, That is $\left[\frac{d(Re\lambda(\tau))}{d\tau}\right]_{\tau=\tau_0} \neq 0.$

If we suppose that $\lambda(\tau) = \eta(\tau) + i\omega_1(\tau)$ is the eigenvalue of equation (19) such that $\eta(\tau_0) = 0$ and $\omega_1(\tau_0) = \omega_1^{\sim} > 0.$ τ_0 define in equation (28).

If we use $\lambda(\tau)$ in equation (19), and take the derivative of that equation with respect to τ, then we get the following:

$$\left[2\lambda+\psi_1+\gamma_1 e^{-\lambda\tau}-\tau(\gamma_1\lambda+\gamma_2)e^{-\lambda\tau}\right]\frac{d\lambda}{d\tau}=\lambda(\gamma_1\lambda+\gamma_2)e^{-\lambda\tau} . \quad (30)$$

$$\left[\frac{d\lambda}{d\tau}\right]^{-1}=\frac{2\lambda+\psi_1}{-\lambda(\lambda^2+\psi_1\lambda+\psi_2)}+\frac{\gamma_1}{\lambda(\gamma_1\lambda+\gamma_2)}-\frac{\tau}{\lambda} . \quad (31)$$

Since for $\tau=\tau_0$, *and* $\lambda=i\tilde{\omega}_1$, we have

$$\left[\frac{d\lambda}{d\tau}\right]^{-1}_{\tau=\tau_0}=\frac{\psi_1+2i\tilde{\omega}_1}{\psi_1(\tilde{\omega}_1)^2-i\tilde{\omega}_1[\psi_2-(\tilde{\omega}_1)^2]}+\frac{\gamma_1}{-\gamma_1\tilde{\omega}_1+i\,\gamma_2\tilde{\omega}_1}-\frac{\tau_0}{i\tilde{\omega}_1} .$$

Now since

$$sign\left[\frac{d(Re\lambda)}{d\tau}\right]_{\tau=\tau_0}=sign\left[Re(\frac{d\lambda}{d\tau})^{-1}\right]_{\lambda=i\tilde{\omega}_1} . \quad (32)$$

It is clear that :

$$Re\left[\frac{\psi_1+2i\tilde{\omega}_1}{\psi_1(\tilde{\omega}_1)^2-i\tilde{\omega}_1[\psi_2-(\tilde{\omega}_1)^2]}\right]=\frac{2[\psi_2-(\tilde{\omega}_1)^2]-{\psi_1}^2}{\psi_1^2(\tilde{\omega}_1)^2+[\psi_2-(\tilde{\omega}_1)^2]^2}$$

$$Re\left[\frac{\gamma_1}{-\gamma_1\tilde{\omega}_1+i\,\gamma_2\tilde{\omega}_1}\right]=\frac{\gamma_1^2}{\tilde{\omega}_1(\gamma_1^2+\gamma_2^2\tilde{\omega}_1)}$$

$$Re\left[\frac{\tau_0}{i\tilde{\omega}_1}\right]=zero$$

Hence, we have

$$Re\left[\frac{d\lambda}{d\tau}\right]^{-1}_{\tau=\tau_0}=\frac{1}{\Psi_0}[-(e_1+2\tilde{\omega}_1)]$$

where

$$\Psi_0=\psi_1^2(\tilde{\omega}_1)^2+[\psi_2-(\tilde{\omega}_1)^2]^2+\tilde{\omega}_1(\gamma_1^2+\gamma_2^2\tilde{\omega}_1)>0$$

and e_1 is given in equation (26)

Thus, we get $\left[\frac{d(Re\lambda(\tau))}{d\tau}\right]_{\tau=\tau_0}>0$ under the condition (29).

This result shows that if τ passes through τ_0, then the roots of the characteristic equation(19) cross the imaginary axis from left to right. As a result, system (2) loses stability and when $\tau=\tau_0$, then a Hopf bifurcation is apparent.

4. The Direction and Stability of the Hopf Bifurcation

Sometimes external factors that influence the dynamic behavior can cause sudden changes in solutions. These changes are called bifurcation. We demonstrate in the preceding section that system (2) exhibits a Hopf bifurcation near the endemic equilibrium point E_1. In this section, we will study the properties of these changes, such as what is the direction of the solution under this change, whether its solution increasing or decreasing, and whether its solution stable or not. By using normal form theory and the center manifold theory, for more details see [25].

Theorem 3.

(i) Suppose that α_2 determines the direction of the Hope bifurcation.
If $\alpha_2 > 0$, then the Hopf bifurcation is supercritical and if $\alpha_2 < 0$, then the Hopf bifurcation is subcritical.

(ii) Suppose that ℓ_2 determines the stability of the bifurcating solution.
If $\ell_2 < 0$, then the bifurcating periodic solutions are stable, and if $\ell_2 > 0$, then the bifurcating periodic solutions are unstable.

(iii) Suppose that T_2 determines the period of the bifurcating periodic solution.
If $T_2 > 0$, then the period increase and if $T_2 < 0$, then the period decrease

where α_2, ℓ_2, and T_2 are given as follows:

$$\left.\begin{aligned} C_1(0) &= \frac{i}{2\omega_1^{\sim}\tau_0}\left(g_{11}\ g_{20} - 2|g_{11}|^2 - \frac{|g_{02}|^2}{3}\right) + \frac{g_{21}}{2}, \\ \alpha_2 &= -\frac{Re\{C_1(0)\}}{Re\left\{\frac{d\lambda}{d\tau}(\tau_0)\right\}}, \\ \ell_2 &= 2Re\{C_1(0)\}, \\ T_2 &= \frac{-Im\{C_1(0)\} + \alpha_2\ Im\left\{\frac{d\lambda}{d\tau}(\tau_0)\right\}}{\omega_1^{\sim}\tau_0}. \end{aligned}\right\} \tag{33}$$

and g_{11}, g_{20}, g_{02} and g_{21} are given in the proof of the theorem.

Proof. Let $u_1(t) = S(t) - S_1$, $u_2(t) = I(t) - I_1$ *and* $\tau = \tau_0 + \sigma$ where τ_0 is defined by equation (28) and $\sigma \in R$ then $\sigma = 0$ is the Hopf bifurcation value of the system (2). The functional differential equation in $C([-1,0], R^2)$ for the system (2) as:

$$u'(t) = L_\sigma(u_t) + h(\sigma, u_t). \tag{34}$$

Here, $u(t) = (u_1(t), u_2(t))^T \in C = C([-1,0]), R^2)$ and $L_\sigma: C \to R^2$, $h: R \times C \to R^2$ are given respectively :

$$L_\sigma(\varphi) = (\tau_0 + \sigma)(N_1\varphi(0) + N_2\varphi(-1)) . \tag{35}$$

and the nonlinear term is

$$h(\sigma, \varphi) = (\tau_0 + \sigma)\ \begin{pmatrix} H_1 \\ H_2 \end{pmatrix},$$

where

$$N_1 = \begin{bmatrix} h_{10}^{(1)} & h_{01}^{(1)} \\ h_{100}^{(2)} & h_{010}^{(2)} \end{bmatrix} = \begin{bmatrix} -(a_1 + \mu) & -a_2 \\ a_1 & a_2 - (\mu + \alpha) \end{bmatrix},$$

$$N_2 = \begin{bmatrix} 0 & 0 \\ 0 & h_{001}^{(2)} \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & a_3 \end{bmatrix}$$

with a_1, a_2, a_3 are define in the $j(E_1)$, while

$$H_1 = \sum_{i+k\geq 2} \frac{1}{i!k!} h_{ik}^{(1)} \varphi_1^i(0)\varphi_2^k(0),$$

$$H_2 = \sum_{i+k+n\geq 2} \frac{1}{i!\,k!\,n!} h_{ikn}^{(2)}\ \varphi_1^i(0)\varphi_2^k(0)\breve{\varphi}_2^n\ (-1)$$

Where $(v) = (\varphi_1(v), \varphi_2(v))^t \in C([-1,0], R^2)$, $-1 \leq v \leq 0$, with

$$h_{ik}^{(1)}\varphi_1^i(0)\varphi_2^k(0)=\left.\frac{\partial^{i+k}h^{(1)}}{\partial\varphi_1^i\varphi_2^k}\right|_{(\varphi_1,\varphi_2)=(0,0)},$$

$$h_{ikn}^{(2)}\,\varphi_1^i(0)\varphi_2^k(0)\,\breve{\varphi}_2^n\,(-1)=\left.\frac{\partial^{i+k+n}h^{(2)}}{\partial\varphi_1^i\varphi_2^k\breve{\varphi}_2^n}\right|_{(\varphi_1,\varphi_2,\breve{\varphi}_2\ )=(0,0,-1)}$$

By the Riesz representation theorem, there exists a matrix $\mathcal{M}\,(\upsilon,\sigma)$ for $-1\leq\upsilon\leq 0$ such that

$$L_\sigma\varphi=\int_{-1}^{0}d\,\mathcal{M}(\upsilon,\sigma)\varphi(\upsilon)\ \ for\ \ \varphi\epsilon C \tag{36}$$

In fact, choosing

$$\mathcal{M}\,(\upsilon,\sigma)=(\tau_0+\sigma)\,(N_1\delta(\upsilon)-N_1\delta(\upsilon+1)) \tag{37}$$

where $\delta(\upsilon)=\begin{cases}1 & \upsilon=0\\ 0 & \upsilon\neq 0\end{cases}$

For $\varphi\ \in C([-1,0]),R^2)$, define

$$A(\sigma)\varphi(\upsilon)=\begin{cases}\frac{d\varphi(\upsilon)}{d\upsilon}\ , & -1\leq\upsilon<0\,,\\ \int_{-1}^{0}d\,\eta(\upsilon,\sigma)\varphi(\upsilon), & \upsilon=0,\end{cases} \tag{38}$$

And

$$R(\sigma)\varphi(\upsilon)=\begin{cases}0, & -1\leq\upsilon<0\ ,\\ H(\sigma,\varphi), & \upsilon=0.\end{cases} \tag{39}$$

Hence, the system (34) is equivalent to operator differential equation

$$u'(t)=A(\sigma)u_t+R(\sigma)u_t \tag{40}$$

Where $u_t(\upsilon)=u(t+\upsilon)\,,-1\leq\upsilon\leq 0$

For $\Psi\in C^1([-1,0]),(R^2)^*)$, define the adjoint operator $A^*$of $A(0)$

$$A^*\Psi(\mathfrak{H})=\begin{cases}-\frac{d\Psi(\mathfrak{H})}{d\mathfrak{H}}, & 0<\mathfrak{H}\leq 1\ ,\\ \int_{-1}^{0}d\,\eta^T(\mathcal{E},0)\Psi(-\mathcal{E}), & \mathfrak{H}=0\,.\end{cases} \tag{41}$$

For $\varphi\ \in C([-1,0]),R^2)$ and $\Psi\in C^1([-1,0]),(R^2)^*)$. we define the bilinear inner product

$$\langle\Psi(\mathfrak{H}),\varphi(\upsilon)\rangle=\overline{\Psi}(0)\varphi(0)-\int_{\upsilon=-1}^{0}\int_{\varsigma=0}^{\upsilon}\overline{\Psi}^T(\varsigma-\upsilon)d\eta(\upsilon)\varphi(\varsigma)d\varsigma, \tag{42}$$

where $\eta(\upsilon)=\eta(\upsilon,0)$.

It is clear that $A(0)$ and A^* are adjoint operators. Let $q(\upsilon)=(1,d_1)^Te^{i\tau_0\upsilon\omega_1^{\sim}}$ be the eigenvectors of A corresponding to $i\omega_1^{\sim}\tau_0$ and let $q^*(\mathfrak{H})=D(1,d_2)^Te^{-i\tau_0\mathfrak{H}\omega_1^{\sim}}$ be the

eigenvectors of A^* corresponding to $-\mathrm{i}\omega_1^{\sim}\tau_0$. Thus, for $\sigma = 0$ by a simple computation, we obtain
$d_1 = \frac{i\omega_1^{\sim} - h_{10}^{(1)}}{h_{01}^{(1)}},\ d_2 = -\frac{h_{10}^{(1)} + i\omega_1^{\sim}}{h_{100}^{(2)}}.$

From bilinear inner product (42), we obtain

$$\langle q^*(\mathfrak{H}), q(v)\rangle = \bar{D}\left[1 + \tau_0 d_1 \bar{d}_2\, h_{001}^{(2)}\, e^{-i\tau_0\omega_1^{\sim}}\right]$$

Let, $D = \left[1 + \tau_0 \bar{d}_1 d_2 h_{001}^{(2)} e^{-i\tau_0\omega_1^{\sim}}\right]^{-1}$, (43)

here, $\bar{D}$ represents the conjugate complex number of D such that $\langle d_1, d_2\rangle = 1$ and $\langle d_1 , \bar{d}_2\rangle = 0$.

Next, according to the algorithm in Hassard et al. [25], we can determine the expression of g_{20} , g_{11}, g_{02} *and* g_{21} as follows :

$$\left.\begin{aligned} g_{20} &= 2\tau_0\bar{D}\left(\mathcal{g}_1 + \mathcal{g}_5\bar{d}_2\right)\\ g_{11} &= \tau_0\bar{D}\left(\mathcal{g}_2 + \mathcal{g}_6\bar{d}_2\right)\\ g_{02} &= 2\tau_0\bar{D}\left(\mathcal{g}_3 + \mathcal{g}_7\bar{d}_2\right)\\ g_{21} &= 2\tau_0\bar{D}\left(\mathcal{g}_4 + \mathcal{g}_8\bar{d}_2\right)\end{aligned}\right\} \quad (44)$$

where

$\mathcal{g}_1 = h_{11}^{(1)}\ d_1 + h_{02}^{(1)}\ d_{12}^2,$

$\mathcal{g}_2 = h_{11}^{(1)}\ \left(d_1 + \bar{d}_1\right) + 2h_{02}^{(1)}\ d_1\bar{d}_1,$

$\mathcal{g}_3 = h_{11}^{(1)}\ \bar{d}_1 + h_{02}^{(1)}\ d_1^2,$

$$\begin{aligned}\mathcal{g}_4 = h_{11}^{(1)}\Big(& d_1 w_{11}^{(1)}(0) + \tfrac{1}{2}\ \bar{d}_1\ w_{20}^{(1)}(0) + \tfrac{1}{2}\ w_{20}^{(2)}(0) + w_{11}^{(2)}(0)\Big)\\ &+ h_{02}^{(1)}\Big(\bar{d}_1\ w_{20}^{(2)}(0) + 2d_1\ w_{11}^{(2)}(0)\Big),\end{aligned}$$

$\mathcal{g}_5 = h_{110}^{(2)}\ d_1 + h_{020}^{(2)}\ d_1^2 + h_{002}^{(2)}\ d_1^2\ e^{-2i\omega_1^{\sim}\tau_0},$

$\mathcal{g}_6 = h_{110}^{(2)}\left(d_1 + \bar{d}_1\right) + 2h_{020}^{(2)}\ d_1\bar{d}_1 + 2h_{002}^{(2)}\ d_1\bar{d}_1,$

$\mathcal{g}_7 = h_{110}^{(2)}\ \bar{d}_1 + h_{020}^{(2)}\ \bar{d}_1^2 + h_{002}^{(2)}\ \bar{d}_1^2\ e^{2i\omega_1^{\sim}\tau_0},$

$$\begin{aligned}\mathcal{g}_8 = h_{110}^{(2)}\Big(& d_1 w_{11}^{(1)}(0) + \tfrac{1}{2}\bar{d}_1\ w_{20}^{(1)}(0) + \tfrac{1}{2}w_{20}^{(2)}(0) + w_{11}^{(2)}(0)\Big)\\ &+ h_{020}^{(2)}\Big(\bar{d}_1\ w_{20}^{(2)}(0) + 2d_1\ w_{11}^{(2)}(0)\Big),\\ &+ h_{002}^{(2)}\Big(\bar{d}_1\ w_{20}^{(2)}(-1)\ e^{i\omega_1^{\sim}\tau_0} + 2\,d_1\,w_{11}^{(2)}(-1)\ e^{-i\omega_1^{\sim}\tau_0}\Big)\end{aligned}$$

with

$$w_{20}(\theta) = \frac{ig_{20}}{\omega_1^{\sim}\tau_0}\ q(0)\ e^{i\omega_1^{\sim}\tau_0 v} + \frac{i\bar{g}_{02}}{3\omega_1^{\sim}\tau_0}\ \bar{q}(0)\ e^{-i\tau_0\omega_1^{\sim}v} + L_1\ e^{2i\omega_1^{\sim}\tau_0 v} \quad (45)$$

$$w_{11}(\theta) = -\frac{ig_{11}}{\omega_1^{\sim}\tau_0}\ q(0)\ e^{i\omega_1^{\sim}\tau_0 v} + \frac{i\bar{g}_{11}}{\omega_1^{\sim}\tau_0}\bar{q}(0)\ e^{-i\omega_1^{\sim}\tau_0 v} + L_2 \quad (46)$$

$L_1 = \left(L_1^{(1)}, L_1^{(2)}\right)^T$ *and* $L_2 = \left(L_2^{(1)}, L_2^{(2)}\right)^T$ can be found from the following equations:

$$\check{Q}_1 L_1 = 2\tau_0 Q_1. \quad (47)$$

$$\breve{Q}_2\, L_2 = -\tau_0 Q_2\,. \tag{48}$$

where,

$\breve{Q}_1 = \left(2iw_0\tau_0 I - \int_{-1}^{0} d\mathcal{M}(v)\ e^{2iw_0\tau_0 v}\right),$

$\breve{Q}_2 = \left(\int_{-1}^{0} d\mathcal{M}(v)\right),$

$Q_1 = (\mathscr{g}_1 \quad \mathscr{g}_5\ )^T,$

$Q_2 = (\mathscr{g}_2 \quad \mathscr{g}_6\ )^T.$

it is obtained that:

$$\breve{Q}_1 = \begin{pmatrix} 2iw_0 - h_{10}^{(1)} & -h_{01}^{(1)} \\ -h_{100}^{(2)} & 2iw_0 - h_{010}^{(2)} - h_{001}^{(2)} e^{-2iw_0\tau_0\theta} \end{pmatrix}$$

$$\breve{Q}_2 = \begin{pmatrix} -h_{10}^{(1)} & -h_{01}^{(1)} \\ -h_{100}^{(2)} & -h_{010}^{(2)} - h_{001}^{(2)} \end{pmatrix}.$$

Hence, $L_1^{(j)} = \frac{2V_j}{V}, j = 1,2,$ where $\mathrm{V} = determinant$ of $(\breve{Q}_1)$ and V_j is the value of the determinant O_j, where O_j is found by replacing the j^{th} column vector of $\breve{Q}_1$ by Q_1 for $j = 1,2,$. Similarly, $L_2^{(j)} = \frac{2\ \bar{V}_J}{\bar{V}}, j = 1,2,$ where $\bar{V} = determinant\ (\breve{Q}_2\ )$ and $\bar{V}_J$ is the value of the determinant E_i, where E_i is founded by replacing the j^{th} column vector of $\breve{Q}_2$ by Q_2 for $\mathrm{j} = 1,2.$

Hence, $w_{20}(\theta)$ *and* $w_{11}(\theta)$ can be found by using equations (45) - (48). Thus, we can obtain the expressions that are given in equation (36) depending on those given in equation (44) and the proof is completed.

5. Numerical Simulation and Discussion

In this section, we shall use numerical simulation to illustrate the results of our analysis. The following hypothetical parameters have been chosen throughout this section.

$$\begin{array}{c} \mathrm{A} = 0.35 \quad , \ \beta = 0.001 \quad , \ K = 0.5 \quad , \ \mu = 0.1, n = 0.3 \\ r = 0.2 \quad , \quad \tau = 1.9 \quad , \ P = 0 \quad , \alpha = 0.003. \end{array} \tag{49}$$

Matlab is used to draw each of the obtained trajectories for the system (2). Equation's (49) parameters are used to solve system (2) numerically and to confirm our conclusions.
It is observed that for the data given by equation (49) then the trajectories of the system (2) approach to UIEP as shown in Figure (1).

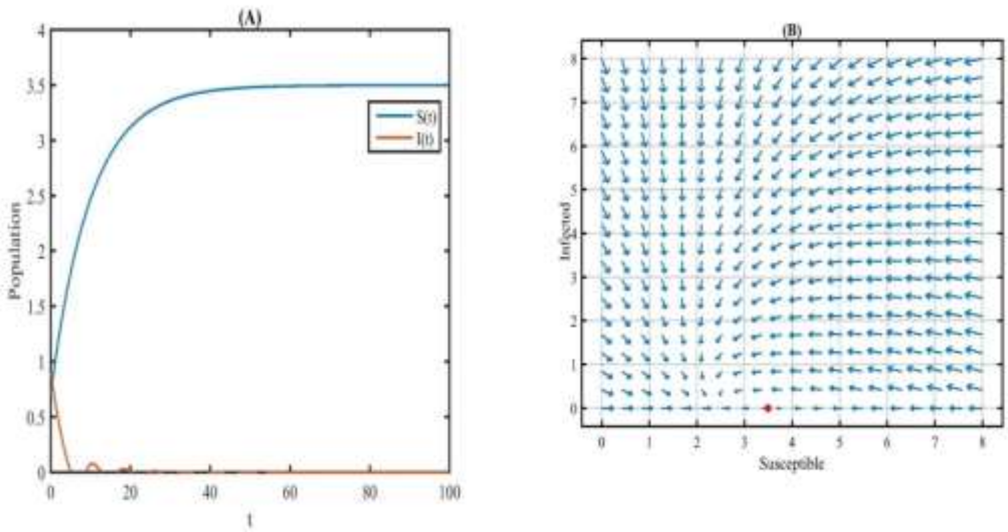

Figure 1: The system's (2) trajectories using the information provided by equation (49).
(A) The trajectories of system (2) approach to UIEP.(B) Phase portrait of the trajectories of the system (2) in which the system approaches UIEP(Red point is the equilibrium point).

It is observed that for given data by equation (49) with $\mathrm{P} = 0.1$ system (2) has a globally asymptotically stable to (EEP) as shown in Figure (2).

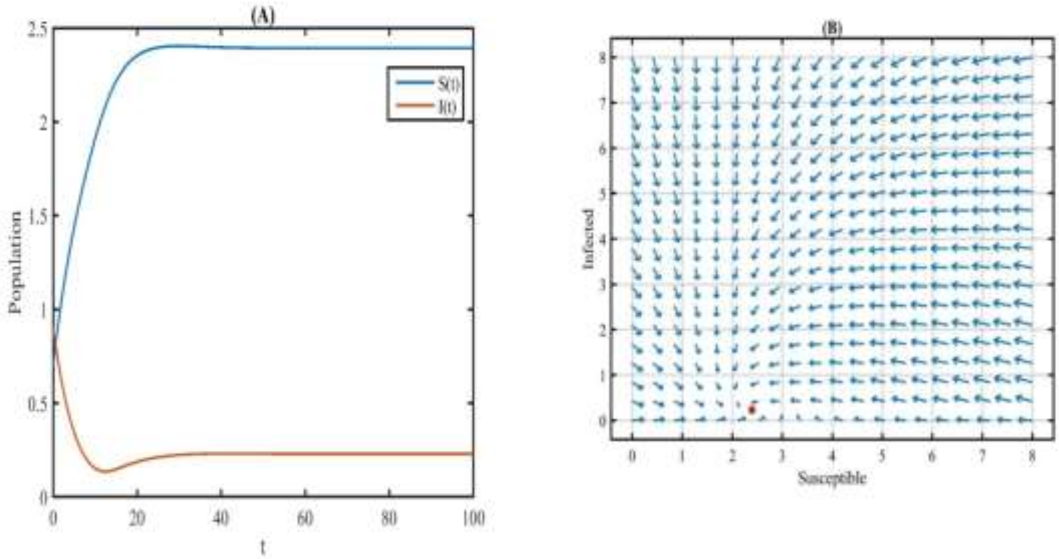

Figure 2: The system's (2) trajectories using the information provided by equation (49). with $\mathrm{P} = 0.1$.

(A) The trajectories of of system (2) approach to EEP. (B) Phase portrait of the trajectories of the system (2) in which the system approaches EEP(Red point is the equilibrium point).

Now, we discuss the effect of the time delay on the system behavior near the EEP point.
For $\tau = 9 < \tau_0 = 10.5$ and $p = 0.3$ with the set of data in equation (49) EEP is still globally asymptotically stable as shown in Figure (3).

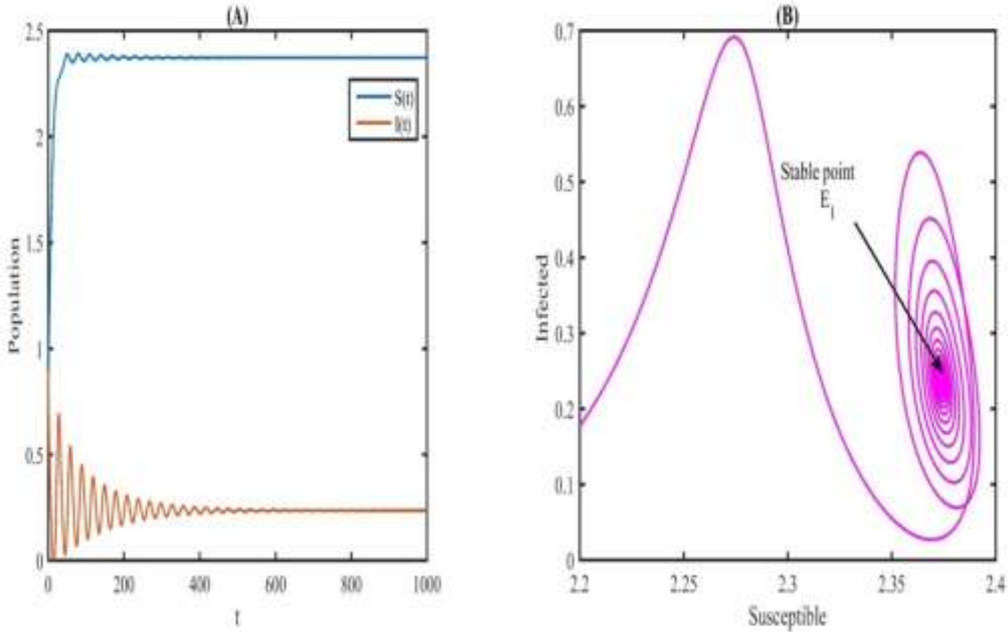

Figure 3: The system's (2) trajectories using the information provided by equation (49) with $p = 0.3$ and $\tau = 9$. (A) The trajectories of system (2) approach EEP. (B) 2D phase plot for globally asymptotically stable EEP.

On the other hand, for $\tau_0 = 10.5$ and $P = 0.3$ with the set of data in equation (49) a Hopf bifurcation occurs at EEP as shown in Figure (4).

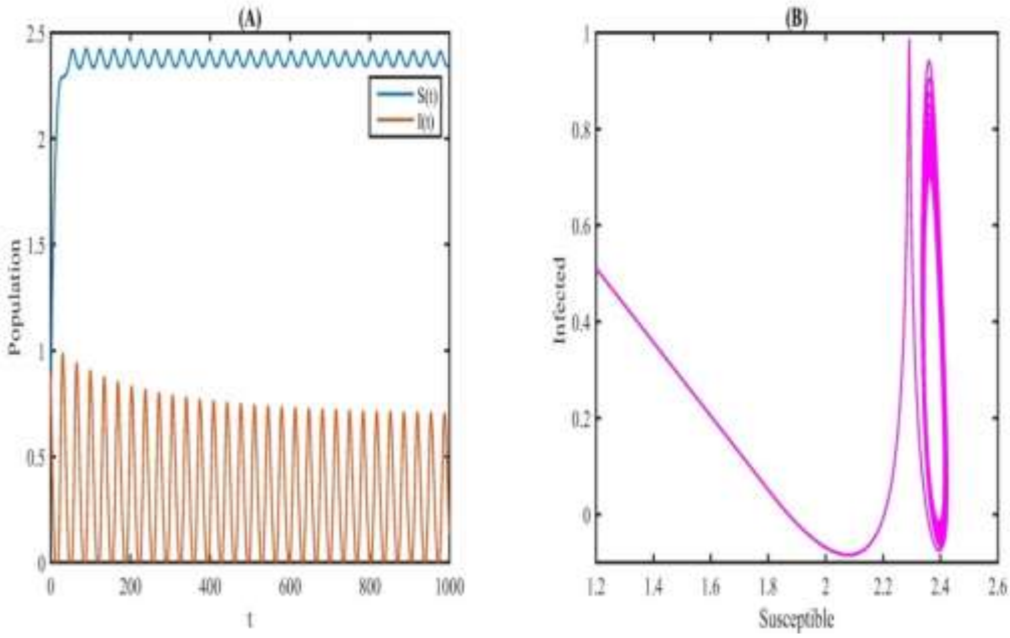

Figure 4: The system's (2) trajectories using the information provided by equation (49) with $P = 0.3$ and
$\tau = 10.5$. (A) The existence of periodic solution neer EEP. (B) 2D periodic solution.

It is observed that for the given data by equation (49) with $p = 0.3$ and
$\tau = 11 > \tau_0 = 10.5$ with the set of data in equation (49) EEP approaches asymptotically to the periodic as shown in Figure (5).

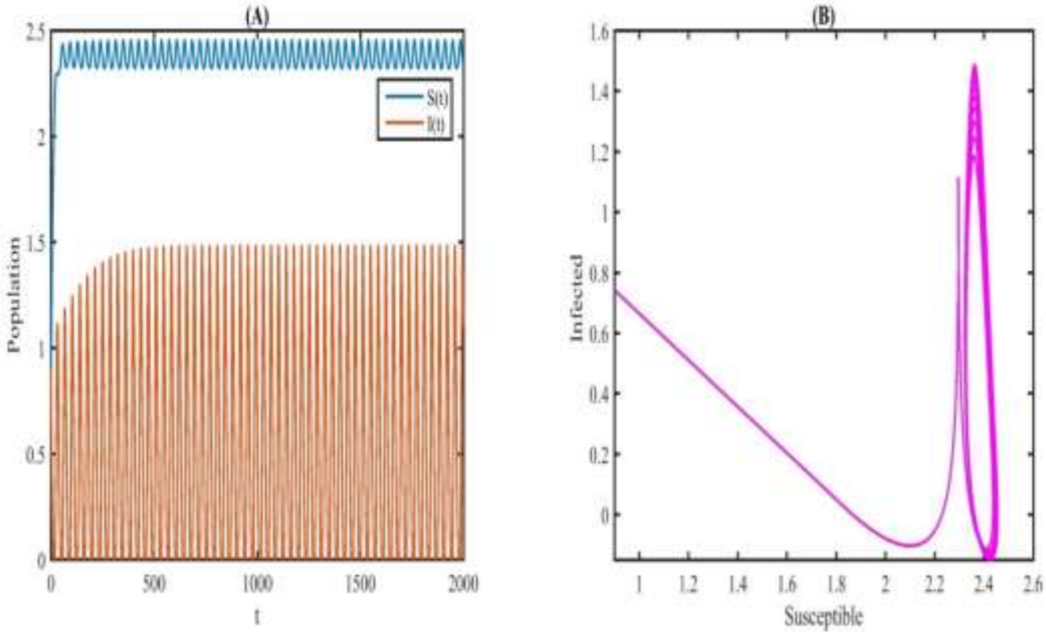

Figure 5: The system's (2) trajectories using the information provided by equation (49)with $\mathrm{p} = 0.3$ and
$\tau = 11$. (A) The existence of periodic solution neer EEP. (B) 2D periodic solution.

6.Conclusion

An epidemic model that involves a time delay for the treatment period has been proposed and studied. The suggested system has two equilibrium points, namely UIEP and EEP. The boundedness of the system has been studied. It is observed that the UIEP is absolutely stable for all $\tau \geq 0$. While the EEP is asymptotically stable for $\tau \in [0, \tau_0)$ and when $\tau = \tau_0$ a Hopf bifurcation occurs. However, the periodic dynamics appear and the point is unstable for $\tau > \tau_0$. Analytically, the periodic dynamics direction and stability have been studied by using normal form and center manifold theory, as well as we study them numerically using MATLAB. For $\tau = 9 < \tau_0 = 10.5$ and $P = 0.3$ with the set of data in equation (49), the EEP is still globally asymptotically stable. While, for $\tau = \tau_0 = 10.5$, a Hopf bifurcation is demonstrated near EEP .

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