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Strongly Cancellation Modules

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Abstract

Let M be an R-module. We introduce in this paper the concept of strongly cancellation module as a generalization of cancellation modules. We give some characterizations about this concept, and some basic properties. We study the direct sum and the localization of this kind of modules. Also we prove that every module over a PID is strongly module and we prove every locally strong module is strongly module.

Keywords: Cancellation modules, quasi fully -cancellation modules, strongly cancellation modules.

المقاسات قوية الحذف

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الخلاصة

ليكن M مقاسا علىR. في هذا البحث قدمنا المقاسات قوية الحذف كأعمام لمفهوم مقاسات الحذف. اعطينا بعض التشخيصات لهذا المفهوم وبعض الخواص الاساسية .درسنا الجمع المباشر والتموضع لهذا النوع من المقاسات. وكذلك اعطينا العلاقة بين هذا النوع من المقاسات والمقاسات الحرة.

الكلمات المفتاحية: مقاسات الحذف،مقاسات شبه الحذف التامة، المقاسات قوية الحذف.

1. Introduction

Let M be an R-module, where R is a commutative ring with unity. The concept of cancellation ideal was introduced by Gilmer in [1]. Also D.D.Anderson and D.F.Anderson studied this kind of ideals in [2], where an ideal I of a ring R is said to be cancellation if for each ideals A,B of R, IA = IB implies A = B. A.S.Mijbass in [3] gave a generalization of this concept namely cancellation module (Weakly cancellation modules) where an R-module M is called cancellation(Weakly cancellation) module if whenever I and J are two ideals of R such that IM = JM implies I = J ($I+ann_RM = J+ann_RM$). Inaam ,M.A. Hadi and Alaa,A.Elewi in [4], introduced the concept of fully cancellation module , where an R-module M is called fully cancellation module if for each ideal I of R and for each submodules N,K of M such that IN = IK implies N = K.In [4] , Inaam ,M.A. Hadi and Alaa,A.Elewi, introduced the concept of naturally cancellation modules, where an R-module M is called naturally cancellation if whenever N, K,W are submodules of M, N.W = N.K implies W = K, where N.K = (N:M)(K:M)M. And M is called quasi-fully cancellation if for each ideal I of R and for each submodules N,K of M such that IN = IK implies $N+ ann_MI = K+ ann_MI$) [5].

In section two we introduce the concept of strongly cancellation module if for each ideals I and J of a ring R such that IN = JN then I = J for every submodule N of M.Also we give the basic properties of these classes of modules, such as every free module over principle ideals domain is strongly cancellation module.

In section three we give some characterizations of strongly cancellation modules. And in section four we give the trace of this kind of modules.

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2. Strongly cancellation modules

An R-module M is called cancellation if whenever I and J are two ideals of R, IM = JM implies I = J [5].

Now, we give a new concept.

Definition 2-1: An R-module M is called strongly cancellation module if for each ideals I and J of R such that IN = JN then I = J for every submodule N of M.

Now, we have the following examples and remarks.

Examples and Remarks 2-2

(1) Every strongly cancellation module is cancellation

Proof It is clear by taking N = M.

(2) The converse of (1) is not true since if we take M = Q$\oplus$ Z as Z-module which is cancellation module [3, Ex.4.8]. But M is not strongly cancellation since if we take N = Q$\oplus$(0),I=(tZ) and J = (m Z) are two ideals of Z, where m, t$\in$Z and t $\neq$ m, then, IN = JN = N. But I $\neq$J. Also, the module M = $Z_2\oplus Z$ as Z-module is cancellation module [3, Ex.4.7], but it is not strongly cancellation since if we take N = $Z_2\oplus(0)$,I = 2Z and J = 4Z . Now, IN = JN = $(\bar{0})\oplus(0)$, but I $\neq$ J, thus M is cancellation module but it is not strongly cancellation module.

(3) It is easy to see that Z as Z-module is cancellation module.

Also, it is strongly cancellation since if we take I = (k) and J = (m) are two ideals of Z, where k, m$\in$Z. Now, let N be a submodule of Z,N = (n) , where n$\in$Z , and supposes IN = JN. Thus (k)(n) = (m)(n) then knZ = mnZ. Therefore kn = mna, for some a$\in$Z .Also, knb = mn, for some b$\in$Z. Thus kn = knba, so ba = 1 in this case we have two cases either a = b = -1 or a = b = 1 ,if a = b = -1, then nk = mk thus m = n. Also, if a = b = 1 thus m= n.

(4) Every submodule of a strongly cancellation module is strongly cancellation.

Proof Let N be submodule of strongly cancellation module M and let K be a submodule of N. Now, suppose that IK = JK, where I and J are two ideals of R. But M is strongly cancellation thus I = J. Thus K is strongly cancellation.

(5) If f:$M_1 \to M_2$ be an epimorphism and M_1 is strongly cancellation module then M_2 is not necessary strongly cancellation module.

For example: $\pi : Z \to \dfrac{Z}{<4>} \cong Z_4$, as we know that Z is strongly cancellation, but one can easily show that Z_4 is not strongly cancellation module.

(6) Let f:M$\to$N be a homomorphism ,where N,M are two R-modules. If N is strongly cancellation module then M is strongly cancellation module.

Proof Let I,J be two ideals of R and K be a submodule of M. Suppose IK = JK and then f(IK) = f(JK). Thus If(K) = Jf(K) , but N is strongly cancellation module and f(K) is a submodule of N, therefore I = J and hence M is strongly cancellation module.

(7) If $M_1 \cong M_2$ then M_1 is strongly cancellation R-module iff M_2 is strongly cancellation R-module.

Proof Since $M_1 \cong M_2$,then there exists an isomorphism f:$M_1 \to M_2$. Now, suppose IN = JN, where I, J are two ideals of R and N is a submodule of M_2. As f is onto, there exists a submodule K of M_1 , such that f(K) = N and hence If(K) = Jf(K). But f is a homomorphism, thus f(IK) = f(JK) and since f is one-to-one then IK = JK. But M_1 is strongly cancellation module, therefore I = J.

The proof of the converse is similarly.

Before we give another remark, lets remember the definition of ann(M) for a module M on a ring R.ann(M) ={x$\in$R:xM =0}. And we say that M is faithful if ann(M) = 0.

(8) Every strongly cancellation module is faithful.

Proof Since M is strongly cancellation module then M is cancellation module (by (1)) and hence M is faithful by [3], Remark(1-4).

Recall that: An element x in an R-module is called a torsion element if r m=0, for some non zero element r$\in$R [6].

Lemma 2-3 Let M be a module over a ring R such that every submodule N of M is generated by non torsion element, then M is strongly cancellation module.

Proof Suppose that IN = JN, where I, J are two ideals of R and N=(x), where x is non torsion element. Thus I(x) = J(x), then for every $a \in I$, $ax \in J(x)$ hence ax = bx where $b \in J$. This is implies (a-b) x=0, but x is non torsion element, thus a-b=0, hence a=b. Therefore $I \leq J$. By similar way, one can easily show that $J \leq I$. Thus I = J.

An ideal I of a ring R is called cancellation ideal if AI = BI then A = B, where A, B are two ideals of R [1, p.66].

Lemma 2-4 Let M be strongly cancellation R-module and I is an ideal of R. If the submodule IM is strongly cancellation R-module, then I is cancellation ideal.

Proof Supposes AI = BI, where A, B are two ideals of R, then AIN = BIN ,for every submodule N of M. But IM is strongly cancellation R-module by hypothesis, so that A = B. Thus I is cancellation ideal.

3. Characterizations for strongly cancellation module

In this section we give characterizations of strongly cancellation module

Theorem 3-1 Let M be an R-module .The following statements are equivalent:-

1. M is a strongly cancellation module.
2. If $IN \subseteq JN$ for every submodule N of M and I,J are two ideals of R, then $I \leq J$.
3. If $(x) N \subseteq JN$ then $x \in J$, where $x \in R$ and J is an ideal of R.
4. $(IN_R:N)=I$ for every ideal I of R and for every submodule N of M, where $(IN:_R N)=\{ x \in R: x N \subseteq IN\}$

Proof (1)⇒(2) Let M be strongly cancellation module and suppose $IN \subseteq JN$, thus JN = IN+JN=(I+J)N. But M is strongly cancellation module, hence J = I+J and therefore $I \leq J$.

(2)⇒(3) Suppose that $(x)N \subseteq JN$, where $x \in R$. By (2), $(x) \subseteq J$ and hence $x \in J$.

(3)⇒(4) Let $x \in (IN:_R N)$, where N is a submodule of M and I is an ideal of R, then $x.N \subseteq IN$ and hence $(x)N \subseteq IN$. By (3) $x \in I$, thus $(IN:_R N) \subseteq I$.

Now, let $y \in I$, hence $yN \subseteq IN \Rightarrow y \in (IN:_R N)$, thus $I \subseteq (IN:_R N)$ and therefore $(IN:_R N) = I$.

(4)⇒(1) suppose that IN = JN for every submodule N of M and for every ideals I,J of R, then $J \subseteq (IN:_R N)$. By (4) $(IN:_R N)=I$, hence $J \subseteq I$. Similarly $I \subseteq J$ and thus I = J. So M is strongly cancellation module.

[3, Lemma 1-10] proved that M is cancellation module if and only if (I:J) = (IM:JM) for every ideals I,J of R.

Now, we will prove the following theorem:

Theorem 3-2 An R-module M is strongly cancellation iff (I:J) = (IN:JN) for every submodule N of M and for every ideals I,J of R.

Proof (⇒) Let $y \in (I:_R J)$, then $yJ \subseteq I$ and hence $yJN \subseteq IN$. Therefore $y \in (IN:_R JN)$.

Now, let $x \in (IN:_R JN)$, thus $xJN \subseteq IN$. Then $xJ \subseteq I$ by [7, Th.2.5] and hence $x \in (I:_R J)$. Thus $(IN:_R JN) \subseteq (I:_R J)$. Therefore $(IN:_R JN)= (I:_R J)$.

(⇐) Now, suppose $(IN:_R JN)= (I:_R J)$, for every ideals I,J of R and for every submodule N of M , and $AN \subseteq BN$ where A,B are two ideals of R. Thus (BN:AN) =(B:A) , then $A \leq B$ and by Th3-1 we have M is strongly cancellation module.

In [3, Prop 3.1], if M is a cancellation R-module, then $M_P \neq 0$ for each maximal ideal P of R. Since every strongly cancellation module is cancellation module, hence $M_P \neq 0$ for each maximal ideal P of R.

An R-module M is called locally cancellation module if $I_p N_p = J_p N_p$ then $I_p = J_p$, for every maximal ideal p of R ,where I,J are two ideals of R ans N is a submodule of M,[8].

Theorem 3-3 Let M be an R-module. If M is locally strongly cancellation module. Then M is strongly cancellation module.

Proof Let I and J are two ideals of R and suppose IN = JN, whrere N is a submodule of M, then $(IN)_p = (JN)_p$, for every maximal ideal p of R. Thus we get $I_p N_p = J_p N_p$, and since the locally cancellation of M is strongly. Then $I_p = J_p$ for every maximal ideal p of R. And by [7, lemma 3-13] , we get I = J. Thus M is strongly cancellation module.

Recall that An R-module M is called quasi cancellation if AM = BM then A = B, where A and B are finitely generated ideals of R [8, p.32].

It is clear that every strongly cancellation R-module is a quasi-cancellation, but the converse is not true since [2, p.884] proved that the quasi cancellation ideal is not cancellation and thus by (Examples &Remarks 2.2), we have the quasi cancellation module is not strongly cancellation module.

$4. The trace of strongly cancellation module

If M is strongly cancellation module then the homomorphic image of M is not necessary strongly cancellation (Examples &Remarks 2.2).

We well study the converse of this relation and we will prove that the homomorphic image of a module M in another module is strongly cancellation then M is also strongly cancellation module.

Theorem 4-1 Let M be a free module over a principle ideals domain. Then M is a strongly module.

Proof Let I and J are two ideals of R. Now, suppose that $IN \subseteq JN$, where N is a submodule of M, thus N is a free module[submodule of free is free[6]. Let $\{x_1, x_1, \ldots.\}$ be a basis for N, then for every $t \in I$, we have $tx_1 \in JN$ where $x_1 \in N$. Then $tx_1 = \sum_{i=1}^{n} b_i x_i$, then we get b_i for every $i \neq 1$, then $tx_1 = b_1 x_1$ thus $t = b_1$ then $t \in J$ therefore $I \subseteq J$. Thus by Th 3.1 M is strongly cancellation module.

Let M be an R- module, where R is an integral domain. The torsion submodule T (M) is define by $T(M)=\{m \in M : \exists 0 \neq r \in R, rm=0\}$.If T(M) = M then M is said to be torsion module [6, p.142].

Corrollary 4-2 Every finitely generated module M over a principle ideal domain R with $T(M) \neq M$ is strongly cancellation module.

Proof: Since M is finitely generated then $\frac{M}{T(M)}$ is finitely generated. Thus, by [8, Th 6, p.99] $\frac{M}{T(M)}$ is a free module and hence $\frac{M}{T(M)}$ strongly cancellation by [Thm 4.1]. Let $\pi : M \rightarrow \frac{M}{T(M)}$ be the natural projection map and by [Ex&Rem 2.2(6)] M is strongly cancellation module.

The following example shows that not every projective module is strongly cancellation.

Example 4.3 Let $M= Z_6$ and $R = Z_6$. It's clear that Z_6 is a free as Z_6 –module and by [6] every free is projective module. Thus Z_6 as Z_6 –module is projective, but by [3, Ex 1.43] M is not cancellation module. Hence it is not strongly cancellation.

Theorem 4-4 If M and N are two modules over a ring R and the submodule $L=\sum_{\lambda \in \Lambda} \varphi_\lambda (M)$ of N, where the summation is taken on any subset of Hom(M,N) satisfies the property strongly cancellation , then M is strongly cancellation.

Proof Suppose IK = JK where I,J are two ideals of R and K is a submodule of M. Then $\varphi_\lambda (IK)=\varphi_\lambda (JK)$ where $\varphi_\lambda : M \rightarrow N$ is a homomorphism for every $\lambda \in \Lambda$, thus $\sum_{\lambda \in \Lambda} \varphi_\lambda (IK)= \sum_{\lambda \in \Lambda} \varphi_\lambda (JK)$.

But $\varphi_\lambda (IK)= I\varphi_\lambda (K)=\varphi_\lambda (JK)= J \varphi_\lambda (K)$. Put $\sum_{\lambda \in \Lambda} \varphi_\lambda (K)=L$, thus IL = JL but L is a submodule satisfies the property of strongly cancellation .Thus I = J. Hence M is strongly cancellation module.

In the module $M = Z_2 \oplus Z$ as Z- module, $ann(Z_2) =2Z$ and $ann(Z) = 0$,thus $ann(Z_2)+ann(Z) \neq Z$ and we show that in EX. and Re. 2.2 that M is not cancellation . Thus the following shows whatever $M_1 \oplus M_2$ is strongly module.

Proposition 4-5 Let M_1 and M_2 be two R-modules such that ann M_1+ann M_2 = R. Then $M = M_1 \oplus M_2$ is strongly cancellation module iff M_1 and M_2 are strongly cancellation module.

Proof (⇒) It is clear.

(⇐) Let I,J be two ideals of R, let N be a submodule of M. As R = ann M_1+ann M_2 then N = $N_1 \oplus N_2$ for some N_1 is a submodule of M_1 , and N_2 is a submodule of M_2 [9,Prop4.2]. Assume IN =JN ,then $I(N_1 \oplus N_2) = J(N_1 \oplus N_2)$,hence $IN_1 \oplus JN_2 = JN_1 \oplus JN_2$. It follows that $IN_1 = JN_1$ and $IN_2 = JN_2$. Since M_1 and M_2 are strongly cancellation modules, then I =J.

Corrollary 4-6 If M is an R-module such that $T_M(R)$ is a cancellation ideal then M is strongly cancellation. Where $T_M(R) = \sum_{\lambda\in\Lambda} \varphi_\lambda(M)$ and the summation is taken for all φ_λ in Hom(M,R).

Proof It follows directly by theorem 4.5

An ideal A of a ring R is said to be pure if A∪B = AB for all ideal B of R [10]

Corrollary 4-7 Let M be a projective R-module and $T_M(R)$ contains a nonzero divisor then M is strongly cancellation and hence $T_M(R)$ =R.

Proof Since M is projective, then $T_M(R)$ is pure ideal [11]. Thus by [3, Cro 8.1] $T_M(R)$ is a strongly cancelation ideal. Hence M is strongly cancellation (by previous corollary) then by [3, Th.5.1] $T_M(R)$ =R.

Corrollary 4-8 Let M be an R-module

1. If $M = \oplus_{i\in I} M_i$ and M_i is strongly cancellation for every i∈I then M is strongly cancellation module.
2. If M contains a submodule N such that every submodule $\frac{M}{N}$ is cyclic and (N:K) =0 for every submodule K of M such that N≤K, then M is strongly cancellation module.

Proof

1. Let $\pi : M \to M_i$ be the natural projection map. Then by [Ex. and Re. 2.2(6)] M is strongly cancellation.
2. Let $\frac{K}{N} \subseteq \frac{M}{N}$. By hypothesis $\frac{K}{N}$ is cyclic, so $\exists k + N \in \frac{K}{N}$ such that $\frac{K}{N}$ =<k+N>. Let r∈R and r(k+N)=0, hence rk∈N. It follows rK⊆N. Thus r∈(N:K)=0. It follows that k+N is a non-torsion element and hence by Lemma 2.3, $\frac{M}{N}$ is strongly cancellation. Now let $\pi : M \to \frac{M}{N}$ be the natural epimorphism, hence by [Ex. and Re. 2.2] M is strongly cancellation.

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