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Convergence of Forked Sequence to a Fixed Point in Modular Function Spaces

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Abstract

In this paper, we present a five-step iterative scheme, at the beginning, it seems complicated and difficult to implement, but in fact it's not. This scheme is constructed for (λ, ρ)- firmly nonexpansive mappings in modular function spaces. Two different ρ-convergences result has been proved for double schemes under consideration. In our study there is a comparison between these cases through answering the question "which one is faster?" Finally, numerical examples are given by using MATLAB software program.

Keywords: Modular spaces, Double sequence, Firmly nonexpansive, Nonexpansive, Iterative scheme, Fixed point.

تقارب المتتابعة المتشعبة الى نقطة صامدة في فضاءات الدالة المعيارية

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الخلاصة

في هذه الورقة ، قدمنا مخطط تكراري من خمس خطوات، للوهلة الأولى ، يبدو أنه متداخل ويصعب تنفيذه ، لكنه في الحقيقة ليس كذلك. تم انشاء هذا المخطط لتطبيقات (ρ ، λ) –غير الموسعة بحزم في فضاءات الدالة المعيارية.في عملنا هذا تم اثبات نتيجتي تقارب مختلفتين للمخطط المزدوج قيد الدراسة. هنالك مقارنة بين هذه الحالات من خلال الاجابة على السؤال ايهما اسرع؟ واخيراً تم اعطاء امثلة عددية وهذه النتائج باستخدام برنامج MATLAB.

1. Introduction

When proving the existence of a fixed point, finding the value of that point is not easy because some functions are complicated, so researchers use iterative scheme to calculate the required fixed point. Over time researcher have worked to develop various iterative schemes to find a faster way to reach the fixed point, many iterative schemes appeared, including Picard, Mann, Ishikawa, Noor, Abbas and others made several improvements to iterative sequences to ensure faster access to the fixed point, see [1]. The fixed point theory plays an important roles in many fields. For instance, it be used in the field of the differential equations in appropriate function spaces. Moreover, many existence theorems in statistics and physics

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have been reduced to fixed point theorems [2]. The concept of the modular space that has been used in this work can be found in [3]. Khamsi and Kozlowski (1990) introduced the concepts fixed point and modular space together with nonexpansive mapping [4]. Ruiz et al. [5] have discussed the ρ-firmly nonexpansive mapping in Banach spaces and this concept has been involved in work of Khan [2] the concept of $(\lambda, \rho) -$ firmly nonexpansive mapping in modular spaces. Recently, the present researchers have presented results in this field for a multi-step iterative sequence adopting $(\lambda, \rho) -$ firmly nonexpansive mappings (multi-valued and single-valued), see [6] and [7].

Now, let E be a non-empty convex subset of L_p where $T: E \longrightarrow 2^E$ and $\rho_P^T(f) = \{g \in T: \rho(f-g) = dist_\rho(f, Tf)\}$. The sequence $\{f_n\}$ by the following iterative process

$$f_1 \in E.$$
$$h_n = (1-\beta_n)f_n + \beta_n u_n.$$
$$g_n = v_n.$$
$$J_n = (1-\alpha_n)g_n + \alpha_n w_n.$$
$$f_{n+1} = m_n, n\epsilon N \quad (1)$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ in (0,1), $u_n \in P_\rho^T(f_n), v_n \in P_\rho^T(h_n),\ w_n \in P_\rho^T(g_n),\ m_n \in P_\rho^T(J_n)$.

Let $T: E \longrightarrow E$, and E be a non-empty convex subset of L_p. We introduced the sequence $\{f_n\}$by the following algorithm.

$$f_1 \in E.$$
$$h_n = (1-\beta_n)f_n + \beta_n T f_n.$$
$$g_n = Th_n.$$
$$J_n = (1-\alpha_n)g_n + \alpha_n T g_n.$$
$$f_{n+1} = TJ_n, n\epsilon N \quad (2)$$

where $\{\alpha_n\}$and $\{\beta_n\}$ are sequences in (0,1).

Some researchers have worked on this field in Hilbert spaces, Banach spaces, and modular spaces [8, 9, and 10]. In 2002, Moore [11] introduced the idea of double sequence iterative in Hilbert spaces and prove the Mann double sequence iteration scheme strongly convergence to fixed point in pseudo contractive map by using the equation $T_k x = (1-\alpha_k)w + \alpha_k T x$ where $w, x \in E$ and $\alpha_k \in (0,1)$. Razani and Moradi [12] studied the double sequence in modular spaces. By using ρ-contractive mapping based on the above equation and presented some example to show the main results. In 2020, Gopinath et al. [8] introduced the double sequence S-iteration in Banach spaces by Lipchitz pseudo contractive map depending on the above equation.

In this paper, we present new double multi step iterative sequences (as in (4) and (5) below) and study its convergence with some other related results and examples.

2. Preliminaries

This section includes the basic definitions and lemmas which are needed for this work.

Definition 2.1 [9]: Let $\rho: M \longrightarrow [0, \infty]$ possesses the following properties:

1- $\rho(0) = 0$ if and only if, $f = 0, \rho - a.e. (a.e.$ means almost everywhere)

2- $\rho(\alpha f) = \rho(f)$, for every scalar $\alpha \in \mathbb{C}\ or\ \mathbb{R}$.

3- $\rho(\alpha x + \beta y) \leq \rho(x) + \rho(y)$ for every $\alpha, \beta \geq 0$ with $\alpha + \beta = 1$.

Where ρ is called a convex modular.

Definition 2.3 [13, 14]: If ρ is a convex modular in M, then L_p is called modular function spaces

$$L_p = \{f \in M: \rho(\lambda f) \longrightarrow 0 \; as \; \lambda \longrightarrow 0\}.$$

The modular space L_p can be equipped with an F-norm defined by
$\|f\|_\rho = \inf\{\alpha > 0 : \rho(\frac{f}{\alpha}) \leq \alpha\}$.
If ρ is a convex modular F-norm if $\|f\|_\rho = \inf\{\alpha > 0 : \rho(\frac{f}{\alpha}) \leq 1\}$, then F-norm is called the Luxemburg norm.

Definition 2.4 [15, 16]: Let $\rho \in \mathcal{R}$
1- A sequence $\{f_n\}$ is ρ-convergent to f if $\rho(f_n - f) \longrightarrow 0$ as $n \to \infty$.
2-A sequence $\{f_n\}$ is ρ-Cauchy sequence if $\rho(f_n - f_m) \longrightarrow 0$ as $n, m \to \infty$.
3-A set $B \subset L_p$ is called ρ-closed if for any $f_n \in L_p$ the convergence $\rho(f_n - f) \longrightarrow 0$ and f belongs to B.
4-A set $B \subset L_p$ is called ρ-bounded if $\rho -$ diameter is finite $\rho -$ diameter define as $\mathfrak{H}_p(B) = \sup\{\rho(f - g), f \in B, g \in B\} < \infty$.
5-A set $B \subset L_p$ is called strongly ρ-bounded if there exists $\beta > 1$ such that $M_p(B) = \sup\{\rho(\beta(f - g)), f \in B, g \in B\} < \infty$.
6-A set $B \subset L_p$ is called ρ-compact, if for every $f_n \in B$, there exists a subsequence $\{f_{n_k}\}$ and f in $\rho(f_{n_k} - f) \to 0$.
7-A set $B \subset L_p$ is called $\rho - a.e$, closed, if every $f_n \in B$, which $\rho - a.e$, converges to some f, then f in B.
8-A set $B \subset L_p$ is called$\rho - a.e$, -compact, if every $f_n \in B$, there exists a subsequence $\{f_{n_k}\}$ $\rho - a.e$ -converges to some f in B.
9-Let f in L_p and $B \sqsubset L_p$, the ρ-distance between f and B is defined as
$dist_p(f, B) = \inf\{\rho(f - g), g \in B\}$.

Definition 2.4 [4]: The map $T: E \longrightarrow E$, where $E \subset L_p$, is said to be ρ-nonexpansive mapping if $\rho(Tf - Tg) \leq \rho(f - g)$, for all f, g in E.

Definition 2.5 [2]: The map $T: E \longrightarrow E$ be is said to be $(\lambda, \rho) -$ firmly nonexpansive mapping if $\rho(Tf - Tg) \leq \rho[(1 - \lambda)(f - g) + \lambda(Tf - Tg)]$, for all f, g in E and $\lambda \in (0,1)$.

Lemma 2.6 [2]: Every (λ, ρ)- firmly nonexpansive mapping is a ρ-nonexpansive mapping.

Now, let L_ρ be a modular space, and $\mathbb{N}$ be the set of natural number, then we define the function $\omega: \mathbb{N} \times \mathbb{N} \longrightarrow L_\rho$ by $\omega(n, m) = f_{n,m} \in L_\rho$.

Definition 2.7 [12]: The double sequence $\{f_{n,m}\}$ is said to be strongly ρ-convergence to z if for any $\epsilon > 0$ where $N, L > 0$ such that $\rho(f_{n,m} - z) < \epsilon$ for $n > N, m > L$ if for all $n, r > N$, $m, t > L$ then $\rho(f_{n,r} - f_{m,t}) < \epsilon$.

Definition 2.8 [8]: The double sequence $\{f_{n,m}\}$ is said to be ρ-Cauchy if for each $\rho(f_{n,m} - z_n) \longrightarrow 0$ and $\rho(z_n - z) \longrightarrow 0$, then $\rho(f_{n,m} - z) \longrightarrow 0$ as $n, m \longrightarrow \infty$.

Lemma 2.9 [17, 18]: Let $\{\rho_n\}$ a non-negative sequence such that

$$\rho_{n+1} \leq (1 - \theta_n)\rho_n + \zeta_n,$$

where $\{\theta_n\}$ sequence in $(0,1)$ and $\{\zeta_n\}$ sequence in real number such that

i- $\lim_{n\to\infty} \theta_n = 0$ and $\sum_{n=1}^{\infty} \theta_n = \infty$.
ii-$\lim sup_{n\to\infty} \frac{\zeta_n}{\theta_n} \leq 0$ or $\sum_{n=1}^{\infty}|\zeta_n| < \infty$ then $\lim_{n\to\infty} \rho_n = 0$.

3. Main Results

By using the iterative scheme that introduced in equation (2), will study forked type of sequences as following, let $T: E \longrightarrow E$, define $T_k: E \longrightarrow E$,where E is a non-empty convex subset of L_p, then we have the following equation

$$T_k f = (1 - \eta_k)Tf + \eta_k w, \tag{3}$$

where η_k in (0,1) and $f, w \in E$.
We introduce the sequence $\{f_{k,n}\}$ by the following algorithm

$$u_{k,n} = \frac{1}{n+1} T_k f_{k,n}.$$
$$h_{k,n} = (1 - \beta_n) f_{k,n} + \beta_n u_{k,n}.$$
$$g_{k,n} = T_k h_{k,n}.$$
$$J_{k,n} = (1 - \alpha_n) g_{k,n} + \alpha_n T_k g_{k,n}.$$
$$f_{k,n+1} = T_k J_{k,n}, n, k \epsilon N$$

(4)

where $\{\alpha_n$, $\{\beta_n\}$ and $\{\eta_k\}$ are sequences in (0,1) and $T_k f$ via equation (3). The sequence $\{f_{k,n}\}, k \geq 0, n \geq 0$ is generated by an arbitrary $f_{0,0} \in E$

$$f_{k,n}^1 = (1 - \gamma_{n.0}) T_k f_{k,n} + \gamma_{n,0} f_{k,n}$$
$$f_{k,n}^2 = (1 - \gamma_{n.1}) T_k {f_{k,n}}^1 + \gamma_{n,1} f_{k,n}$$
$$f_{k,n}^3 = (1 - \gamma_{n.2}) T_k {f_{k,n}}^2 + \gamma_{n,2} f_{k,n}$$
$$\vdots$$
$$\vdots$$
$$f_{k,n}^m = (1 - \gamma_{n.m-1}) T_k {f_{k,n}}^{m-1} + \gamma_{n,m-1} f_{k,n}$$
$$f_{k,n+1} = (1 - \gamma_{n.m}) T_k {f_{k,n}}^m + \gamma_{n,m} f_{k,n}. \tag{5}$$

Where $\gamma_{n,i}$ is real sequence in (0,1).
Suppose $\gamma_{n,m}, \eta_k$ in equations (3) and (5), so the following three condition are satisfied.
i- $\lim_{n\to\infty} \gamma_{n,m} = \lim_{k\to\infty} \eta_k = 0$.
ii- For all $f \in E, c \in R^+, \rho\big(c(f - w)\big) \leq v < \infty$, where $w, v \in E$.
iii- T_k has unique fixed point and $F_p(T) \neq \emptyset$.
Then, we prove the following theorem.

Theorem 3.1: Let $\rho \in \mathcal{R}$ be ρ- complete, convex modular spaces, E subset of L_ρ which is ρ-closed, ρ-bounded and convex, $T: E \longrightarrow E$ is (λ, ρ)-firmly nonexpansive mapping and $T_k: E \longrightarrow E$, then $\{f_{k,n}\}$ in equation (4) is ρ-strong convergence to fixed point s of T in E and $\rho(Tf_{k,n} - f_{k,n}) \longrightarrow 0$.
Proof: To prove T_k is (λ, ρ)-firmly non-expansive mapping, let f, g and w in E.
By (3), and T is (λ, ρ)-firmly nonexpansive mapping, we get

$$\rho(T_k f - T_k g) = \rho((1 - \eta_k)Tf + \eta_k w - (1 - \eta_k)Tg - \eta_k w)$$
$$\leq (1 - \eta_k)\rho(Tf - Tg).$$

By condition (i) $\lim_{k\to\infty} \eta_k = 0$.

$$\rho(T_k f - T_k g) \leq \rho(Tf - Tg)$$
$$\leq \rho((1 - \lambda)(f - g) + \lambda(Tf - Tg))$$
$$\leq \rho((1 - \lambda)(f - g) + \lambda(\frac{1}{(1-\eta_k)} T_k f - \frac{\eta_k}{(1-\eta_k)} w - \frac{1}{(1-\eta_k)} T_k g + \frac{\eta_k}{(1-\eta_k)} w)$$
$$\leq \rho((1 - \lambda)(f - g) + \lambda(T_k f - T_k g),$$

where T_k is (λ,ρ)-firmly nonexpansive mapping. Now, by condition (iii) let s_k be unique fixed point of T_k in E, to prove $\rho(f_{k,n}-s_k)\longrightarrow 0$ as $n\longrightarrow\infty$.
By equation (5) and Lemma 2.6, we get

$$\begin{aligned}\rho(f_{k,n+1}-s_k) &\le (1-\gamma_{n.m})\rho(T_k {f_{k,n}}^m - s_k)+\gamma_{n,m}\rho(f_{k,n}-s_k)\\ &\le (1-\gamma_{n.m})\rho({f_{k,n}}^m - s_k)+\gamma_{n,m}\rho(f_{k,n}-s_k)\\ &\le (1-\gamma_{n.m})[(1-\gamma_{n.m-1})\rho({f_{k,n}}^{m-1}-s_k)+\gamma_{n,m-1}\rho(f_{k,n}-s_k)]+\gamma_{n,m}\rho(f_{k,n}-s_k)\\ &\vdots\\ &\le \gamma_{n,m}\rho\big(f_{k,n}-s_k\big)+(1-\gamma_{n.m})\big[\gamma_{n,m-1}\rho\big(f_{k,n}-s_k\big)+(1-\gamma_{n.m-1})\big[\gamma_{n,m-2}\rho\big(f_{k,n}-s_k\big)\\ &\quad +(1-\gamma_{n.m-2})\big[\gamma_{n,m-3}\rho\big(f_{k,n}-s_k\big)+\cdots+(1-\gamma_{n.1})\big[\gamma_{n,0}\rho\big(f_{k,n}-s_k\big)\big]\big]\\ &\quad +\cdots\big]\big].\end{aligned}$$

So, $\rho(f_{k,n+1}-s_k)\le\mu_n\rho(f_{k,n}-s_k)$,
where, $\mu_n=\gamma_{n,m}+(1-\gamma_{n.m})[\gamma_{n.m-1}+(1-\gamma_{n.m-1})\left[\gamma_{n,m-2}+(1-\gamma_{n.m-2})\left[\gamma_{n,m-3}+\cdots+(1-\gamma_{n.1})\left[\gamma_{n,0}\right]\right]+\cdots\right]$.
By condition (i) $\lim_{n\to\infty}\gamma_{n,m}=0$, and Lemma 2.9 $\rho(f_{k,n}-s_k)\longrightarrow 0$. Now, to prove s_k is fixed point to T in E. By equation (3)
$T_k s_k=(1-\eta_k)Ts_k+\eta_k w$.
Since s_k is fixed point to T_k in E, then $s_k=(1-\eta_k)Ts_k+\eta_k w$. Using condition (ii), to

$$\rho(s_k-Ts_k)=\rho\left(s_k-\frac{1}{(1-\eta_k)}s_k+\frac{\eta_k}{(1-\eta_k)}w\right)=\rho\left(\frac{\eta_k}{(1-\eta_k)}(w-s_k)\right)\le\frac{\eta_k}{(1-\eta_k)}v.$$

Using condition (i), $\lim_{k\to\infty}\eta_k=0$, then $\rho(s_k-Ts_k)\longrightarrow 0$ as $k\longrightarrow\infty$, hence $\{s_k\}$ is approximate fixed point sequence in T. We have to prove s_k is ρ-Caushy

$$\begin{aligned}(s_m-s_n) &= (1-\eta_m)Ts_m+\eta_m w-(1-\eta_n)Ts_n-\eta_n w\\ &= (\eta_m-\eta_n)w-\eta_m(Ts_m-Ts_n)+(\eta_n-\eta_m)Ts_n-(Ts_n-Ts_m)\\ \rho(s_m-s_n) &\le (\eta_m-\eta_n)\rho(w)-\eta_m\rho(Ts_m-Ts_n)+(\eta_n-\eta_m)\rho(Ts_n)-\rho(Ts_n-Ts_m)\\ &\le (\eta_m-\eta_n)\rho(w)-\eta_m\rho(Ts_m-Ts_n)+(\eta_n-\eta_m)\rho(Ts_n).\end{aligned}$$

Since T is (λ,ρ)-firmly nonexpansive mapping and by Lemma 2.6.
$\rho(s_m-s_n)\le\frac{(\eta_m-\eta_n)}{1+\eta_m}\rho(w)+\frac{(\eta_n-\eta_m)}{1+\eta_m}\rho(Ts_n)$, by using condition (i) $\lim_{k\to\infty}\eta_k=0$, so $\{s_k\}$ is ρ-Caushy sequence.
Since L_ρ is ρ-complete, there exists s in E such that $\rho(s_k-s)\longrightarrow 0$ as $k\longrightarrow\infty$ and T is (λ,ρ)-firmly nonexpansive mapping and by Lemma 2.6
$\rho(Ts_k-Ts)\le\rho(s_k-s)$ implies that $\rho(Ts_k-Ts)\longrightarrow 0$ as $k\longrightarrow\infty$
By using ρ-Caushy sequence, the sequence $\{f_{k,n}\}$ convergence to s. To prove s fixed point of T in E.

$$\begin{aligned}\rho(Ts-s) &\le \rho(Ts-Ts_k)+\rho(Ts_k-s_k)+\rho(s_k-s)\\ &\le \rho(s-s_k)+\rho(Ts_k-s_k)+\rho(s_k-s).\end{aligned}$$

So, $\rho(Ts-s)\longrightarrow 0$, s fixed point of T.
Finally, to prove $\rho(Tf_{k,n}-f_{k,n})\longrightarrow 0$ as $k,n\longrightarrow\infty$.

$$\begin{aligned}\text{So, } \rho\big(Tf_{k,n}-f_{k,n}\big) &\le \rho\big(Tf_{k,n}-Ts_k\big)+\rho(Ts_k-s_k)+\rho(s_k-f_{k,n})\\ &\le \rho\big(f_{k,n}-s_k\big)+\rho(Ts_k-s_k)+\rho\big(s_k-f_{k,n}\big)\end{aligned}$$

Then $\rho(Tf_{k,n}-f_{k,n})\longrightarrow 0$, the proof is complete.
When the value of $w=0$, it is possible to discuss this case, so the equation (3) become the following form.

$$T_k f=(1-\eta_k)Tf \tag{6}$$

Suppose α_n,β_n and η_k in equations (4) and (6), the following three conditions are satisfied
i-$\sum_{n=0}^{\infty}\alpha_n=\infty$ and $\lim_{n\to\infty}\alpha_n=0$.
ii-$\lim_{n\to\infty}\beta_n=\lim_{k\to\infty}\eta_k=0$.

iii-$F_\rho(T) \neq \emptyset$.

Based on equations (4) and (6) in addition to the above three condition, we prove the following theorem.

Theorem 3.2: Let $\rho \in \mathcal{R}$ is ρ- complete, convex modular spaces, E subset of L_ρ is ρ-closed, ρ-bounded and convex, $T: E \longrightarrow E$ is (λ, ρ)-firmly nonexpansive mapping and $T_k: E \longrightarrow E$, then $\{f_{k,n}\}$ in equation (4) is ρ-strongly convergence to fixed point s of T in E, where $\{\alpha_n\}, \{\beta_n\}$ and, $\{\eta_k\}$ be real sequence in (0,1).

Proof: Let s_k is the fixed point of T_k, by (4), (6) and Lemma2.6

$$\rho(u_{k,n} - s_k) = \frac{1}{n+1}\rho(T_k f_{k,n} - s_k) \leq (1-\eta_k)\rho(f_{k,n} - s_k) \quad (7)$$

So, by (4) and (7)

$$\rho(h_{k,n} - s_k) \leq (1-\beta_n)\rho(f_{k,n} - s_k) + \beta_n \rho(u_{k,n} - s_k)$$
$$\leq [(1-\beta_n) + (1-\eta_k)\beta_n]\rho(f_{k,n} - s_k). \quad (8)$$

Similarity, by using (4), (8) and Lemma 2.6.

$$\rho(g_{k,n} - s_k) \leq \rho(T_k h_{k,n} - s_k) \leq [(1-\beta_n)(1-\eta_k) + (1-\eta_k)^2\beta_n]\rho(f_{k,n} - s_k). \quad (9)$$

By the same way, using (4), (9) and Lemma 2.6.

$$\rho(J_{k,n} - s_k) \leq (1-\alpha_n)\rho(g_{k,n} - s_k) + \alpha_n \rho(T_k g_{k,n} - s_k)$$
$$\leq [(1-\alpha_n) + \alpha_n(1-\eta_k)]\rho(g_{k,n} - s_k)$$
$$\leq [(1-\alpha_n) + \alpha_n(1-\eta_k)][(1-\beta_n)(1-\eta_k) + (1-\eta_k)^2\beta_n]\rho(f_{k,n} - s_k). \quad (10)$$

By (4), (10) and Lemma 2.6.

$$\rho(f_{k,n+1} - s_k) \leq \rho(T_k J_{k,n} - s_k) \leq (1-\eta_k)\rho(J_{k,n} - s_k) \leq \mu_n \rho(f_{k,n} - s_k) \quad (11)$$

and $\mu_n = [(1-\alpha_n)(1-\beta_n)(1-\eta_k)^2 + \alpha_n(1-\beta_n)(1-\eta_k)^3 + (1-\alpha_n)(1-\eta_k)^3\beta_n + \alpha_n\beta_n(1-\eta_k)^4]$.

Through Lemma 2.9, the first and second conditions above become clear $\rho(f_{k,n} - s_k) \longrightarrow 0$.

Now, by equation (3)

$T_k s_k = (1-\eta_k)Ts_k$. Since s_k is the fixed point of T_k, Then $s_k = (1-\eta_k)Ts_k$.

$\rho(Ts_k - s_k) = \rho(Ts_k - (1-\eta_k)Ts_k) \leq \eta_k \rho(Ts_k)$

By condition (ii), $\rho(Ts_k - s_k) \longrightarrow 0$, then $\{s_k\}$ is an approximate fixed point sequence of T.

So, $(s_m - s_n) = (1-\eta_m)Ts_m - (1-\eta_n)Ts_n$

$$= (\eta_n - \eta_m)Ts_n - \eta_m(Ts_m - Ts_n) - (Ts_n - Ts_m)$$

$$\rho(s_m - s_n) \leq (\eta_n - \eta_m)\rho(Ts_n) - \eta_m\rho(Ts_m - Ts_n) - \rho(Ts_n - Ts_m)$$
$$\leq (\eta_n - \eta_m)\rho(Ts_n) - \eta_m\rho(Ts_m - Ts_n).$$

Since T is (λ, ρ)-firmly nonexpansive mapping and by Lemma 2.6.

$\rho(s_m - s_n) \leq \frac{(\eta_n - \eta_m)}{1+\eta_m}\rho(Ts_n)$, by using condition (ii), $\lim_{k\to\infty}\eta_k = 0$, then $\{s_k\}$ is ρ-Caushy sequence.

Since L_ρ is ρ-complete, there exists s in E such that. $\rho(s_k - s) \longrightarrow 0$ as $\longrightarrow \infty$, T is (λ, ρ)-firmly nonexpansive mapping and by Lemma 2.6.

$\rho(Ts_k - Ts) \leq \rho(s_k - s)$, so $\rho(Ts_k - Ts) \longrightarrow 0$.

By using ρ-Caushy sequence, the sequence $\{f_{k,n}\}$ convergence to s. In the rest of proof, we show that s is fixed point of T in E.

$$\rho(Ts - s) \leq \rho(Ts - Ts_k) + \rho(Ts_k - s_k) + \rho(s_k - s)$$
$$\leq \rho(s - s_k) + \rho(Ts_k - s_k) + \rho(s_k - s)$$

So, $\rho(Ts - s) \longrightarrow 0$, s fixed point of T.

4. Comparison Results

Definition 4.1 [19]: Let $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ by two iterative scheme sequences converging to the same fixed point s, and let $\lim_{n\to\infty}\frac{\rho(a_n-s)}{\rho(b_n-s)}=L$, then

1- If $L=0$ then $\{a_n\}_{n=1}^{\infty}$ converges faster than $\{b_n\}_{n=1}^{\infty}$ to fixed point s.

2- If $1<L<\infty$ then $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ they reach to the fixed point at the same speed.

Theorem 4.2: Let $\rho\in\Re$, E be a non-empty ρ-bounded, ρ-closed and convex $E\subset L_p$ and $T\colon E\longrightarrow E$, be (λ,ρ)-firmly nonexpansive multivalued mapping $T_k\colon E\longrightarrow E$ be (λ,ρ)-firmly nonexpansive multivalued mapping, let $\{\alpha_n\},\{\beta_n\},\{\eta_k\}$ be real sequences in (0,1), then the iterative scheme in (4) by using equation $T_kf=(1-\eta_k)Tf$ faster of iterative scheme in (4) by using equation $T_kf=(1-\eta_k)Tf+\eta_k w$.

Proof: By using equations (3) and (4), convexity of ρ, Definitions 2.4 and 2.5 and Lemma 2.6, implies that

$$\rho(f_{k,n+1}-s_k)=\rho(T_kJ_{k,n}-s_k)\le(1-\eta_k)\rho(J_{k,n}-s_k)+\eta_k\rho(w-s_k)$$
$$\le(1-\eta_k)[(1-\alpha_n)\rho(g_{k,n}-s_k)+\alpha_n\rho(T_kg_{k,n}-s_k)]+\eta_k\rho(w-s_k)$$
$$\le[(1-\eta_k)(1-\alpha_n)+\alpha_n(1-\eta_k)^2]\rho(g_{k,n}-s_k)+[\alpha_n\eta_k(1-\eta_k)+\eta_k]\rho(w-s_k)$$
$$\le[(1-\eta_k)^2(1-\alpha_n)+\alpha_n(1-\eta_k)^3]\rho(h_{k,n}-s_k)+[(1-\eta_k)(1-\alpha_n)\eta_k+\alpha_n\eta_k(1-\eta_k)^2+\alpha_n\eta_k(1-\eta_k)+\eta_k]\rho(w-s_k)$$
$$\le\mu_n\rho(f_{k,n}-s_k)+\psi_n\rho(w-s_k).$$

Where

$$\mu_n=[(1-\alpha_n)(1-\beta_n)(1-\eta_k)^2+\alpha_n(1-\beta_n)(1-\eta_k)^3+(1-\alpha_n)(1-\eta_k)^3\beta_n+\alpha_n\beta_n(1-\eta_k)^4].$$

and

$$\psi_n=[(1-\alpha_n)(1-\eta_k)\eta_k+\alpha_n(1-\eta_k)^2\eta_k+\eta_k(1-\eta_k)\alpha_n+\eta_k(1-\eta_k)^2(1-\alpha_n)\beta_n+\eta_k\alpha_n\beta_n(1-\eta_k)^3+\eta_k].$$

Then

$$\rho(f_{k,n}-s_k)\le(\mu_n)^{n+1}(f_{k,0}-s_k)+(1+\mu_n+(\mu_n)^2+\cdots.+(\mu_n)^n)\psi_n\rho(w-s_k). \quad (12)$$

By the same of previous proof and by using (6), we get

$$\rho(f_{k,n}-s_k)\le(\mu_n)^{n+1}(f_{k,0}-s_k). \quad (13)$$

By definition 4.1 and equations (12) and (13). The proof is completed.

Below we present an example illustrating the previous theorem

Example 4.3: Let $L_\rho=\mathbb{R}$, the set of real number, ρ be absolute value and $T\colon E\longrightarrow E$, $E=[0,\infty)$, T be define by $Tf=\frac{f}{4}$, $T_k\colon E\longrightarrow E$, T_k define by (6), and the double sequence define by (4), the fixed point of T is $s=0$, where $\alpha_n=\beta_n=0.5$, $\eta_k=\frac{k}{k+40}$, and let $k=100$, and by using (6) then the iterative scheme will become

$u_{100,n}=\frac{1}{n+1}\left(\frac{2}{7}\right)\frac{f_{100,n}}{4}$, $h_{100,n}=0.5f_{100,n}+0.5u_{100,n}$, $g_{100,n}=\left(\frac{2}{7}\right)\frac{h_{100,n}}{4}$,

$J_{100,n}=0.5g_{100,n}+0.5\left(\frac{2}{7}\right)\frac{g_{100,n}}{4}$, $f_{100,n+1}=\left(\frac{2}{7}\right)\frac{J_{100,n}}{4}$, $n\epsilon N$.

Table1 and Figure1 show the numerical results with some step, when $f_{100,1}=1.5$.

Also, see Table2 and Figure2 when $u_{100,n}=\frac{1}{n+1}\left(\frac{2}{7}\right)\frac{f_{100,n}}{4}$, $h_{100,n}=0.8f_{100,n}+0.2u_{100,n}$, $g_{100,n}=\left(\frac{2}{7}\right)\frac{h_{100,n}}{4}$.

$J_{100,n}=0.8g_{100,n}+0.2\left(\frac{2}{7}\right)\frac{g_{100,n}}{4}$, $f_{100,n+1}=\left(\frac{2}{7}\right)\frac{J_{100,n}}{4}$, $n\epsilon N$.

Table 1: Shown $f_{k,n}$ in (3) by using equation (5), where α_n= β_n= 0.5, $\eta_k = \frac{k}{k+40}$, $k =$ 100 with $f_{100,n} = 1.5$.

n	$f_{100,n}$	$u_{100,n}$	$h_{100,n}$	$g_{100,n}$	$J_{100,n}$
1	1.5	0.053571	0.77679	0.055485	0.029724
2	0.0021231	5.0551e-005	0.0010868	7.7632e-005	4.1588e-005
3	2.9706e-006	5.3046e-008	1.5118e-006	1.0799e-007	5.785e-008
4	4.1322e-009	5.9031e-011	2.0956e-009	1.4969e-010	8.0189e-011
5	5.7278e-012	6.8188e-014	2.898e-012	2.07e-013	1.1089e-013
6	7.9209e-015	8.0825e-017	4.0008e-015	2.8577e-016	1.5309e-016
7	1.0935e-017	9.7636e-020	5.5164e-018	3.9403e-019	2.1109e-019
8	1.5078e-020	1.1966e-022	7.5987e-021	5.4277e-022	2.9077e-022
9	2.0769e-023	1.4835e-025	1.0459e-023	7.4705e-025	4.0021e-025
10	2.8586e-026	1.8562e-028	1.4386e-026	1.0276e-027	5.5048e-028
11	3.932e-029	2.3405e-031	1.9777e-029	1.4126e-030	7.5677e-031
12	5.4055e-032	2.9701e-034	2.7176e-032	1.9412e-033	1.0399e-033
13	7.4279e-035	3.7897e-037	3.7329e-035	2.6663e-036	1.4284e-036
14	1.0203e-037	4.8585e-040	5.1257e-038	3.6612e-039	1.9614e-039
15	1.401e-040	6.2544e-043	7.0362e-041	5.0258e-042	2.6924e-042
16	1.9232e-043	8.0805e-046	9.6562e-044	6.8973e-045	3.695e-045
17	2.6393e-046	1.0473e-048	1.3249e-046	9.4633e-048	5.0696e-048
18	3.6212e-049	1.3613e-051	1.8174e-049	1.2981e-050	6.9543e-051
19	4.9674e-052	1.7741e-054	2.4926e-052	1.7804e-053	9.5378e-054
20	6.8127e-055	2.3173e-057	3.418e-055	2.4414e-056	1.3079e-056

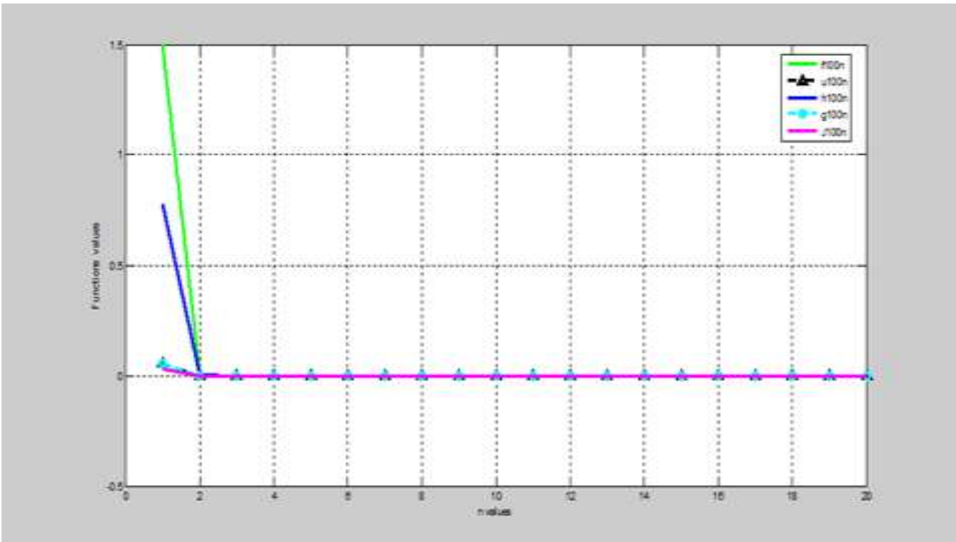

Figure 1: The function $f_{100,n}$, $u_{100,n}$, $h_{100,n}$, $g_{100,n}$ and $J_{100,n}$, where α_n= β_n= 0.5, $\eta_k = \frac{k}{k+40}$, $k = 100$ with $f_{100,n} = 1.5$.

Table 2: Shown $f_{k,n}$ in (3) by using equation (5), where α_n= β_n= 0.2, $\eta_k = \frac{k}{k+40}$, $k =$ 100 with $f_{100,n} = 1.5$.

n	$f_{100,n}$	$u_{100,n}$	$h_{100,n}$	$g_{100,n}$	$J_{100,n}$
1	1.5	0.053571	1.2107	0.08648	0.070419
2	0.0050299	0.00011976	0.0040479	0.00028914	0.00023544
3	1.6817e-005	3.003e-007	1.3514e-005	9.6527e-007	7.86e-007
4	5.6143e-008	8.0204e-010	4.5075e-008	3.2196e-009	2.6217e-009
5	1.8726e-010	2.2293e-012	1.5026e-010	1.0733e-011	8.7395e-012
6	6.2425e-013	6.3699e-015	5.0067e-013	3.5762e-014	2.9121e-014
7	2.08e-015	1.8572e-017	1.6678e-015	1.1913e-016	9.7002e-017
8	6.9287e-018	5.499e-020	5.554e-018	3.9671e-019	3.2304e-019
9	2.3074e-020	1.6481e-022	1.8492e-020	1.3209e-021	1.0756e-021
10	7.6826e-023	4.9887e-025	6.1561e-023	4.3972e-024	3.5806e-024
11	2.5576e-025	1.5224e-027	2.0491e-025	1.4636e-026	1.1918e-026
12	8.513e-028	4.6775e-030	6.8197e-028	4.8712e-029	3.9666e-029
13	2.8333e-030	1.4455e-032	2.2695e-030	1.6211e-031	1.32e-031
14	9.4287e-033	4.4899e-035	7.552e-033	5.3943e-034	4.3925e-034
15	3.1375e-035	1.4007e-037	2.5128e-035	1.7948e-036	1.4615e-036
16	1.0439e-037	4.3863e-040	8.3603e-038	5.9716e-039	4.8626e-039
17	3.4733e-040	1.3783e-042	2.7814e-040	1.9867e-041	1.6177e-041
18	1.1555e-042	4.3441e-045	9.253e-043	6.6093e-044	5.3818e-044
19	3.8442e-045	1.3729e-047	3.0781e-045	2.1986e-046	1.7903e-046
20	1.2788e-047	4.3496e-050	1.0239e-047	7.3136e-049	5.9554e-049

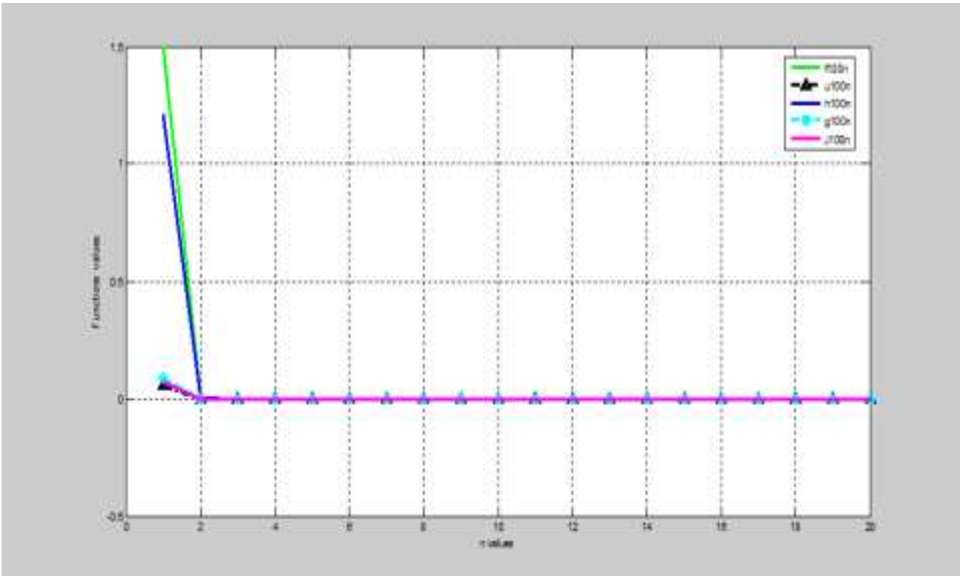

Figure 2: The function $f_{100,n}$, $u_{100,n}$, $h_{100,n}$, $g_{100,n}$ and $J_{100,n}$, where α_n= β_n= 0.2, $\eta_k = \frac{k}{k+40}$, $k = 100$ with $f_{100,n} = 1.5$.

In the above example, it is clear that the iterative scheme presented in equation (4) approaches the fixed point at a record speed. In addition, when α_n= β_n= 0.5 it is faster to reach the fixed point when α_n= β_n= 0.2.

5-Conclusions

A new five step iterative scheme have been presented in this paper with double sequence in (λ, ρ)-firmly nonexpansive mapping of modular function spaces. As well as a new formula for the T_k function has been defined through Theorem 4.2, where it has been shown the special case of equation (6) is faster to reach the fixed point than equation (3). In addition, as the value of n increases than the double sequence $\{f_{k,n}\}$ approaches the fixed point, as shown in example 4.3. It is worth nothing that we aspire to obtain results related to what Tarsh and Abed presented in [20].

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References

[1] K. Ullah, and M.Arshad," Numerical Reckoning Fixed Point for Suzuki Generalized Nonexpansive Mapping via new Iteration Process" *Faculty of Science and Mathematics,* vol.32, no.1, pp 187-196, 2018.

[2] S.H. Khan, "Approximating Fixed Point of (λ, ρ)- Firmly Nonexpansive Mappings in Modular Function Spaces" *Arab Journal of Mathematics,* vol.7, pp281–287, 2018.

[3] H. Nakano, (1950) "Modular Semi-ordered Spaces" *Tokyo Mathematical Book Series,* Maruzen Co. Ltd, Tokyo, Japan, 1950.

[4] M.A. Khamsi, and W. Kozlowski, "Fixed Point Theory in Modular Function Spaces" *Journal of Nonlinear Analysis,* vol.1990, pp 1-35, 1990.

[5] D. Ruiz, G.L. Acedo, and V.M. Marquez, "Firmly Nonexpansive Mapping" *Journal of Nonlinear and Convex Analysis,* vol.15, no.1, pp-61-87, 2014.

[6] B.B. Salman, and S.S. Abed, "New Accelerated Iterative Algorithm for (λ, ρ)-Quasi Firmly Nonexpansive Multivalued Mappings", *International Journal of Nonlinear Analysis Applications,* vol. 14, no.1, 1825–1833, 2023.

[7] B.B. Salman, and S.S. Abed, "A New Iterative Sequence of (λ,ρ)-Firmly Nonexpansive Multivalued Mappings in Modular Function Spaces" Mathematical Modeling of Engineering Problems, vol. 10, no. 1, pp. 212-219, 2022 .

[8] S. Gopinath, J. Gnanaraj, and S. Lalithambigai, "A Double Sequence Hybrid s-Iteration Scheme for Fixed Point of Lipchitz Pseud-contraction in Banach space" Palestine Journal of Mathematics, vol.9, no.1, pp 470-475, 2020.

[9] 9 S.S. Abed, and K.E. Abdul Sada, "Common Fixed Points in Modular Spaces " Ibn Al-Haitham Journal for Pure and Applied Science,no.1822, pp 501-509, 2017.

[10] A.J. Kadhim, "New Common Fixed Points for Total Asymptotically Nonexpansive Mapping in CAT (0) space" *Baghdad Science Journal*, vol.18, no.4, pp 1286-1293, 2021.

[11] C. Moore, "A Double Sequence Iterative Process for Fixed Points of Continuous Pseudo-contractions" Journal of Computer and Mathematics with Application, no.43, pp-1585-1589, 2002.

[12] R. Moradi, and A. Razaani, "Double Sequence Iteration for a Strongly Contractive Mapping in the Modular Space" Iranian Journal of Mathematical Sciences and Informatics, vol.11, no.2, pp-119-130, 2016.

[13] M.A. Khamsi, "Nonlinear Semi Groups in Modular Function Spaces" University of Texas at El Paso, El Paso, Tx 79968-0514, 1994.

[14] A.A. Abdou, M.A. Khamsi and A.R. Khan, " Convergence of Ishikawa Iterates of Two Mappings in Modular Function Spaces" Journal Fixed Point Theory and Applications, vol.1, no.74, pp 1-10, 2014.

[15] M.F. Abduljabbar, and S.S. Abed, "The convergence of Iteration Scheme to Fixed Points in Modular Spaces" *Iraqi Journal of Science,* vol.60, no.10, pp 2196-2201, 2019.

[16] S.S. Abed, and M.F. Abdul Jabber, "Approximating Fixed Points in Modular Spaces" *Karbala International Journal of Modern Science*, vol.6, no.2, pp121-128, 2020.

[17] A. N. Abed, and S.S. Abed, "Convergence and Stability of Iterative Scheme for a Monotone Total Asymptotically Non-expansive Mapping" Iraqi Journal of Science, vol.63, no.1, pp 241-250, 2022.
[18] Z.M. Hasan, and S.S. Abed, "Strongly Convergence of Two Iterations for a Common Fixed Point with an Application" Ibn Al-Haitham Journal for Pure and Applied Science ,vol.32, no.3, pp 83-94, 2019.
[19] V. Karakaya, etc all, "Convergence Analysis for a New Faster Iteration Method" *Istanbul Commerce University Journal of Science,* vol.15, no.30, pp35-53, 2016.
[20] S.S. Abed, and N.S. Taresh, "On Stability of Iterative Sequences with Error" Mathematics Journal, vol.7, no.765, 2019.