Convergence of Forked Sequence to a Fixed Point in Modular Function Spaces

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Received: 27/1/2023       Accepted: 7/6/2023       Published: 30/6/2024

Abstract
In this paper, we present a five-step iterative scheme, at the beginning, it seems complicated and difficult to implement, but in fact it's not. This scheme is constructed for (\(\lambda, \rho\))-firmly nonexpansive mappings in modular function spaces. Two different \(\rho\)-convergences result has been proved for double schemes under consideration. In our study there is a comparison between these cases through answering the question "which one is faster?" Finally, numerical examples are given by using MATLAB software program.

Keywords: Modular spaces, Double sequence, Firmly nonexpansive, Nonexpansive, Iterative scheme, Fixed point.

1. Introduction
When proving the existence of a fixed point, finding the value of that point is not easy because some functions are complicated, so researchers use iterative scheme to calculate the required fixed point. Over time researcher have worked to develop various iterative schemes to find a faster way to reach the fixed point, many iterative schemes appeared, including Picard, Mann, Ishikawa, Noor, Abbas and others made several improvements to iterative sequences to ensure faster access to the fixed point, see [1]. The fixed point theory plays an important roles in many fields. For instance, it be used in the field of the differential equations in appropriate function spaces. Moreover, many existence theorems in statistics and physics

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have been reduced to fixed point theorems [2]. The concept of the modular space that has been used in this work can be found in [3]. Khamsi and Kozlowski (1990) introduced the concepts fixed point and modular space together with nonexpansive mapping [4]. Ruiz et al. [5] have discussed the \( p \)-firmly nonexpansive mapping in Banach spaces and this concept has been involved in work of Khan [2] the concept of \( (\lambda, \rho) \) — firmly nonexpansive mapping in modular spaces. Recently, the present researchers have presented results in this field for a multi-step iterative sequence adopting \( (\lambda, \rho) \) — firmly nonexpansive mappings (multi-valued and single-valued), see [6] and [7].

Now, let \( E \) be a non-empty convex subset of \( L_p \) where \( T: E \to 2^E \) and \( \rho_p^T(f) = \{ g \in T: \rho(f - g) = dist_p(f, T)f \} \). The sequence \( \{f_n\} \) by the following iterative process

\[
f_1 \in E.
\]

\[
h_n = (1 - \beta_n)f_n + \beta_n u_n.
\]

\[
g_n = v_n.
\]

\[
f_n+1 = m_n, n \in N
\]

where \( \{\alpha_n\} \) and \( \{\beta_n\} \) in \( (0,1) \), \( u_n \in \rho_p^T(f_n), v_n \in \rho_p^T(h_n), w_n \in \rho_p^T(g_n), m_n \in \rho_p^T(J_n) \).

Let \( T: E \to E \), and \( E \) be a non-empty convex subset of \( L_p \). We introduced the sequence \( \{f_n\} \) by the following algorithm.

\[
f_1 \in E.
\]

\[
h_n = (1 - \beta_n)f_n + \beta_n T f_n.
\]

\[
g_n = T h_n.
\]

\[
f_n = (1 - \alpha_n)g_n + \beta_n T g_n.
\]

\[
f_{n+1} = T J_n, n \in N
\]

where \( \{\alpha_n\} \) and \( \{\beta_n\} \) are sequences in \( (0,1) \).

Some researchers have worked on this field in Hilbert spaces, Banach spaces, and modular spaces [8, 9, and 10]. In 2002, Moore [11] introduced the idea of double sequence iterative in Hilbert spaces and prove the Mann double sequence iteration scheme strongly convergence to fixed point in pseudo contractive map by using the equation \( T_kx = (1 - \alpha_k)w + \alpha_k T x \) where \( w, x \in E \) and \( \alpha_k \in (0,1) \). Razani and Moradi [12] studied the double sequence in modular spaces. By using \( \rho \)-contractive mapping based on the above equation and presented some example to show the main results. In 2020, Gopinath et al. [8] introduced the double sequence S-iteration in Banach spaces by Lipschitz pseudo contractive map depending on the above equation.

In this paper, we present new double multi-step iterative sequences (as in (4) and (5) below) and study its convergence with some other related results and examples.

2. Preliminaries

This section includes the basic definitions and lemmas which are needed for this work.

**Definition 2.1** [9]: Let \( \rho: M \to [0, \infty) \) possesses the following properties:

1. \( \rho(0) = 0 \) if and only if, \( f = 0 \), \( \rho = a \cdot e \). (\( a \cdot e \) means almost everywhere)

2. \( \rho(\alpha f) = \rho(f) \), for every scalar \( \alpha \in \mathbb{C} or \mathbb{R} \).

3. \( \rho(\alpha x + \beta y) \leq \rho(x) + \rho(y) \) for every \( \alpha, \beta \geq 0 \) with \( \alpha + \beta = 1 \).

Where \( \rho \) is called a convex modular.
Definition 2.3 [13, 14]: If \( \rho \) is a convex modular in \( M \), then \( L_p \) is called modular function spaces
\[
L_p = \{ f \in M : \rho(\lambda f) \to 0 \text{ as } \lambda \to 0 \}.
\]
The modular space \( L_p \) can be equipped with an F-norm defined by
\[
\|f\|_\rho = \inf\{\alpha > 0 : \rho(\frac{f}{\alpha}) \leq \alpha\}.
\]
If \( \rho \) is a convex modular F-norm if \( \|f\|_\rho = \inf\{\alpha > 0 : \rho(\frac{f}{\alpha}) \leq 1\} \), then F-norm is called the Luxemburg norm.

Definition 2.4 [15, 16]: Let \( \rho \in \mathcal{R} \)
1- A sequence \( \{f_n\} \) is \( \rho \)-convergent to \( f \) if \( \rho(f_n - f) \to 0 \) as \( n \to \infty \).
2- A sequence \( \{f_n\} \) is \( \rho \)-Cauchy sequence if \( \rho(f_n - f_m) \to 0 \) as \( n, m \to \infty \).
3- A set \( B \subset L_p \) is called \( \rho \)-closed if for any \( f_n \in L_p \) the convergence \( \rho(f_n - f) \to 0 \) and \( f \) belongs to \( B \).
4- A set \( B \subset L_p \) is called \( \rho \)-bounded if \( \rho \) – diameter is finite \( \rho \) – diameter define as \( \mathcal{D}_\rho(B) = \sup\{\rho(f - g), f, g \in B, g \in B\} < \infty \).
5- A set \( B \subset L_p \) is called strongly \( \rho \)-bounded if there exists \( \beta > 1 \) such that \( M_\rho(B) = \sup\{\rho(\beta(f - g)), f, g \in B, g \in B\} < \infty \).
6- A set \( B \subset L_p \) is called \( \rho \)-compact, if for every \( f_n \in B \) , there exists a subsequence \( \{f_{n_k}\} \) and \( f \) in \( \rho(f_n - f) \to 0 \).
7- A set \( B \subset L_p \) is called \( \rho - a.e. \), closed, if every \( f_n \in B \) , which \( \rho - a.e. \), converges to some \( f \), then \( f \in B \).
8- A set \( B \subset L_p \) is called \( \rho - a.e. \), compact, if every \( f_n \in B \) , there exists a subsequence \( \{f_{n_k}\} \) \( \rho - a.e. \) -converges to some \( f \) in \( B \).
9- Let \( f \in L_p \) and \( B \subset L_p \) , the \( \rho \)-distance between \( f \) and \( B \) is defined as \( \text{dist}_\rho(f, B) = \inf\{\rho(f - g), g \in B\} \).

Definition 2.4 [4]: The map \( T: E \to E \) , where \( E \subset L_p \), is said to be \( \rho \)-nonexpansive mapping if \( \rho(Tf - Tg) \leq \rho(f - g) \), for all \( f, g \) in \( E \).

Definition 2.5 [2]: The map \( T: E \to E \) be is said to be \( (\lambda, \rho) \) – firm nonexpansive mapping if \( \rho(Tf - Tg) \leq \rho[(1 - \lambda)(f - g) + \lambda(Tf - Tg)] \), for all \( f, g \) in \( E \) and \( \lambda \in (0,1) \).

Lemma 2.6 [2]: Every \( (\lambda, \rho) \)- firm nonexpansive mapping is a \( \rho \)-nonexpansive mapping.

Now, let \( L_\rho \) be a modular space, and \( \mathbb{N} \) be the set of natural number, then we define the function \( \omega: \mathbb{N} \times \mathbb{N} \to L_\rho \) by \( \omega(n, m) = f_{n,m} \in L_\rho \).

Definition 2.7 [12]: The double sequence \( \{f_{n,m}\} \) is said to be strongly \( \rho \)-convergence to \( z \) if for any \( \epsilon > 0 \) where \( N, L > 0 \) such that \( \rho(f_{n,m} - z) < \epsilon \) for \( n > N, m > L \) if for all \( n, r > N, m, t > L \) then \( \rho(f_{n,r} - f_{m,t}) < \epsilon \).

Definition 2.8 [8]: The double sequence \( \{f_{n,m}\} \) is said to be \( \rho \)-Cauchy if for each \( \rho(f_{n,m} - z_n) \to 0 \) and \( \rho(z_n - z) \to 0 \), then \( \rho(f_{n,m} - z) \to 0 \) as \( n, m \to \infty \).

Lemma 2.9 [17, 18]: Let \( \{\rho_n\} \) a non-negative sequence such that
\[
\rho_{n+1} \leq (1 - \theta_n)\rho_n + \zeta_n,
\]
where \( \{\theta_n\} \) sequence in \( (0,1) \) and \( \{\zeta_n\} \) sequence in real number such that
3. Main Results

By using the iterative scheme that introduced in equation (2), will study forked type of sequences as following, let $T: E \to E$, define $T_k: E \to E$, where $E$ is a non-empty convex subset of $L_p$, then we have the following equation

$$T_k f = (1 - \eta_k) T f + \eta_k w,$$

where $\eta_k$ in $(0,1)$ and $f, w \in E$.

We introduce the sequence $\{f_{k,n}\}$ by the following algorithm

$$u_{k,n} = \frac{1}{n+1} T_k f_{k,n},$$
$$h_{k,n} = (1 - \beta_n) f_{k,n} + \beta_n u_{k,n},$$
$$g_{k,n} = T_k h_{k,n},$$
$$T_k f_{k,n+1} = T_k f_{k,n} + \alpha_n T_k g_{k,n} - \alpha_n T_k f_{k,n}.$$

(4)

where $\{\alpha_n\}, \{\beta_n\}$ and $\{\eta_k\}$ are sequences in $(0,1)$ and $T_k f$ via equation (3). The sequence $\{f_{k,n}\}, k \geq 0, n \geq 0$ is generated by an arbitrary $f_{0,0} \in E$

$$f^1_{k,n} = (1 - \gamma_{n,0}) T_k f_{k,n} + \gamma_{n,0} f_{k,n},$$
$$f^2_{k,n} = (1 - \gamma_{n,1}) T_k f_{k,n}^1 + \gamma_{n,1} f_{k,n}^2,$n
$$f^3_{k,n} = (1 - \gamma_{n,2}) T_k f_{k,n}^2 + \gamma_{n,2} f_{k,n}^3,$n

where $\gamma_{n,i}$ is real sequence in $(0,1)$.

Suppose $\gamma_{n,m}, \eta_k$ in equations (3) and (5), so the following three condition are satisfied.

i- $\lim_{n \to \infty} \gamma_{n,m} = \lim_{k \to \infty} \eta_k = 0$.

ii- For all $f, g, w \in R^+$, $\rho(c(f - w)) \leq \nu < \infty$, where $w, v \in E$.

iii- $T_k$ has unique fixed point and $F_p(T) \neq \emptyset$.

Then, we prove the following theorem.

Theorem 3.1: Let $\rho \in \mathcal{R}$ be $\rho$- complete, convex modular spaces, $E$ subset of $L_p$ which is $\rho$-closed, $\rho$-bounded and convex, $T: E \to E$ is $(\lambda, \rho)$-firmly nonexpansive mapping and $T_k: E \to E$, then $\{f_{k,n}\}$ in equation (4) is $\rho$-strong convergence to fixed point $s$ of $T$ in $E$ and $\rho(T_k f_{k,n} - f_{k,n}) \to 0$.

Proof: To prove $T_k$ is $(\lambda, \rho)$-firmly non-expansive mapping, let $f, g$ and $w$ in $E$.

By (3), and $T$ is $(\lambda, \rho)$-firmly nonexpansive mapping, we get

$$\rho(T_k f - T_k g) = \rho((1 - \eta_k) T f + \eta_k w - (1 - \eta_k) T g - \eta_k w) \leq (1 - \eta_k) \rho(T f - T g).$$

By condition (i) $\lim_{k \to \infty} \eta_k = 0$.

$$\rho(T_k f - T_k g) = \rho(T f - T g)$$

$$\leq \rho((1 - \lambda)(f - g) + \lambda(T f - T g))$$

$$\leq \rho(((1 - \lambda)(f - g) + \lambda((1 - \eta_k) T_k f + \eta_k w - \frac{1}{1-\eta_k} T_k g + \frac{\eta_k}{1-\eta_k} w).$$

$$\leq \rho((1 - \lambda)(f - g) + \lambda(T_k f - T_k g).$$

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where $T_k$ is $(\lambda, \rho)$-firmly nonexpansive mapping. Now, by condition (iii) let $s_k$ be unique fixed point of $T_k$ in $E$, to prove $\rho(f_{k,n} - s_k) \rightarrow 0$ as $n \rightarrow \infty$.

By equation (5) and Lemma 2.6, we get

$$\rho(f_{k,n+1} - s_k) \leq (1 - \gamma_{n,m})\rho(T_k f_{k,m} - s_k) + \gamma_{n,m}\rho(f_{k,n} - s_k)$$
$$\leq (1 - \gamma_{n,m})\rho(f_{k,m} - s_k) + \gamma_{n,m}\rho(f_{k,n} - s_k)$$
$$\leq (1 - \gamma_{n,m})[(1 - \gamma_{m-1})\rho(f_{k,m-1} - s_k) + \gamma_{m-1}\rho(f_{k,n} - s_k)] + \gamma_{n,m}\rho(f_{k,n} - s_k)$$
$$\leq \gamma_{n,m}\rho(f_{k,n} - s_k) + (1 - \gamma_{n,m})[\gamma_{n,m-1}\rho(f_{k,m} - s_k) + (1 - \gamma_{m-1})\rho(f_{k,n} - s_k)] + (1 - \gamma_{m-2})[\gamma_{n,m-2}\rho(f_{k,m-2} - s_k) + \gamma_{m-2}\rho(f_{k,n} - s_k)] + \cdots$$
$$+ (1 - \gamma_{n})[\gamma_{n,0}\rho(f_{k,n} - s_k)]$$

So, $\rho(f_{k,n+1} - s_k) \leq \mu_n\rho(f_{k,n} - s_k)$.

where,

$$\mu_n = \gamma_{n,m} + (1 - \gamma_{n,m})\gamma_{n,m-1} + \gamma_{n,m-2}$$

By condition (i) $\lim_{n \rightarrow \infty} \gamma_{n,m} = 0$, and Lemma 2.9 $\rho(f_{k,n} - s_k) \rightarrow 0$. Now, to prove $s_k$ is fixed point to $T$ in $E$. By equation (3)

$$T_k s_k = (1 - \eta_k)T s_k + \eta_k w.$$ 

Since $s_k$ is fixed point to $T_k$ in $E$, then $s_k = (1 - \eta_k)T s_k + \eta_k w$. Using condition (ii), to $\rho(s_k - T s_k) = \rho(s_k - \frac{1}{1-\eta_k}s_k + \frac{\eta_k}{1-\eta_k}w) = \rho\left(\frac{\eta_k}{1-\eta_k}(w - s_k)\right) \leq \frac{\eta_k}{1-\eta_k} \nu.$

Using condition (i), $\lim_{k \rightarrow \infty} \eta_k = 0$, then $\rho(s_k - T s_k) \rightarrow 0$ as $k \rightarrow \infty$, hence $\{s_k\}$ is approximate fixed point sequence in $T$. We have to prove $s_k$ is $\rho$-Caushy

$$(s_m - s_n) = (1 - \eta_m)T s_m + \eta_m w - (1 - \eta_n)T s_n - \eta_n w$$
$$= (\eta_m - \eta_n)w - \eta_m(T s_m - T s_n) + (\eta_n - \eta_m)T s_n - (T s_n - T s_m)$$
$$\rho(s_m - s_n) \leq (\eta_m - \eta_n)\rho(w) - \eta_m\rho(T s_m - T s_n) + (\eta_n - \eta_m)\rho(T s_n) - \rho(T s_n - T s_m)$$
$$\leq (\eta_m - \eta_n)\rho(w) - \mu_n\rho(T s_m - T s_n) + (\eta_n - \eta_m)\rho(T s_n).$$

Since $T$ is $(\lambda, \rho)$-firmly nonexpansive mapping and by Lemma 2.6.

$$\rho(s_m - s_n) \leq \frac{\eta_m - \eta_n}{1 + \eta_m} \rho(w) + \frac{\eta_n - \eta_m}{1 + \eta_m} \rho(T s_n),$$

by using condition (i) $\lim_{k \rightarrow \infty} \eta_k = 0$, so $\{s_k\}$ is $\rho$-Caushy sequence.

Since $L_{\rho}$ is $\rho$-complete, there exists $s$ in $E$ such that $\rho(s_k - s) \rightarrow 0$ as $k \rightarrow \infty$ and $T$ is $(\lambda, \rho)$-firmly nonexpansive mapping and by Lemma 2.6 $\rho(T s_k - T s) \leq \rho(s_k - s)$ implies that $\rho(T s_k - T s) \rightarrow 0$ as $k \rightarrow \infty$

By using $\rho$-Caushy sequence, the sequence $\{f_{k,n}\}$ convergence to $s$. To prove $s$ fixed point of $T$ in $E$.

$$\rho(T s - s) \leq \rho(T s - T s_k) + \rho(T s_k - s_k) + \rho(s_k - s)$$
$$\leq \rho(T s_k - s_k) + \rho(s_k - s).$$

So, $\rho(T s - s) \rightarrow 0$, $s$ fixed point of $T$.

Finally, to prove $\rho(T f_{k,n} - f_{k,n}) \rightarrow 0$ as $k, n \rightarrow \infty$.

$$\rho(T f_{k,n} - f_{k,n}) \leq \rho(T f_{k,n} - T s_k) + \rho(T s_k - s_k) + \rho(s_k - f_{k,n})$$
$$\leq \rho(f_{k,n} - s_k) + \rho(T s_k - s_k) + \rho(s_k - f_{k,n})$$

Then $\rho(T f_{k,n} - f_{k,n}) \rightarrow 0$, the proof is complete.

When the value of $w = 0$, it is possible to discuss this case, so the equation (3) become the following form.

$$T_k f = (1 - \eta_k)T f$$

(6)

Suppose $\alpha_n$, $\beta_n$ and $\eta_k$ in equations (4) and (6), the following three conditions are satisfied

i- $\Sigma_{n=0}^{\infty} \alpha_n = \infty$ and $\lim_{n \rightarrow \infty} \alpha_n = 0$.

ii- $\lim_{n \rightarrow \infty} \beta_n = \lim_{k \rightarrow \infty} \eta_k = 0$. 

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Based on equations (4) and (6) in addition to the above three condition, we prove the following theorem.

**Theorem 3.2**: Let $\rho \in \mathcal{R}$ is $\rho$- complete, convex modular spaces, $E$ subset of $L_{\rho}$ is $\rho$-closed, $\rho$-bounded and convex, $T: E \to E$ is $(\lambda, \rho)$-firmly nonexpansive mapping and $T_k: E \to E$, then \{f_{k,n}\} in equation (4) is $\rho$-strongly convergence to fixed point $s$ of $T$ in $E$, where \{\alpha_n\}, \{\beta_n\} and \{\eta_k\} be real sequence in $(0,1)$.

**Proof**: Let $s_k$ is the fixed point of $T_k$, by (4), (6) and Lemma 2.6
\[\rho(u_{k,n} - s_k) = \frac{1}{n+1}\rho(T_k f_{k,n} - s_k) \leq (1 - \eta_k)\rho(f_{k,n} - s_k) \] (7)
So, by (4) and (7)
\[\rho(h_{k,n} - s_k) \leq (1 - \beta_n)\rho(f_{k,n} - s_k) + \beta_n\rho(u_{k,n} - s_k) \]
\[\leq [(1 - \beta_n) + (1 - \eta_k)\beta_n]\rho(f_{k,n} - s_k). \] (8)

Similarity, by using (4), (8) and Lemma 2.6.
\[\rho(g_{k,n} - s_k) \leq [(1 - \beta_n)(1 - \eta_k) + (1 - \eta_k)^2\beta_n]\rho(f_{k,n} - s_k). \] (9)

By the same way, using (4), (9) and Lemma 2.6.
\[\rho(j_{k,n} - s_k) \leq [(1 - \alpha_n) + \alpha_n(1 - \eta_k)]\rho(g_{k,n} - s_k) \]
\[= [(1 - \alpha_n) + \alpha_n(1 - \eta_k)][(1 - \beta_n)\rho(f_{k,n} - s_k)]. \] (10)

By (4), (10) and Lemma 2.6.
\[\rho(f_{k,n+1} - s_k) \leq \rho(T_k f_{k,n} - s_k) \leq (1 - \eta_k)\rho(f_{k,n} - s_k) \] (11)
and
\[\mu_n = [(1 - \alpha_n)(1 - \beta_n)(1 - \eta_k)^2 + \alpha_n(1 - \beta_n)(1 - \eta_k)^3 + (1 - \alpha_n)(1 - \eta_k)^3\beta_n + \alpha_n\beta_n(1 - \eta_k)^4]. \]

Through Lemma 2.9, the first and second conditions above become clear $\rho(f_{k,n} - s_k) \to 0$.

Now, by equation (3)
\[T_k s_k = (1 - \eta_k)T s_k . \]

Since $s_k$ is the fixed point of $T_k$, Then $s_k = (1 - \eta_k)T s_k$.

\[\rho(T s_k - s_k) = \rho(T s_k - (1 - \eta_k)T s_k) \leq \eta_k\rho(T s_k) \]
By condition (ii), $\rho(T s_k - s_k) \to 0$, then \{s_k\} is an approximate fixed point sequence of $T$.

So, \[(s_m - s_n) = (1 - \eta_m)T s_m - (1 - \eta_n)T s_n \]
\[= (1 - \eta_m)T s_m - \eta_m(T s_m - T s_n) - (T s_n - T s_m) \]
\[\rho(s_m - s_n) \leq \eta_m - \eta_m\rho(T s_m - T s_n) - \rho(T s_n - T s_m) \]
\[\leq (\eta_m - \eta_m\rho(T s_n - T s_m)). \]

Since $T$ is $(\lambda, \rho)$-firmly nonexpansive mapping and by Lemma 2.6.
\[\rho(s_m - s_n) \leq \frac{(\eta_m - \eta_m)}{1 + \eta_m}\rho(T s_n), \]
by using condition (ii), $\lim_{k \to \infty} \eta_k = 0$, then \{s_k\} is $\rho$-Cauchy sequence.

Since $L_{\rho}$ is $\rho$-complete, there exists $s$ in $E$ such that. $\rho(s_k - s) \to 0$ as $\to \infty$, $T$ is $(\lambda, \rho)$-firmly nonexpansive mapping and by Lemma 2.6.
\[\rho(T s_k - T s) \leq \rho(s_k - s), \]
so $\rho(T s_k - T s) \to 0$.

By using $\rho$-Cauchy sequence, the sequence \{f_{k,n}\} convergence to $s$. In the rest of proof, we show that $s$ is fixed point of $T$ in $E$.
\[\rho(T s - s) \leq \rho(T s - s_k) + \rho(s_k - s) \]
\[\leq \rho(s - s_k) + \rho(T s_k - s_k) + \rho(s_k - s) \]
So, $\rho(T s - s) \to 0$, $s$ fixed point of $T$. 

4. Comparison Results

**Definition 4.1** [19]: Let \( \{a_n\}_{n=1}^\infty \) and \( \{b_n\}_{n=1}^\infty \) by two iterative scheme sequences converging to the same fixed point \( s \), and let \( \lim_{n \to \infty} \frac{\rho(a_{n+s})}{\rho(b_{n+s})} = L \), then

1- If \( L = 0 \) then \( \{a_n\}_{n=1}^\infty \) converges faster than \( \{b_n\}_{n=1}^\infty \) to fixed point \( s \).

2- If \( 1 < L < \infty \) then \( \{a_n\}_{n=1}^\infty \) and \( \{b_n\}_{n=1}^\infty \) they reach to the fixed point at the same speed.

**Theorem 4.2:** Let \( \rho \in \mathbb{R} \), \( E \) be a non-empty \( \rho \)-bounded, \( \rho \)-closed and convex \( E \subseteq L_p \) and \( T: E \to E \) be \((\lambda, \rho)\)-firmly nonexpansive multivalued mapping \( T_k: E \to E \) by \((\lambda, \rho)\)-firmly nonexpansive multivalued mapping, let \( \{a_n\}, \{b_n\}, \{\eta_k\} \) be real sequences in \((0,1)\), then the iterative scheme in \((4)\) by using equation \( T_k f = (1 - \eta_k) Tf + \eta_k w \).

**Proof:** By using equations \((3)\) and \((4)\), convexity of \( \rho \), Definitions 2.4 and 2.5 and Lemma 2.6, implies that

\[
\rho(f_{k,n+1} - s_k) = \rho(T_k f_{k,n} - s_k) \leq (1 - \eta_k) \rho(f_{k,n} - s_k) + \eta_k \rho(w - s_k) \\
\leq (1 - \eta_k) \left[ (1 - \alpha_n) \rho(f_{k,n} - s_k) + \alpha_n \rho(T_k f_{k,n} - s_k) \right] + \eta_k \rho(w - s_k) \\
\leq \left[ (1 - \eta_k)(1 - \alpha_n) + \alpha_n(1 - \eta_k)^2 \right] \rho(f_{k,n} - s_k) + \left[ \eta_k(1 - \eta_k) + \eta_k \right] \rho(w - s_k) \\
\leq \left[ (1 - \eta_k)^2(1 - \alpha_n) + \alpha_n(1 - \eta_k)^3 \right] \rho(h_{k,n} - s_k) + \left[ (1 - \eta_k)(1 - \alpha_n) \eta_k + \alpha_n \eta_k(1 - \eta_k) + \eta_k \right] \rho(w - s_k) \\
\leq \mu_n \rho(f_{k,n} - s_k) + \psi_n \rho(w - s_k).
\]

Where

\[
\mu_n = [(1 - \alpha_n)(1 - \beta_n)(1 - \eta_k)^2 + \alpha_n(1 - \beta_n)(1 - \eta_k)^3 + (1 - \alpha_n)(1 - \eta_k)^3 \beta_n + \alpha_n \beta_n(1 - \eta_k)^4].
\]

and

\[
\psi_n = [(1 - \alpha_n)(1 - \eta_k) \eta_k + \alpha_n (1 - \eta_k)^2 \eta_k + \eta_k (1 - \eta_k) \alpha_n + \eta_k (1 - \eta_k)^2 (1 - \alpha_n) \beta_n + \eta_k \alpha_n \beta_n (1 - \eta_k)^3 + \eta_k].
\]

Then

\[
\rho(f_{k,n} - s_k) \leq (\mu_n)^{n+1} (f_{k,0} - s_k) + (1 + \mu_n + (\mu_n)^2 + \cdots + (\mu_n)^n) \psi_n \rho(w - s_k). \tag{12}
\]

By the same of previous proof and by using \((6)\), we get

\[
\rho(f_{k,n} - s_k) \leq (\mu_n)^{n+1} (f_{k,0} - s_k). \tag{13}
\]

By definition 4.1 and equations \((12)\) and \((13)\). The proof is completed.

Below we present an example illustrating the previous theorem

**Example 4.3:** Let \( L \rho = \mathbb{R} \), the set of real number, \( \rho \) be absolute value and \( T: E \to E \), \( E = [0, \infty) \), \( T \) be define by \( Tf = \frac{f}{4} \), \( T_k: E \to E \), \( T_k \) define by \((6)\), and the double sequence define by \((4)\), the fixed point of \( T \) is \( s = 0 \), where \( \alpha_n = \beta_n = 0.5 \), \( \eta_k = \frac{k}{k+40} \), and let \( k = 100 \), and by using \((6)\) then the iterative scheme will become

\[
u_{100,n} = \frac{1}{n+1} \left( \frac{2}{7} \right) f_{100,n} \quad h_{100,n} = 0.5 f_{100,n} + 0.5 u_{100,n} \quad g_{100,n} = \left( \frac{2}{7} \right) h_{100,n} \quad J_{100,n} = 0.5 g_{100,n} + 0.5 \left( \frac{2}{7} \right) g_{100,n} \quad f_{100,n+1} = \left( \frac{2}{7} \right) J_{100,n} \quad n \in \mathbb{N}.
\]

Table1 and Figure1 show the numerical results with some step, when \( f_{100,0} = 1.5 \).

Also, see Table2 and Figure2 when \( u_{100,n} = \frac{1}{n+1} \left( \frac{2}{7} \right) f_{100,n} \quad h_{100,n} = 0.8 f_{100,n} + 0.2 u_{100,n} \quad g_{100,n} = \left( \frac{2}{7} \right) h_{100,n} \quad J_{100,n} = 0.8 g_{100,n} + 0.2 \left( \frac{2}{7} \right) g_{100,n} \quad f_{100,n+1} = \left( \frac{2}{7} \right) J_{100,n} \quad n \in \mathbb{N}.
\]

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Table 1: Shown $f_{k,n}$ in (3) by using equation (5), where $\alpha_n = \beta_n = 0.5$, $\eta_k = \frac{k}{k+40}$, $k = 100$ with $f_{100,n} = 1.5$.

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Figure 1: The function $f_{100,n}$, $u_{100,n}$, $h_{100,n}$, $g_{100,n}$ and $J_{100,n}$, where $\alpha_n = \beta_n = 0.5$, $\eta_k = \frac{k}{k+40}$, $k = 100$ with $f_{100,n} = 1.5$. 
Table 2: Shown $f_{k,n}$ in (3) by using equation (5), where $\alpha_n = \beta_n = 0.2$, $\eta_k = \frac{k}{k+40}$, $k = 100$ with $f_{100,n} = 1.5$.

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Figure 2: The function $f_{100,n}$, $u_{100,n}$, $h_{100,n}$, $g_{100,n}$ and $f_{100,n}$, where $\alpha_n = \beta_n = 0.2$, $\eta_k = \frac{k}{k+40}$, $k = 100$ with $f_{100,n} = 1.5$.

In the above example, it is clear that the iterative scheme presented in equation (4) approaches the fixed point at a record speed. In addition, when $\alpha_n = \beta_n = 0.5$ it is faster to reach the fixed point when $\alpha_n = \beta_n = 0.2$. 

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5-Conclusions
A new five step iterative scheme have been presented in this paper with double sequence in $(\lambda, \rho)$-firmly nonexpansive mapping of modular function spaces. As well as a new formula for the $T_k$ function has been defined through Theorem 4.2, where it has been shown the special case of equation (6) is faster to reach the fixed point than equation (3). In addition, as the value of $n$ increases than the double sequence $\{f_{k,n}\}$ approaches the fixed point, as shown in example 4.3. It is worth nothing that we aspire to obtain results related to what Tarsh and Abed presented in [20].

6-Acknowledgment
I appreciate the efforts of previous researchers in this field.

References

