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On Commutativity of Prime and Semiprime Γ- Rings with Reverse Derivations

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Abstract

Let M be a weak Nobusawa Γ-ring and γ be a non-zero element of Γ. In this paper, we introduce concept of k-reverse derivation, Jordan k-reverse derivation, generalized k-reverse derivation, and Jordan generalized k-reverse derivation of Γ-ring, and γ-homomorphism, anti-γ-homomorphism of M. Also, we give some commutattivity conditions on γ-prime Γ-ring and γ-semiprime Γ-ring .

Keywords: Gamma ring, γ-prime gamma ring, γ-semiprime gamma ring, γ-Lie ideal, k- reverse derivation, generalized k- reverse derivation.

حول الحلقات الاولية وشبه الاولية من النمط Γ– مع المشتقات المعكوسة

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الخلاصة

لتكن M حلقة من النمط – Γ نوبوساوا ضعيفة وγ عنصر غير صفري في Γ. في هذا البحث , سنقدم تعاريف المشتقات المعكوسة من النمط –k و جوردان المشتقات المعكوسة –k, وتعميم المشتقات المعكوسة –k وجوردان تعميم المشتقات المعكوسة – k, و مفهوم التشاكل من النمط –γ.ايضا , سنعطي بعض الشروط الابدالية لحلقة اولية وشبه اولية من النمط –γ.

1. Introduction

The concepts of a Γ-ring were first introduced by Nobusawa in 1964, this Γ-ring is generalized by Barnes. Let M and Γ be two additive abelain groups. M is called a Γ-ring in the sense of Barnes [1] if there exists a mapping of $M \times \Gamma \times M \rightarrow M$ satisfying these two conditions for all $a, b, c \in M$, $\alpha,\beta \in \Gamma$:

(i) $(a + b)\alpha c = a\alpha c + b\alpha c$, $a(\alpha + \beta)b = a\alpha b + a\beta b$, $a\alpha(b + c) = a\alpha b + a\alpha c$

(ii) $(a\alpha b)\beta c = a\alpha(b\beta c)$

In addition, if there exists a mapping of $\Gamma \times M \times \Gamma \rightarrow \Gamma$ such that the following axioms hold for all $a, b, c \in M$, $\alpha, \beta \in \Gamma$:

(iii) $(a\alpha b)\beta c = a(\alpha b\beta)c$

(iv) $a\alpha b = 0$ for all $a, b \in M$ implies $\alpha = 0$ where $\alpha \in \Gamma$.

Then M is said to be a Γ-ring in the sense of Nobusawa [2]. If a Γ-ring M in the sense of Barnes satisfies only the condition (iii), then it is called weak Nobusawa Γ-ring [3]. We assume that all gamma rings in this paper are weak Nobusawa Γ-ring unless otherwise specified.

Let M be Γ-ring, M is called a Γ- prime gamma ring if $a\Gamma M\Gamma b = 0$ with $a, b \in M$ implies $a = 0$ or $b = 0$ [4], and M is called a Γ- semiprime gamma ring if $a\Gamma M\Gamma a = 0$ with $a \in M$ implies $a = 0$ [4]. $C_\Gamma =$

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$\{ c \in M \mid c\alpha m = m\alpha c$, for all $m \in M, \alpha \in \Gamma \}$ is called center of Γ-ring M and $C_\gamma = \{ c \in M \mid c\gamma m = m\gamma c$, for all $m \in M \}$ with $\gamma \in \Gamma$ is γ - center of Γ-ring M. An additive mapping $d : M \rightarrow M$ is called a derivation if $d(a\alpha b) = d(a)\alpha b + a\alpha d(b)$ for all $a, b \in M, \alpha \in \Gamma$, [5] and d is said to be a reverse derivation of Γ-ring M if $d(a\alpha b) = d(b)\alpha a + b\alpha d(a)$ for all $a, b \in M, \alpha \in \Gamma$ [6]. In 2000, Kandamar [7] firstly introduced the notion of a k-derivation for a gamma ring in the sense of Barnes a. Also Chakraborty and Paul [8] introduced the notion of generalized k- derivations for gamma rings.

In this work, we define k-reverse derivation, Jordan k-reverse derivation, generalized k-reverse derivation, Jordan generalized k-reverse derivation for a gamma ring in the sense of Barnes and γ-homomorphism, and anti-γ-homomorphism of Γ-ring M. Also we obtain some commutattivity conditions on γ-prime Γ-ring and γ-semiprime Γ-ring with k- reverse derivations and generalized k-reverse derivations.

2. K-reverse derivations on γ-prime and γ–semiprime gamma rings.

We give the following definitions that have firstly defined by Arslan and Kandamar.

Definition 2.1. [9, 10]

Let M be a Γ-ring, γ be a non-zero element of Γ and I be an additive subgroup of M :

M is called a γ-commutative if $a\gamma b = b\gamma a$ for all $a,b \in M$. And M is called a γ-prime gamma ring if there exists a non-zero element γ in Γ such that $a\gamma M\gamma b = 0$ with $a, b \in M$ implies $a = 0$ or $b = 0$. Also M is a γ-semiprime gamma ring if there exists a non-zero element γ in Γ such that $a\gamma M\gamma a = 0$ with $a \in M$ implies $a = 0$. I is called a γ-subring of M if $a\gamma b \in I$ for all $a, b \in I$. And I is called a γ- left ideal (resp. γ- right ideal) of M if $m\gamma a \in I$ (resp. $a\gamma m \in I$) for all $m \in M, a \in I$. If I is both γ-left and γ-right ideal then I is said to be a γ-ideal of M. I is said to be a γ- Lie ideal of M if $[a, m]_\gamma = a\gamma m - m\gamma a \in I$ for all $m \in M$ and $a \in I$.

Definition 2.2. Let M be a Γ-ring and γ be a non-zero element of Γ. An additive mapping $f: M\rightarrow M$ is called a γ-centralizing if $[a, f(a)]_\gamma \in C_\gamma$ for all $a \in M$ and it is called a γ-commuting if $[a, f(a)]_\gamma = 0$ for all $a \in M$.

Now, we introduce the following definitions:

Definition 2.3. Let M be a Γ-ring and $d: M \rightarrow M$, $\mathrm{k}: \Gamma\rightarrow\Gamma$ be two additive maps, then d is called a k-reverse derivation if $d(a\alpha b) = d(b)\alpha a + a\mathrm{k}(\alpha)b + b\alpha d(a)$ for all $a, b \in M, \alpha \in \Gamma$. And d is called a Jordan k-reverse derivation if $d(a\alpha a) = d(a)\alpha a + a\mathrm{k}(\alpha)a + a\alpha d(a)$ for all $a \in M, \alpha \in \Gamma$.

Example 2.4. Let R be an associative ring with 1 of characteristic equal 2 and d be a reverse derivation of R. Consider $M = M_{1\times 2}(R)$ and $\Gamma = \left\{\begin{pmatrix} n \\ 0 \end{pmatrix} : n \text{ is an integer number}\right\}$. Then M is a Γ-ring. Let $N = \{(x \quad x): x \in R\}$, then N is a subring of M. Let K be an additive map and $D: N \rightarrow N$ defined by

$D(x \quad x) = (d(x) + x \quad d(x) + x)$ for $x \in R$.

Then d is a k-reverse derivation on M.

Definition 2.5. Let M be a Γ-ring and γ be a non-zero element of Γ. An additive mapping $f : M\rightarrow M$ is called a γ-homomorphism if $f(a\gamma b) = f(a)\gamma f(b)$ for all $a, b \in M$ and it is called anti- γ-homomorphism if $f(a\gamma b) = f(b)\gamma f(a)$ for all $a, b \in M$. Also it is called a γ-strong commutativity preserving if $[f(a), f(b)]_\gamma = [a, b]_\gamma$ for all $a, b \in M$.

Arslan and Kandamar [10] introduced the relation between Γ-rings and rings up to γ and give some commutative properties between them.

It is clear that if d is a k-reverse derivation (resp. Jordan k-reverse derivation) of Γ-ring M and $\mathrm{k}(\gamma) = 0$, then d is a reverse derivation (resp. Jordan reverse derivation) of the ring $(M , +, ._\gamma)$. And if f is a generalized k- reverse derivation (resp. Jordan generalized k- reverse derivation) of Γ-ring M associated with a non-zero k- reverse derivation d of M such that $k(\gamma) = 0$, then f is a generalized reverse derivation(resp. Jordan generalized reverse derivation) of the ring $(M , +, ._\gamma)$ associated with a non-zero reverse derivation d of the ring $(M , +, ._\gamma)$. Also a γ-homomorphism (resp. anti- γ-homomorphism) of a Γ-ring M is a homomorphism (resp. anti-homomorphism) of ring $(M , +, ._\gamma)$. We give some results:

Theorem 2.6. Let M be a γ-prime gamma ring and d_1, d_2 be non-zero $\mathrm{k}_1, \mathrm{k}_2$- reverse derivations of M such that $k_1(\gamma) = k_2(\gamma) = 0$ respectively. If char$M \neq 2$ and d_1d_2 is $\mathrm{k}_1\mathrm{k}_2$-reverse derivation of M, then $d_1 = 0$ or $d_2 = 0$.

Proof. By the hypothesis $d_1 \neq 0$, $d_2 \neq 0$ and d_1d_2 are reverse derivations of the prime ring $(M , +, ._\gamma)$. Also the characteristic of the ring $(M, +, ._\gamma)$ is different from 2. Therefore by [11, Theorem2] one of the reverse derivations d_1 and d_2 is zero in the ring$(M , +, ._\gamma)$.

Corollary 2.7. Let M be a γ-prime gamma ring of characteristic not 2 and d be a 0-reverse derivation of M such that $d^2 = 0$. Then $d = 0$.
Proof. Suppose M be a γ-prime gamma ring, then M is a Γ-prime gamma ring by [10, Lemma(2.4)]. Since $d^2 = 0$ is a reverse derivation on M, we get $d = 0$ by Theorem(2.6) .
Theorem 2.8. Let M be a Γ-ring and d be a k-reverse derivation of M such that $k(\gamma)= 0$ and $d^3 \neq 0$. Then the γ-subring generated by $d(m)$ for all m in M contains a non-zero γ-ideal of M.
Proof. Since d is a reverse derivation of the ring $(M, +, ._\gamma)$ and $d^3 \neq 0$, we have the subring generated by $d(m)$ for all m in M contains a non-zero ideal of $(M, +, ._\gamma)$ by [11, Theorem1]. Therefore the γ-subring generated by $d(m)$ for all m in M contains a non-zero γ-ideal of M.
Theorem 2.9. Let M be a γ-prime gamma ring and U be a non-zero γ-right ideal of M. Suppose d is a non-zero k–reverse derivation of M such that $k(\gamma) = 0$. Then M is a γ-commutative if one of the following conditions hold:
(i) d is a γ-commuting on M.
(ii) char $M \neq 2$ and d is a γ-commuting on U.
(iii) d is a γ-centralizing on U.
(iv) char $M \neq 2$ and $[d(a), d(b)]_\gamma = 0$ for all $a, b \in U$.
(v) char $M \neq 2$ and d is a γ-strong commutativity preserving on U.
(vi) U be a γ-ideal of M and $d(U) \subset C_\gamma$.
Proof. (i) By the hypothesis d is a non-zero reverse derivation of a prime ring $(M, +, ._\gamma)$. Since $[a, d(a)] = 0$ for all a in $(M, +, ._\gamma)$, the ring $(M, +, ._\gamma)$ is commutative by [12 , Theorem1]. Therefore the gamma ring M is a γ-commutative since commutativity of $(M, +, ._\gamma)$ requires γ-commutativity of Γ-ring M.
(ii) By the assumption d is a non-zero reverse derivation of prime ring $(M, +, ._\gamma)$, U is a right ideal of ring M, and the characteristic of $(M, +, ._\gamma)$ is different from 2. Also $[a, d(a)] = 0$ for all $a \in U$ in the ring$(M, +, ._\gamma)$. Hence M is commutative as a ring by [13, Theorem4]. Therefore M is γ-commutative.
(iii) By the hypothesis $(M, +, ._\gamma)$ is a prime ring and d is a non-zero reverse derivation of ring M. Since $[a, d(a)]$ is contained in the center of the ring $(M, +, ._\gamma)$ for all $a \in U$, we get the ring $(M, +, ._\gamma)$ is commutative by [14, Theorem(3.1)]. Therefore gamma ring M is a γ-commutative since commutativity of $(M, +, ._\gamma)$ requires γ-commutativity of Γ-ring M.
(iv) By the hypothesis d is a non-zero reverse derivation of prime ring $(M, +, ._\gamma)$, the characteristic of the ring M is different from 2 and $[d(a), d(b)] = 0$ for all a, b in right ideal U of the ring $(M, +, ._\gamma)$. Hence M is commutative as a ring by [13, Theorem5]. Then M is γ-commutative.
(v) Since d is a non-zero reverse derivation of prime ring $(M, +, ._\gamma)$. And characteristic of the ring $(M, +, ._\gamma)$ is different from 2 and $[d(a), d(b)] = [a, b]$ for all $a, b \in U$ in $(M, +, ._\gamma)$. Hence M is commutative as a ring by [13, Theorem6]. We have M is γ-commutative.
(vi) By the hypothesis U is an ideal of prime ring $(M, +, ._\gamma)$ and d is a non-zero reverse derivation of ring M. Since $d(U)$is contained in center of the ring $(M, +, ._\gamma)$. Hence M is commutative as a ring by [15, Theorem1]. Therefore M is γ-commutative.
Corollary 2.10. Let M be a γ-prime gamma ring for all non-zero elements γ in Γ, U a non-zero γ-right ideal of M and d is a non-zero 0– reverse derivation on M. Then M is a Γ-commutative if one of the following conditions hold:
(i) d is a γ-commuting on M for all $\gamma \in \Gamma$.
(ii) Char $M \neq 2$ and d is a γ-commuting on U for all $\gamma \in \Gamma$.
(iii) d is a γ-centralizing on U for all $\gamma \in \Gamma$.
(iv) Char $M \neq 2$ and $[d(a) , d(b)]_\gamma = 0$ for all $a, b \in U, \gamma \in \Gamma$.
(v) Char $M \neq 2$ and d is a γ-strong commutativity preserving on U for all $\gamma \in \Gamma$.
(vi) U a γ-ideal of M and $d(U) \subset C_\gamma$ for all $\gamma \in \Gamma$.
Theorem 2.11 Let M be a γ-prime gamma ring and d be a non-zero k-reverse derivation of M such that $k(\gamma) = 0$, then M is a γ-commutative.
Proof. Since d is a non-zero reverse derivation of a prime ring $(M, +, ._\gamma)$, we get d is a central derivation of $(M, +, ._\gamma)$by [16 , Proposition(3.7)], we have gamma ring M is a γ-commutative since commutativity of $(M, +, ._\gamma)$ requires γ-commutativity of Γ-ring M.
Theorem 2.12. Let M be a γ-prime gamma ring and U be a non-zero γ-right ideal of M. If d is a k-reverse derivation of M such that $k(\gamma) = 0$ which acts as a γ-homomorphism on U or an anti- γ-homomorphism on U, then $d = 0$ on M.

Proof. By assumption U is a non-zero right ideal of a prime ring $(M, +, ._{\gamma})$ and d a reverse derivation of M. Since d acts as a homomorphism on U of ring $(M, +, ._{\gamma})$ or an anti-homomorphism on U of $(M, +, ._{\gamma})$. Therefore $d = 0$ in $(M, +, ._{\gamma})$ by [17, Theorem1].

3. Generalized k-reverse derivations on γ-prime and γ–semiprime gamma rings.

Now, we introduce the following definition.

Definition 3.1. Let M be a Γ-ring. An additive mapping f: $M \to M$ is called a generalized k-reverse derivation if there exists a k-reverse derivation d:$M \to M$ such that $f(a\beta b) = f(b)\beta a + a k(\beta) b + b\beta d(a)$ for all $a, b \in M$, $\beta \in \Gamma$. And f is called a Jordan generalized k-reverse derivation if there exists a Jordan k-reverse derivation d:$M \to M$ such that $f(a\beta a) = f(a)\beta a + a k(\beta) a + a\alpha d(\beta)$ for all $a \in M$, $\beta \in \Gamma$.

Example 3.2. Let R be an associative ring with 1 of characteristic equal 2 and f be a generalized derivation on R, then there exists a reverse derivation d of R into itself such that $f(xy) = f(x)y + xd(y)$ for all $x, y \in R$. Consider $M = M_{1\times 2}(R)$ and $\Gamma = \left\{\binom{n}{0} : n \textit{ is an integer number}\right\}$. Then M is a Γ-ring. Let $N = \{(x \quad x): x \in R\}$, then N is a subring of M. Let K be an additive map and D: $N \to N$, F: $N \to N$ defined by

$D(x \quad x) = (d(x) + x \quad d(x) + x)$ and $F(x \quad x) = (f(x) + x \quad f(x) + x)$ for $x \in R$

Then F is a generalized k-reverse derivation on N associated with k-reverse derivation D of N.

Theorem 3.3. Let M be a γ-prime gamma ring and U be a γ-right ideal of M. If f is a generalized k-reverse derivation of M associated with a non-zero k-reverse derivation d of M such that $k(\gamma) = 0$. Then M is a γ-commutative if one of the following conditions holds:

(i) $U \neq 0$ and f is a γ-commuting on U.

(ii) $U \cap C_{\gamma} \neq 0$ and f is a γ-commuting on U.

Proof. (i) By the hypothesis f is a generalized reverse derivation of a prime ring $(M, +, ._{\gamma})$ associated with a non-zero reverse derivation d of M and U a non-zero right ideal of $(M, +, ._{\gamma})$. Since $[a, f(a)] = 0$ for all $a, b \in U$ in ring$(M, +, ._{\gamma})$. Hence M is commutative as a ring by [14, Theorem(3.3)]. Therefore M is a γ-commutative.

(ii) Since U is a right ideal of $(M, +, ._{\gamma})$ such that $U \cap Z \neq 0$, where Z denotes center of the ring $(M, +, ._{\gamma})$ and f a generalized reverse derivation of a prime ring $(M, +, ._{\gamma})$ associated with a non-zero reverse derivation d of M and since $[a, f(a)] = 0$ for all $a, b \in U$ in ring $(M, +, ._{\gamma})$. Then M is commutative as a ring by [14, Theorem(3.4)]. Hence M is a γ-commutative.

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