DOI: 10.24996/ijs.2024.65.8.25

ISSN: 0067-2904

[0,1]Truncated Exponentiated Exponential Burr type X Distribution with Applications

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Received: 22/1/2023 Accepted: 24/7/2023 Published: 30/8/2024

Abstract

We introduce a new flexible distribution four-parameter which is called [0,1]Truncated Exponentiated Exponential Burr type X distribution ([0,1] TEE-BX). Using binomial series expansion and exponential expansion, the new distribution is expanded by four parameters. We derive moments, moment generating function, quantile function, order statistics and the Rényi entropy. The maximum likelihood estimation method is used to estimate the parameters of the proposed TEE-BX model [0,1]. Finally, using two real-world data sets, the performance of the [0,1] TEE-BX distribution is explored. Based on the certain goodness of fit criteria, we conclude that the [0,1] TEE-BX distribution has a better fit than the other distributions.

Keywords: [0,1]Truncated, moments, order statistics, Rényi entropy, parameter estimation.

[0,1] توزيع بور من النوع X الأسي المعمم المبتور مع التطبيقات

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الخلاصة

نقدم توزيعاً جديداً مرناً من أربعة معلمات يسمى [0,1] توزيع بور من النوع X الأسي المعمم المبتور. باستخدام توسيع السلسلة ذات الحدين والتوسع الأسي ، يتم توسيع التوزيع الجديد بأربعة معلمات. نشتق العزوم , الدالة المولدة للعزوم, دالة الكمية, الإحصاءات المرتبة وريني انتروبي. تم استخدام طريقة MLE لتقدير معلمات نموذج TEE-BX [0,1] . أخيراً, باستخدام مجموعتين من البيانات الواقعية ، تم استكشاف أداء توزيع TEE-BX [0,1]. بناءً على جودة معينة لمعايير الملاءمة ، نستنتج أن توزيع TEE-BX [0,1] له ملاءمة أفضل من التوزيعات الأخرى.

1-Introduction

Statistical distributions can be used to describe and forecast real-world phenomena. Even though many distributions have been developed. There is always room for developing

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distributions that are more flexible or fit to specific real-world scenarios. Thus researchers encourage to seek out and develop new more flexible distributions. Many distributions have been developed, namely, The [0,1] Truncated Generalized Gamma-Generalized Gamma Distribution, the truncated Fréchet generated (TF-G) family, the truncated Weibull generated (TW-G) family, the truncated inverted Kumaraswamy generated (TIK-G) family, the Truncated Weibull power Lomax distribution (TWPL), and the [0,1] Truncated Lomax-Inverted Gamma Distribution (TLIG) [1-6]. Faton Merovci et al.[7] proposed a new distribution in 2016 that generalizes the Burr type X distribution. In 2018, Mundher A. Khaleel et al. [8] used the Burr type X by finding a new distribution called the Exponentiated Generalized Burr type X. In 2021, Ahmed et al.[9] Introduced a new version of the Exponentiated Burr X (EBX) distribution and studied some statistical properties.

The cumulative distribution function (cdf) of the Burr type X (BX) is given as follows[10]

$$G(x,w,\theta) = \left(1 - e^{-(wx)^2}\right)^{\theta} \tag{1}$$

And the probability density function (pdf) of the Burr type X (BX) are given below

$$g(x,w,\theta) = 2w^2\theta x e^{-(wx)^2}\left(1 - e^{-(wx)^2}\right)^{\theta-1} \tag{2}$$

where $x>0$, $w, \theta>0$

Abid et al. [2] and Khalaf and Khaleel [11] used the [0, 1] Truncated method to define and study a new family which is named the [0,1] Truncated Exponentiated Exponential-G Family ([0,1] TEE-G). The cdf and pdf. of the [0,1] TEE-G will become as follows:

$$F(x)_{[0,1]TEE-G} = \frac{\left(1 - e^{-\beta G(x;\xi)}\right)^{\alpha}}{(1 - e^{-\beta})^{\alpha}} \tag{3}$$

and

$$f(x)_{[0,1]TEE-G} = \frac{\beta\alpha g(x;\xi) e^{-\beta G(x;\xi)}\left(1 - e^{-\beta G(x;\xi)}\right)^{\alpha-1}}{(1 - e^{-\beta})^{\alpha}} \tag{4}$$

where $x,\beta,\alpha >0$

This paper is divided as follows: In Section 2, we introduce the PDF and CDF of the [0,1]Truncated Exponentiated Exponential Burr type X distribution ([0,1] TEE-BX). Its statistical properties are introduced in section 3. In Section 4, the parameters are derived and estimated by using the maximum likelihood estimation method. In section 5, the applications are given. Finally, in section 6, the conclusions are presented.

1-1 The [0,1]Truncated Exponentiated Exponential Burr type X distribution ([0,1] TEE-BX)

In this section, we introduce the [0,1] TEE-BX by inserting equation (1) into (3), we obtain the CDF of the [0,1] TEE-BX as follows:

$$F(x)_{[0,1]TEE-BX} = \frac{\left(1 - e^{-\beta\left(1-e^{-(wx)^2}\right)^{\theta}}\right)^{\alpha}}{(1 - e^{-\beta})^{\alpha}} \tag{5}$$

Now we differentiate equation (5), we get

$$f(x)_{[0,1]TEE-BX} = \frac{\beta\alpha 2w^2\theta x e^{-(wx)^2}\left(1-e^{-(wx)^2}\right)^{\theta-1} e^{-\beta\left(1-e^{-(wx)^2}\right)^{\theta}}\left(1-e^{-\beta\left(1-e^{-(wx)^2}\right)^{\theta}}\right)^{\alpha-1}}{\left(1-e^{-\beta}\right)^{\alpha}} \tag{6}$$

$x, \beta, \alpha, w, \theta>0$.

The survival function $S(x,\beta,\alpha,w,\theta)$ and the hazard function $h(x,\beta,\alpha,w,\theta)$ can be found as follows:

$$S(x)_{[0,1]TEE-BX} = 1 - \frac{\left(1 - e^{-\beta\left(1-e^{-(wx)^2}\right)^{\theta}}\right)^{\alpha}}{(1-e^{-\beta})^{\alpha}} \tag{7}$$

And

$$h(x)_{[0,1]TEE-BX} = \frac{\dfrac{\beta\alpha 2w^2\theta x e^{-(wx)^2}\left(1-e^{-(wx)^2}\right)^{\theta-1} e^{-\beta\left(1-e^{-(wx)^2}\right)^{\theta}}\left(1 - e^{-\beta\left(1-e^{-(wx)^2}\right)^{\theta}}\right)^{\alpha-1}}{(1-e^{-\beta})^{\alpha}}}{1 - \dfrac{\left(1 - e^{-\beta\left(1-e^{-(wx)^2}\right)^{\theta}}\right)^{\alpha}}{(1-e^{-\beta})^{\alpha}}} \tag{8}$$

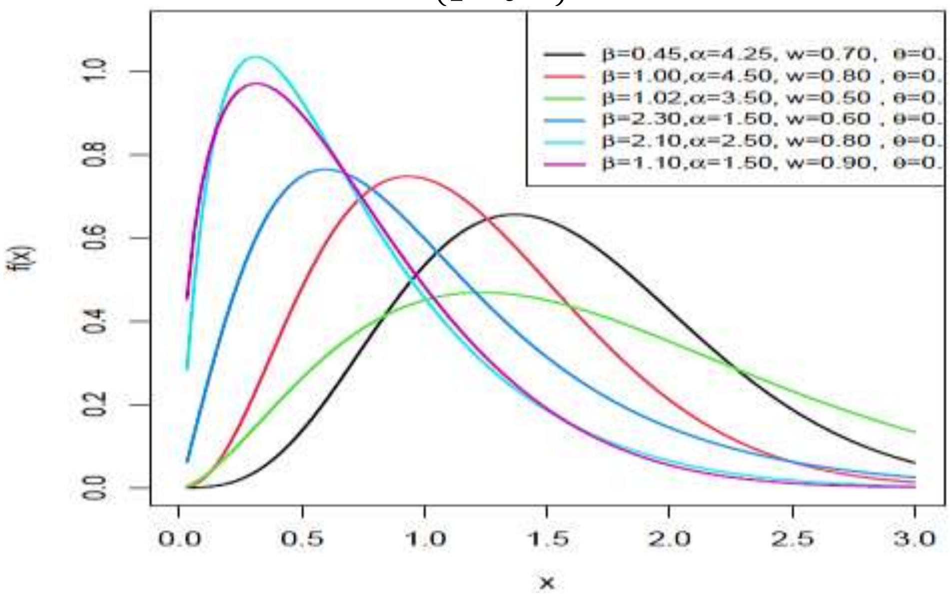


Figure 1: The pdf plots of the [0,1] TEE-BX distribution.

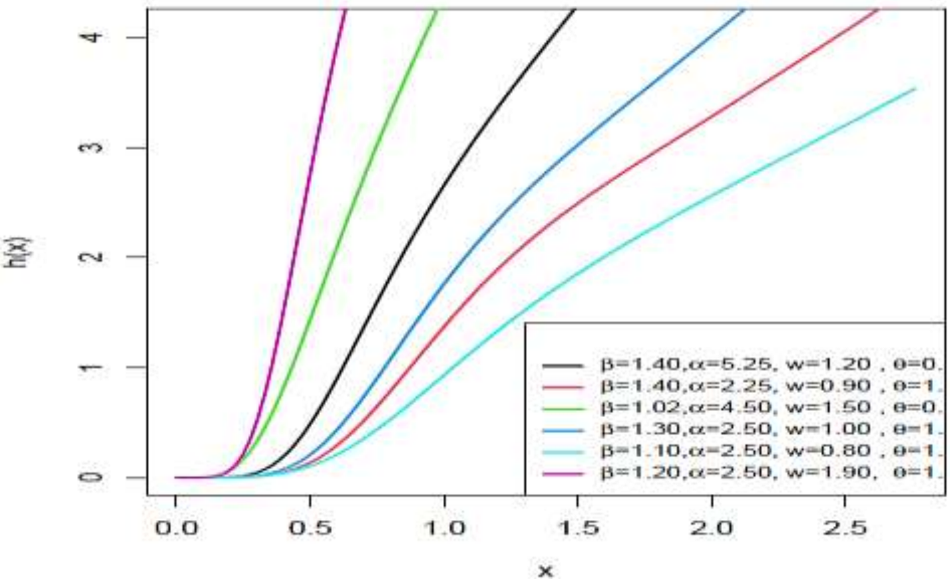


Figure 2: The H(x) plots of the [0,1] TEE-BX distribution.

1-2 Expansion of functions

To explore a few of the new distribution's statistical traits, this section will enlarge the PDF for the [0,1] TEE-BX distribution. By taking equation (4) and simplifying it:

$$f(x)_{[0,1]TEE-G} = \frac{\beta\alpha g(x;\xi)e^{-\beta G(x;\xi)}\left(1-e^{-\beta G(x;\xi)}\right)^{\alpha-1}}{(1-e^{-\beta})^{\alpha}} \quad (9)$$

using the binomial series expansion

$$\left(1-e^{-\beta G(x;\xi)}\right)^{\alpha-1} = \sum_{j=0}^{\infty}(-1)^j\binom{\alpha-1}{j}e^{-\beta jG(x;\xi)} \quad (10)$$

Following some simple steps, we get

$$f(x)^E_{[0,1]TEE-G} = \sum_{j=0}^{\infty}\frac{(-1)^j\beta\alpha}{(1-e^{-\beta})^{\alpha}}\binom{\alpha-1}{j}g(x;\xi)e^{-\beta(1+j)G(x;\xi)} \quad (11)$$

By using the exponential expansion

$$e^{-\beta(1+j)G(x;\xi)} = \sum_{k=0}^{\infty}\frac{(-1)^k}{k!}\left(\beta(1+j)G(x;\xi)\right)^k \quad (12)$$

Then

$$f(x)^E_{[0,1]TEE-G} = \sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\frac{(-1)^{j+k}\beta^{k+1}\alpha(1+j)^k}{k!\,(1-e^{-\beta})^{\alpha}}\binom{\alpha-1}{j}g(x;\xi)\left(G(x;\xi)\right)^k \quad (13)$$

By substituting equations (1) and (2) into equation (13), we obtain

$$f(x)^E_{[0,1]TEE-BX} = \sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\frac{2(-1)^{j+k}\beta^{k+1}\alpha(1+j)^k w^2\theta}{k!\left(1-e^{-\beta}\right)^{\alpha}}\binom{\alpha-1}{j}xe^{-(wx)^2}\left(1-e^{-(wx)^2}\right)^{k\theta+\theta-1} \quad (14)$$

Again, using the expansion

$$\left(1-e^{-(wx)^2}\right)^{k\theta+\theta-1} = \sum_{m=0}^{\infty}(-1)^m\binom{k\theta+\theta-1}{m}e^{-m(wx)^2} \quad (15)$$

Then

$$f(x)^E_{[0,1]TEE-BX} = \sum_{j=k=m=0}^{\infty}\Psi_{j,k,m}\; xe^{-(m+1)(wx)^2} \quad (16)$$

where

$$\Psi_{j,k,m} = \frac{2(-1)^{j+k+m}\beta^{k+1}\alpha(1+j)^k w^2\theta}{k!\,(1-e^{-\beta})^{\alpha}}\binom{\alpha-1}{j}\binom{k\theta+\theta-1}{m}$$

2- Statistical Properties of the [0,1] TEE-BX distribution

This section introduces the moments, moment generating functions, quantile functions, order statistics, and the Renyi entropy of the [0,1] TEE-BX distribution.

2-1 Moments

The rth moment of X is supplied via

$$E(X^r) = \int_0^{\infty} x^r f(x)dx \quad (17)$$

By using equation (16), the moment of the [0,1] TEE-BX is derived as follows:

$$E(X^r)_{[0,1]TEE-BX} = \sum_{j=k=m=0}^{\infty} \Psi_{j,k,m} \int_0^{\infty} x^{r+1} e^{-(m+1)(wx)^2} dx \tag{18}$$

Let

$$y = (wx)^2 \Rightarrow x = \frac{(y)^{\frac{1}{2}}}{w}, \qquad dy = 2w^2 x dx \Rightarrow dx = \frac{dy}{2w(y)^{\frac{1}{2}}}$$

$$E(X^r)_{[0,1]TEE-BX} = \frac{\sum_{j=k=m=0}^{\infty} \Psi_{j,k,m}}{2w^{r+2}} \int_0^{\infty} (y)^{\frac{r}{2}} e^{-(m+1)y} dy \tag{19}$$

Using the definition of complete Gamma function, we get the following:

$$E(X^r)_{[0,1]TEE-BX} = \sum_{j=k=m=0}^{\infty} \Psi_{j,k,m} \frac{\Gamma\left(\frac{r}{2}+1\right)}{2w^{r+2}(m+1)^{\frac{r}{2}+1}} \tag{20}$$

Through equation (20), we can find the first moment ($\acute{\mu}_1$), the second moment ($\acute{\mu}_2$), the third moment ($\acute{\mu}_3$), and the fourth moment ($\acute{\mu}_4$), respectively:

$$\acute{\mu}_1 = E(X) = \sum_{j=k=m=0}^{\infty} \Psi_{j,k,m} \frac{\Gamma\left(\frac{1}{2}+1\right)}{2w^{1+2}(m+1)^{\frac{1}{2}+1}} = \sum_{j=k=m=0}^{\infty} \Psi_{j,k,m} \frac{\sqrt{\pi}}{4w^3(m+1)^{\frac{3}{2}}} \tag{21}$$

$$\acute{\mu}_2 = E(X^2) = \sum_{j=k=m=0}^{\infty} \Psi_{j,k,m} \frac{\Gamma\left(\frac{2}{2}+1\right)}{2w^{2+2}(m+1)^{\frac{2}{2}+1}} = \sum_{j=k=m=0}^{\infty} \Psi_{j,k,m} \frac{1}{2w^4(m+1)^2} \tag{22}$$

$$\acute{\mu}_3 = E(X^3) = \sum_{j=k=m=0}^{\infty} \Psi_{j,k,m} \frac{\Gamma\left(\frac{3}{2}+1\right)}{2w^{3+2}(m+1)^{\frac{3}{2}+1}} = \sum_{j=k=m=0}^{\infty} \Psi_{j,k,m} \frac{3\sqrt{\pi}}{8w^5(m+1)^{\frac{5}{2}}} \tag{23}$$

$$\acute{\mu}_4 = E(X^4) = \sum_{j=k=m=0}^{\infty} \Psi_{j,k,m} \frac{\Gamma\left(\frac{4}{2}+1\right)}{2w^{4+2}(m+1)^{\frac{4}{2}+1}} = \sum_{j=k=m=0}^{\infty} \Psi_{j,k,m} \frac{1}{w^6(m+1)^3} \tag{24}$$

The variance of the [0,1] TEE-BX distribution is obtained by the following formula ($\sigma^2 = \acute{\mu}_2 - \acute{\mu}_1^2$). The skewness (SK) and kurtosis (KU) are defined as follows[12]:

$$\text{SK} = \frac{\mu_3}{\mu_2^{(\frac{3}{2})}} = \frac{\sum_{j=k=m=0}^{\infty} \Psi_{j,k,m} \frac{3\sqrt{\pi}}{8w^5(m+1)^{\frac{5}{2}}}}{\left[\sum_{j=k=m=0}^{\infty} \Psi_{j,k,m} \frac{1}{2w^4(m+1)^2}\right]^{\frac{3}{2}}} = \frac{3}{4}\sqrt{\pi}\sqrt{2}w \left[\frac{\frac{\sum_{j=k=m=0}^{\infty} \Psi_{j,k,m}}{(m+1)^{\frac{5}{2}}}}{\left(\frac{\sum_{j=k=m=0}^{\infty} \Psi_{j,k,m}}{(m+1)^2}\right)^{\frac{3}{2}}}\right] \tag{25}$$

$$KU = \frac{\mu_4}{\mu_2^2} = \frac{\sum_{j=k=m=0}^{\infty} \Psi_{j,k,m} \frac{1}{w^6(m+1)^3}}{\left[\sum_{j=k=m=0}^{\infty} \Psi_{j,k,m} \frac{1}{2w^4(m+1)^2}\right]^2} = 2w^2 \left[\frac{\frac{\sum_{j=k=m=0}^{\infty} \Psi_{j,k,m}}{(m+1)^3}}{\left(\frac{\Psi_{j,k,m}}{(m+1)^2}\right)^2}\right] \tag{26}$$

The moments generating $M_x(t)$ function are given by the moment's formula and Taylor expansion:

$$M_x(t) = E(e^{tx}) = \sum_{l=0}^{\infty} \frac{t^l}{l!} E(x^l) = \sum_{l=0}^{\infty} \frac{t^l}{l!} \left[\sum_{j=k=m=0}^{\infty} \Psi_{j,k,m} \frac{\Gamma\left(\frac{r}{2}+1\right)}{2w^{r+2}(m+1)^{\frac{r}{2}+1}}\right] \tag{27}$$

Table 1: Default values for the first four Moments, Variance, Skewness, and Kurtosis of the [0,1] TEE-BX distribution**.**

β	α	w	θ	$\acute{\mu}_1$	$\acute{\mu}_2$	$\acute{\mu}_3$	$\acute{\mu}_4$	Var(X)	Skewness	Kurtosis
Values of parameter				values of properties						
1	**0.3**	**0.5**	**0.1**	0.0851	0.1178	0.2526	0.6774	0.1105	6.25247	49.0869
			0.2	0.1739	0.2400	0.5113	1.3652	0.2097	4.35148	23.7013
		0.6	**0.1**	0.0709	0.0818	0.1462	0.3266	0.0767	6.27467	49.4848
			0.2	0.1449	0.1666	0.2959	0.6584	0.1457	4.35147	23.7689
	1.3	**0.5**	**0.1**	0.3302	0.4884	1.0699	2.8964	0.3794	3.13477	12.1442
			0.2	0.6066	0.9503	2.1141	5.7572	0.5824	3.37553	6.37563
		0.6	**0.1**	0.2751	0.3392	0.6191	1.3968	0.2636	3.13468	12.1460
			0.2	0.5055	0.6599	1.2234	2.7764	0.4044	2.28246	6.37666
2	**0.3**	**0.5**	**0.1**	0.0521	0.0671	0.1401	0.3710	0.0644	8.09826	82.4444
			0.2	0.1166	0.1438	0.2920	0.7611	0.1303	5.35779	36.9466
		0.6	**0.1**	0.0434	0.0466	0.0811	0.1789	0.0448	8.11065	85.1904
			0.2	0.0972	0.0998	0.1690	0.3670	0.0904	5.36167	37.0707
	1.3	**0.5**	**0.1**	0.2087	0.2828	0.5990	1.5955	0.2393	3.98297	19.9687
			0.2	0.4235	0.5845	1.2274	3.2434	0.4052	0.50895	9.49473
		0.6	**0.1**	0.1739	0.1964	0.3466	0.7694	0.1662	3.98390	19.9844
			0.2	0.3529	0.4059	0.7103	1.5641	0.2814	2.74671	9.49666

We notice in Table 1 that when the values of β, α, and w are constant as the value of θ increases, the moments and variances increase and the skewness and kurtosis values of the [0,1] TEE-BX distribution decrease. When the values of β, α, and θ are constant as the value of w increases, the moments and variances decrease and the skewness and kurtosis values of the [0,1] TEE-BX distribution increase. We also see that when the values of β, w, and θ are constant as the value of α increases, the moments and variances increase and the skewness and kurtosis values of the [0,1] TEE-BX distribution decrease. Moreover, when the values of α, w, and θ are constant as the value of β increases, the moments and variances decrease and the skewness and kurtosis values of the [0,1] TEE-BX distribution increase.

2-2 Quantile Function

By inverting equation (5), we have the quantile function of the [0,1] TEE-BX as follows:

$$Q(u)_{[0,1]TEE-BX} = \left\{-\frac{1}{w^2} ln\left[1-\left[-\frac{1}{\beta} ln\left[1-\left[u\left(1-e^{-\beta}\right)^{\alpha}\right]^{\frac{1}{\alpha}}\right]\right]^{\frac{1}{\theta}}\right]\right\}^{\frac{1}{2}} \quad (28)$$

2-3 Order statistics

The pdf of the rth order statistics for the size n random sample for the [0,1] TEE-BX distribution is obtained as follows [13]:

$$f_{j:n}(x) = \sum_{s=0}^{\infty} \frac{n!}{(j-1)!\,(n-j)!} (-1)^s \binom{n-j}{s} f(x)[F(x)]^{j+s-1} \quad (29)$$

Substituting equation(5) and (6) into equation (29), we get

$$f_{j:n}(x) = \quad (30)$$

$$\sum_{s=0}^{\infty} k(-1)^s \binom{n-j}{s} \left[\frac{\beta\alpha 2w^2\theta x e^{-(wx)^2}\left(1-e^{-(wx)^2}\right)^{\theta-1} e^{-\beta\left(1-e^{-(wx)^2}\right)^{\theta}} \left(1-e^{-\beta\left(1-e^{-(wx)^2}\right)^{\theta}}\right)^{\alpha-1}}{\left(1-e^{-\beta}\right)^{\alpha}} \right.$$

$$*\left[\frac{\left(1-e^{-\beta\left(1-e^{-(wx)^2}\right)^{\theta}}\right)^{\alpha}}{(1-e^{-\beta})^{\alpha}}\right]^{j+s-1}$$

2-4 Entropy Rényi

The Rényi entropy for the [0,1] TEE-BX distribution can be obtained:

$I_R(c)_{[0,1]TEE-BX}=\frac{1}{1-c}log\int_0^{\infty}f(x)^c dx$

Then

$$I_R(c)_{[0,1]TEE-BX}=\frac{1}{1-c}\log\left\{\Psi_{j,k,m}\frac{\Gamma\left(\frac{C+1}{2}\right)}{2w^{c+1}(cm+c)^{\frac{c+1}{2}}}\right\} \tag{31}$$

3- Estimation parameters

According to Jabr and Karam [14], [15], let $X_1, X_2, \ldots, X_n$ be a random sample from a [0,1] TEE-BX distribution that is defined in equation(6) with unidentified parameters β,α,w, and θ. The log- likelihood function $\Phi=(\beta,\alpha,w,\theta)^T$ is given by:

$$l=\log(\Phi)=nlog\beta+nlog\alpha+4nlogw+nlog\theta-n\alpha log\left(1-e^{-\beta}\right)+\sum_{t=1}^{n}logx_t -\sum_{t=1}^{n}(wx_t)^2+(\theta-1)\sum_{t=1}^{n}\log\left(1-e^{-(wx_t)^2}\right) -\beta\sum_{t=1}^{n}\left(1-e^{-(wx_t)^2}\right)^{\theta}+(\alpha-1)\sum_{t=1}^{n}\log\left(1-e^{-\beta\left(1-e^{-(wx_t)^2}\right)^{\theta}}\right) \tag{32}$$

The four parameters' first partial derivatives of the log-likelihood function are

$$\frac{\partial l}{\partial\beta}=\frac{n}{\beta}-\frac{n\alpha e^{-\beta}}{1-e^{-\beta}}-\sum_{t=1}^{n}\left(1-e^{-(wx_t)^2}\right)^{\theta} +(\alpha-1)\left(\sum_{t=1}^{n}\frac{\left(1-e^{-(wx_t)^2}\right)^{\theta}e^{-\beta\left(1-e^{-(wx_t)^2}\right)^{\theta}}}{1-e^{-\beta\left(1-e^{-(wx_t)^2}\right)^{\theta}}}\right) \tag{33}$$

$$\frac{\partial l}{\partial\alpha}=\frac{n}{\alpha}-nlog\left(1-e^{-\beta}\right)+\sum_{t=1}^{n}log\left(1-e^{-\beta\left(1-e^{-(wx_t)^2}\right)^{\theta}}\right) \tag{34}$$

$$\frac{\partial l}{\partial w}=\frac{4n}{w}-\sum_{t=1}^{n}2wx_t^2+(\theta-1)\left(\sum_{t=1}^{n}\frac{2wx_t^2e^{-(wx_t)^2}}{1-e^{-(wx_t)^2}}\right) -\beta\sum_{t=1}^{n}\frac{2\left(1-e^{-(wx_t)^2}\right)^{\theta}\theta wx_t^2e^{-(wx_t)^2}}{1-e^{-(wx_t)^2}} +(\alpha-1)\left(\sum_{t=1}^{n}\frac{2\beta\left(1-e^{-(wx_t)^2}\right)^{\theta}\theta wx_t^2e^{-(wx_t)^2}e^{-\beta\left(1-e^{-(wx_t)^2}\right)^{\theta}}}{\left(1-e^{-(wx_t)^2}\right)\left(1-e^{-\beta\left(1-e^{-(wx_t)^2}\right)^{\theta}}\right)}\right) \tag{35}$$

$$\frac{\partial l}{\partial \theta}=\frac{n}{\theta}+\sum_{t=1}^{n} log\left(1-e^{-(wx_t)^2}\right)-\beta\sum_{t=1}^{n}\left(1-e^{-(wx_t)^2}\right)^{\theta} log\left(1-e^{-(wx_t)^2}\right) + \beta(\alpha-1)\sum_{t=1}^{n}\frac{\left(1-e^{-(wx_t)^2}\right)^{\theta}\log\left(1-e^{-(wx_t)^2}\right)e^{-\beta\left(1-e^{-(wx_t)^2}\right)^{\theta}}}{\left(1-e^{-\beta\left(1-e^{-(wx_t)^2}\right)^{\theta}}\right)} \quad (36)$$

Note that it is challenging to manually calculate equations (33), (34), (35), and (36) to obtain the parameter values the R program is used in this study.

4- Application

To illustrate the performance of the [0,1]Truncated Exponentiated Exponential Burr type X distribution ([0,1] TEE-BX), it is fitted with the two data sets;

Data Set 1: It is discussed in[16], [17], [18] and [19]. The first data contains 40 observations and are listed as: 1.6, 2.0, 2.6, 3.0, 3.5, 3.9,4.5, 4.6, 4.8, 5.0, 5.1, 5.3, 5.4, 5.6, 5.8, 6.0, 6.0, 6.1, 6.3,6.5, 6.5, 6.7, 7.0, 7.1, 7.3, 7.3,7.3, 7.7, 7.7, 7.8, 7.9, 8.0, 8.1, 8.3,8.4, 8.4, 8.5, 8.7, 8.8, 9.0.

Data Set 2: we use the following set of data that is related to the strength of 1.5 cm glass fiber which was previously analyzed by [20], and [21] to fit the models.
(0.55, 0.74, 0.77, 0.81, 0.84, 1.24, 0.93, 1.04, 1.11, 1.13, 1.30, 1.25, 1.27, 1.28,1.29, .48, 1.36, 1.39, 1.42, 1.48, 1.51, 1.49, 1.49, 1.50, 1.50,1.55, 1.52, 1.53, 1.54, 1.55, 1.61, 1.58, 1.59, 1.60, 1.61, 1.63,1.61, 1.61, 1.62, 1.62, 1.67, 1.64, 1.66, 1.66, 1.66, 1.70, 1.68, 1.68, 1.69, 1.70, 1.78, 1.73, 1.76, 1.76, 1.77, 1.89, 1.81, 1.82,1.84, 1.84, 2.00, 2.01, 2.24).

We obtain the maximum likelihood estimates for [0,1]TEE-BX form parameters for the data sets listed above. We Compare the appropriate ability on [0,1]TEE-BX with the following competition distributions:

Table 2: The competing distributions of the [0,1]TEE-BX distribution with their CDF and Statistical criteria and w^{*}, A^{*} , k s with their laws.

Distributions	Abbreviation	CDF
Kumaraswamy Burr type X distribution	KUBX	$F(x)=1-\left[1-\left(1-e^{-(\lambda x)^2}\right)^{\theta\beta}\right]^{\alpha}$
Beta Burr type X distribution	BEBX	$F(x)=pB\left(\left(1-e^{-(\lambda x)^2}\right)^{\theta},\beta,\alpha\right)$
Exponential Generalized Burr type X distribution	EGBX	$F(x)=\left[1-\left[1-\left(1-e^{-(\lambda x)^2}\right)^{\theta}\right]^{\beta}\right]^{\alpha}$
Burr type X distribution	BX	$G(x,w,\theta)=\left(1-e^{-(wx)^2}\right)^{\theta}$
Rayleigh distribution	R	$F(x)=1-e^{-(\theta x)^2}$
Statistical criteria and w^{*}, A^{*}, k s	**Abbreviation**	**Law**
Akaike Information Criteria	AIC	$AIC=-2l+2K$
Consistent Akaike Information Criteria	CAIC	$CAIC=AIC+\frac{2K(K+1)}{n-K-1}$
Bayesian Information Criteria	BIC	$BIC=-2l+klog(n)$
Hanan and Quinn Information Criteria	HQIC	$HQIC=2kln[\ln(n)]-2l$
Cramér-von Mises criterion	w^{*}	$W^{\star}=\frac{1}{12n}+\sum_{i=1}^{n}\left[\frac{2i-1}{2n}-\hat{F}(x_i)\right]^2$

Anderson–Darling criterion	A^*	$A^* = -n - \frac{1}{n}\sum_{i=1}^{n}(2i-1)\left[\log \hat{F}(x_i) + \log\left(1-\hat{F}(x_{n+1-i})\right)\right]$
Kolmogorov–Smirnov criterion	KS	$\mathrm{KS} = max\left\{\frac{i}{n} - \hat{F}(x_i), \hat{F}(x_i) - \frac{i-1}{n}\right\}$

Table 3: Summary statistics for the dataset1 and dataset2

	n	mean	Sd	median	Min	Max	Skew	Kurtosis
Data1	40	6.25	1.96	6.5	1.6	9	-0.64	-0.49
Data2	63	1.51	0.32	1.59	0.55	2.24	-0.88	0.8

Table 4 : The ML estimates of models for both data

	Distribution	**Estimates**			
Data1	[0,1]TEEBX	$\hat{\beta}$=29.4644	$\hat{\alpha}$=0.1551	$\hat{w}$=0.1348	$\hat{\theta}$=11.6777
	KUBX	$\hat{\beta}$=1.6237	$\hat{\alpha}$=1.4793	$\hat{w}$=0.1691	$\hat{\theta}$=1.4234
	BEBX	$\hat{\beta}$=1.634	$\hat{\alpha}$=1.3129	$\hat{w}$=0.1747	$\hat{\theta}$=1.4376
	EGBX	$\hat{\beta}$=1.2071	$\hat{\alpha}$=1.6568	$\hat{w}$=0.1788	$\hat{\theta}$=1.4217
	BX	$\hat{\beta}$=0.1946	$\hat{\alpha}$=2.3992		
	R	$\hat{\beta}$=0.0233			
Data2	[0,1]TEEBX	$\hat{\beta}$=28.6197	$\hat{\alpha}$=0.38305	$\hat{w}$=0.5821	$\hat{\theta}$=8.3171
	KUBX	$\hat{\beta}$= 1.3269	$\hat{\alpha}$=3.9267	$\hat{w}$=0.6842	$\hat{\theta}$=3.1952
	BEBX	$\hat{\beta}$=1.2778	$\hat{\alpha}$=2.8787	$\hat{w}$=0.7199	$\hat{\theta}$=3.3993
	EGBX	$\hat{\beta}$=3.7227	$\hat{\alpha}$=1.0076	$\hat{w}$=0.6778	$\hat{\theta}$=3.9748
	BX			$\hat{w}$=1.0076	$\hat{\theta}$=6.0709
	R				$\hat{\theta}$=0.4212

Table 5: Goodness-of-fit statistics for both data

	Distribution	**-LL**	**AIC**	**CAIC**	**BIC**	**HQIC**	W^*	A^*	**KS**
Data1	**[0,1]TEEBX**	**80.45**	**168.91**	**170.06**	**175.67**	**171.36**	**0.0433**	**0.3459**	**0.1070**
	KUBX	85.24	178.49	179.64	185.25	180.94	0.1460	1.0100	0.1131
	BEBX	85.51	179.02	180.17	185.78	181.46	0.1527	1.0509	0.1130
	EGBX	85.68	179.36	180.50	186.11	181.80	0.1566	1.0744	0.1166
	BX	85.79	175.59	175.91	178.97	176.81	0.1597	1.0931	0.1172
	R	91.88	185.76	185.87	187.45	186.37	0.1428	0.9912	0.2268
Data2	**[0,1]TEEBX**	**15.06**	**38.132**	**38.821**	**46.704**	**41.503**	**0.2338**	**1.2852**	**0.1639**
	KUBX	19.20	46.42	47.11	54.993	49.792	0.4041	2.2172	0.1911
	BEBX	20.41	48.858	49.548	57.431	52.23	0.4456	2.4467	0.2026
	EGBX	19.42	46.855	47.545	55.427	50.227	0.4097	2.2485	0.1980
	BX	24.03	52.08	52.28	56.366	53.766	0.5697	3.1286	0.2134
	R	49.79	101.58	101.64	103.72	102.42	0.4654	2.5538	0.3338

According to Table 5, the [0,1]TEE-BX in data1 and data2 has the minimum values for the AIC, CAIC, BIC, HQIC, W^*, A^* and KS measures, it is compared to other models. As a result of these criteria, we can declare that it is the best fit among the models studied. Figures (3), (4), (5) and (6) support the results of Table 5.

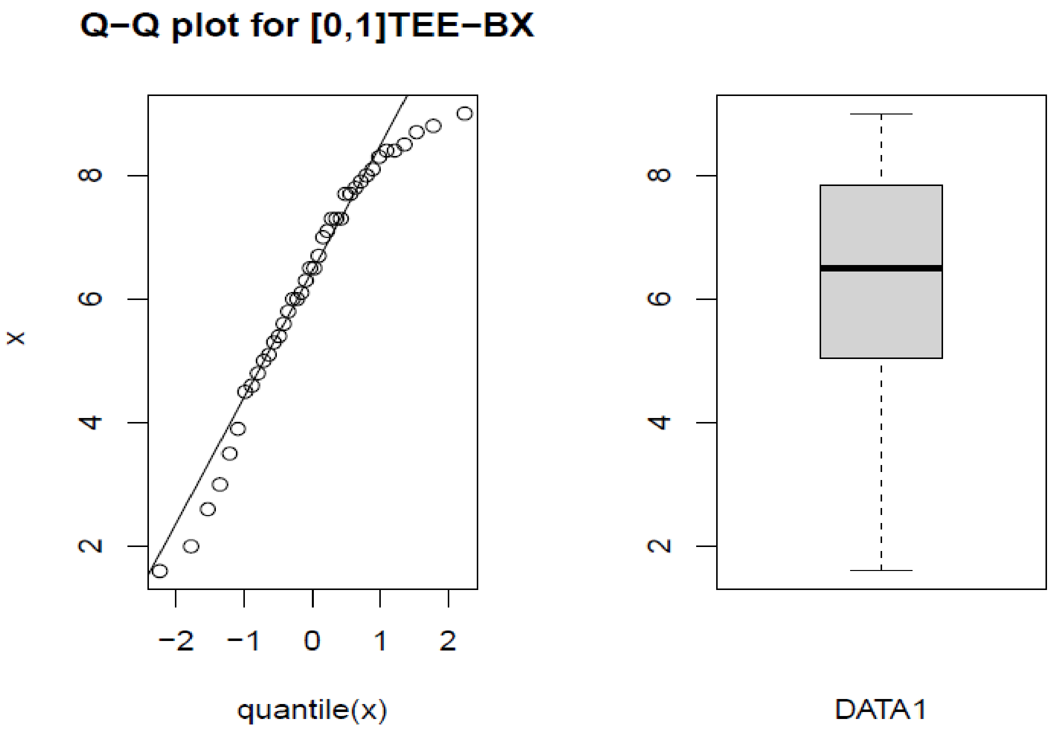
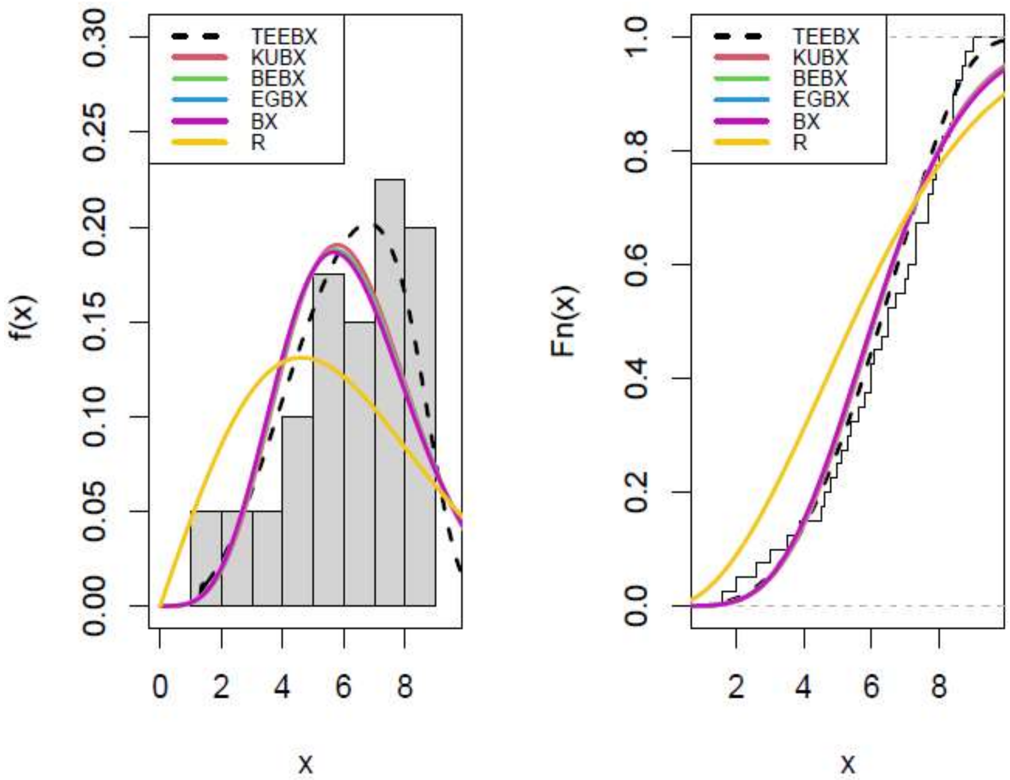


Figure 3: Q-Q plot, and Boxplot of the [0,1] TEE-BX for first data.



(a) **(b)**

Figure 4: (a) Fitted densities and (b) cdf of fitted [0,1] TEE-BX for first data.

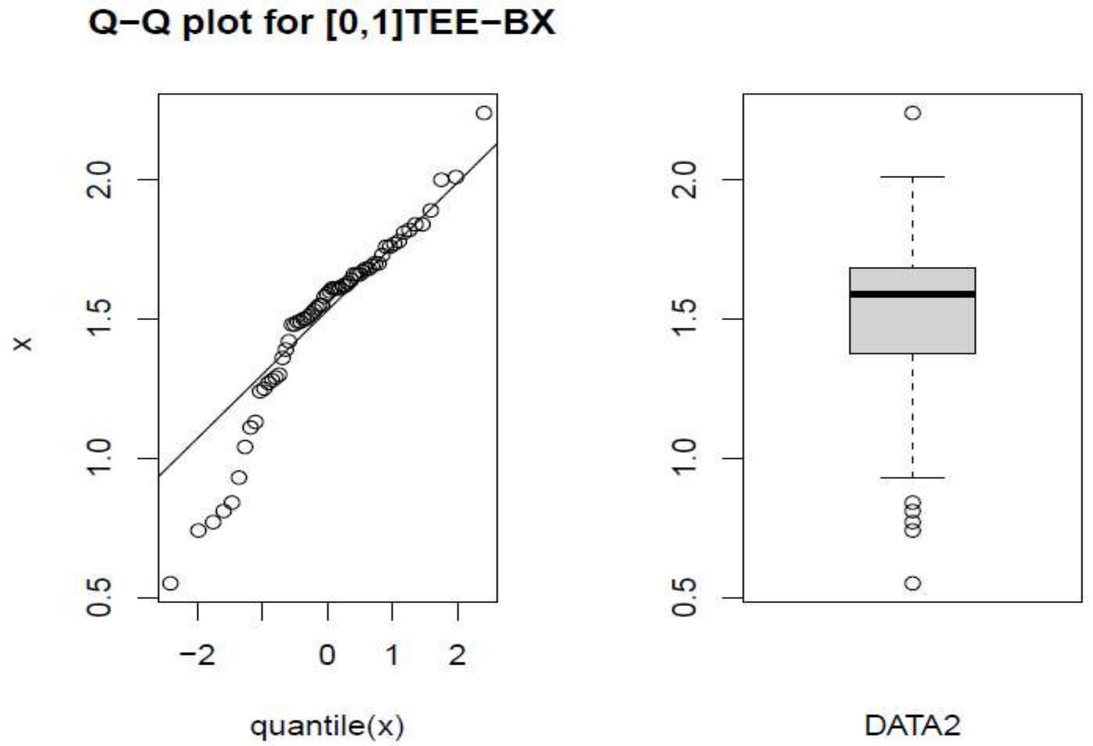
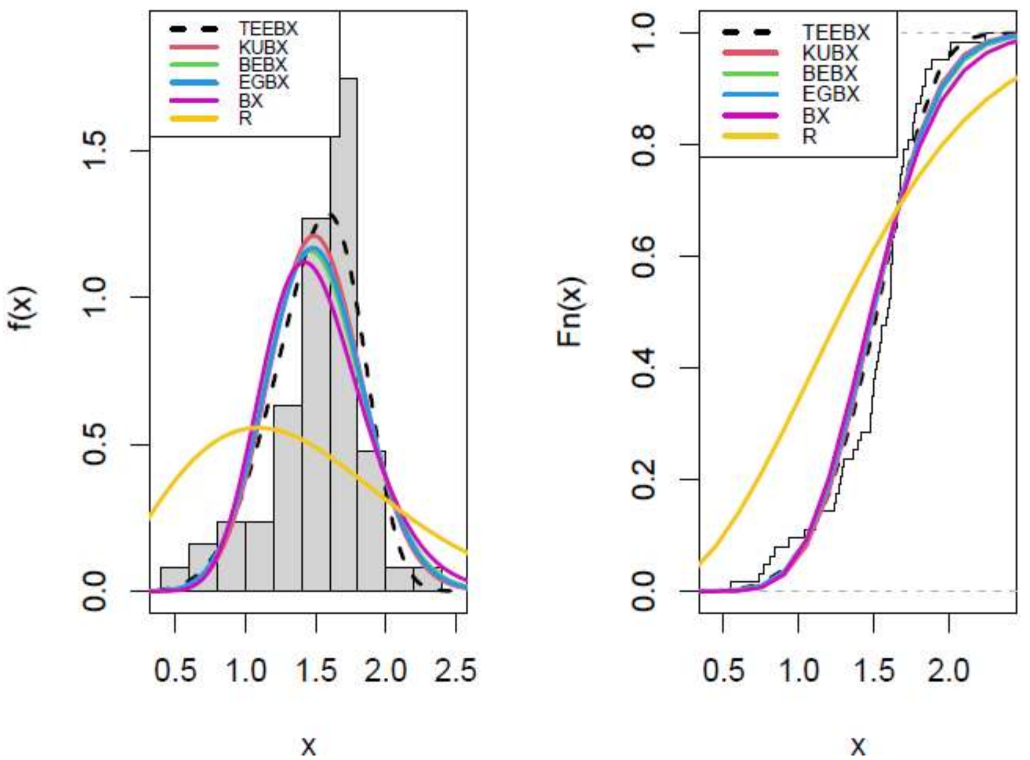


Figure 5: Q-Q plot, and Boxplot of the [0,1] TEE-BX for second data.



(a) **(b)**

Figure 6: (a) Fitted densities and (b) cdf of fitted [0,1] TEE-BX for second data

5- Conclusion

In this paper, we introduce a new distribution with a four-parameters which is named the [0,1]Truncated Exponentiated Exponential Burr type X distribution ([0,1] TEE-BX). The basic statistical properties of the [0,1] TEE-BX distribution such as the moment, moment generating function, quantile function, order statistics, and the Rényi entropy. The [0,1] TEE-

BX parameters are estimated through the maximum likelihood estimation method which provides good estimates of the distribution parameters.

Applications with two real data sets are conducted. It shows that the [0,1] TEE-BX distribution performed better than the KUBX, BEBX, EGBX, BX, and R, We use the R program.

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