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FSFS Neotherian and Artinian Modules

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Abstract
Let M be an $R-$ module, (F,A) be a fuzzy soft module over M, and $(\mathcal{R},A)$ be a fuzzy soft ring over R, then $(F,\mathcal{R},A)$ is called FSFS module if and only if (F,A) is an $(\mathcal{R},A)-$ module. In this paper, we introduce the concept of $FSFS$ Noetherian and $FSFS$ Artinian modules and finally we investigate some basic properties of $FSFS$ Noetherian and $FSFS$ Artinian modules.

Keywords: fuzzy soft ring, fuzzy soft module, fuzzy soft Noetherian, fuzzy sof Artinian, *FSFS* Noetherian, *FSFS* Artinian

المقاسات FSFS النويثيرية والآرتينية

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الخلاصة
ليكن M مقاساً على R ، ولتكن (F,A) مقاساً ضبابياً املساً على M ، و $(\mathcal{R},A)$ حلقة ضبابية ملساء على R. فإن (F,A) يسمى مقاساً $FSFS$ إذذا كان (F,A) مقاساً على $(\mathcal{R},A)$. في بحثنا هذا قدمنا مفهوم المقاسات FSFS النويثيرية والآرتينية وتحققنا من بعض الخواص الأساسية لصنف المقاسات FSFS النويثيرية والآرتينية.

1. Introduction

Molodtsov [1] initiated the theory of soft sets as a new mathematical tool for dealing with uncertainties. Later other authors Maji et al. [2-4] have further studied the theory of soft sets and also introduced the concept of fuzzy soft set, which is a combination of fuzzy set [5] and soft set. Thereafter, Aktas and Cagman [6] have introduced the notion of soft groups. Aygunoglu and Aygun [7] have generalized the concept of Aktas and Cagman [6] and introduce fuzzy soft group. F.Feng et al. [8] gave soft semirings and U.Acar et al. [9] introduced initial concepts of soft rings. Ghosh et al. [10] gave the notion of fuzzy soft rings. The definition of fuzzy modules is given by some authors. [11-13] Qiu- Mei Sun et al. [14] defined soft modules and investigated their basic properties. Gunduz and Bayramov [15] introduced a basic version of fuzzy soft module theory. Alhusiny A. [16] gave the concept of $FSFS$ modules. In this paper the main purpose is to introduce a basic concept of $FSFS$ Noetherian and $FSFS$ Artinian. and finally some of their basic properties.

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2. Preliminaries:

Definition (2.1) [1] Let U be an initial universe set and E be the set of parameters. Let $P(U)$ denotes the power set of U. A pair $(\mathcal{F},E)$ is called a soft set over U, where $\mathcal{F}$ is a mapping given by $\mathcal{F}: E \to P(U)$.

Definition (2.2) [2] Let U be an initial universe set and E be the set of parameters. Let $A \subseteq E$. A pair $(\mathcal{F},A)$ is called fuzzy soft set over U, where $\mathcal{F}$ is a mapping given by $\mathcal{F}: A \to I^U$, where I^U denotes the collection of all fuzzy subsets of U.

Definition (2.3) [17] A binary operation $*: [0,1] \times [0,1] \to [0,1]$ is continuous t-norm if $*$ satisfies the following conditions:

(i) $*$ is commutative and associative,

(ii) $*$ is continuous,

(iii) $a * 1 = a$ for all $a \in [0,1]$,

(iv) $a * b \leq c * d$ whenever $a \leq c, b \leq d$, and $a, b, c, d \in [0,1]$.

Definition (2.4) [17] A binary operation $\diamond: [0,1] \times [0,1] \to [0,1]$ is continuous t-conorm if $\diamond$ satisfies the following conditions:

(i) $\diamond$ is commutative and associative,

(ii) $\diamond$ is continuous,

(iii) $a \diamond 0 = a$ for all $a \in [0,1]$,

(iv) $a \diamond b \leq c \diamond d$ whenever $a \leq c, b \leq d$ and $a, b, c, d \in [0,1]$.

Definition (2.5) [6] Let X be a group and $(\mathcal{F},A)$ be a soft set over X. Then $(\mathcal{F},A)$ is said to be a soft group over X if and only if $\mathcal{F}(a)$ is a subgroup of X for each $a \in A$.

Definition (2.6) [7] Let X be a group and $(\mathcal{F},A)$ be a fuzzy soft set over X. Then $(\mathcal{F},A)$ is said to be a fuzzy soft group over X if and only if for each $a \in A$ and $x, y \in X$,

(1) $\mathcal{F}_a(x \cdot y) \geq \mathcal{F}_a(x) * \mathcal{F}_a(y)$

(2) $\mathcal{F}_a(x^{-1}) \geq \mathcal{F}_a(x)$

Where $\mathcal{F}_a$ is the fuzzy subset of X corresponding to the parameter $a \in A$.

Remark (2.7) $\mathcal{F}_a(x^{-1}) = \mathcal{F}_a(x)$.

Proof: Let $x, y \in X$, such that $x^{-1} = y$ which means that $y^{-1} = x$, since $\mathcal{F}_a(x^{-1}) \geq \mathcal{F}_a(x)$ then $\mathcal{F}_a(y^{-1}) \geq \mathcal{F}_a(y)$, and so $\mathcal{F}_a(x) \geq \mathcal{F}_a(y) = \mathcal{F}_a(x^{-1})$, and thus $\mathcal{F}_a(x^{-1}) = \mathcal{F}_a(x)$.

Definition (2.8) [10] Let f and g be any two fuzzy subset of a ring R. Then $f \circ g$ is a fuzzy subset of R defined by

$(f \circ g)(z) = \begin{cases} sup_{z=x \cdot y}\{min\{f(x), g(y)\}\} \\ 0 \quad otherwise \end{cases}$, if z is expressed as $z = x \cdot y$, where $x, y\, z \in R$.

Definition (2.9) [10] The intersection of two fuzzy soft sets $(\mathcal{F},A)$ and $(\mathcal{G},B)$ over the same Universe U is denoted by $(\mathcal{F},A)\tilde{\cap}(\mathcal{G},B)$ and defined by a fuzzy soft set $(\mathcal{H},C)$ where $C = A \cap B$ and $\mathcal{H}: C \to [0,1]^U$ such that for each $e \in C$,

$$\mathcal{H}(e) = \{(x, \mathcal{H}_e(x)) : x \in U\}$$

Where $\mathcal{H}_e(x) = \mathcal{F}_e(x) * \mathcal{G}_e(x)$ and $\mathcal{H}_e(x), \mathcal{F}_e(x), \mathcal{G}_e(x)$ are the fuzzy subset of U corresponding to the parameter $e \in C$.

Definition (2.10) [10] The union of two fuzzy soft sets $(\mathcal{F},A)$ and $(\mathcal{G},B)$ over the same universe U is denoted by $(\mathcal{F},A)\tilde{\cup}(\mathcal{G},B)$ and defined by a fuzzy soft set $(\mathcal{H},C)$ where $C = A \cup B$ and $\mathcal{H}: C \to [0,1]^U$such that for each $e \in C$,

$$\begin{aligned} \mathcal{H}(e) &= \{(x, \mathcal{F}_e(x)) : x \in U\}, if\ e \in A - B \\ &= \{(x, \mathcal{G}_e(x)) : x \in U\}, if\ e \in B - A \\ &= \{(x, \mathcal{H}_e(x)) : x \in U\}, if\ e \in A \cap B. \end{aligned}$$

Where $\mathcal{H}_e(x) = \mathcal{F}_e(x) \diamond \mathcal{G}_e(x)$ and $\mathcal{H}_e(x), \mathcal{F}_e(x), \mathcal{G}_e(x)$ are the fuzzy subset of U corresponding to the parameter $e \in C$.

Definition (2.11) [4] If (f, A) and $(g, B)$are two soft sets, then (f, A) **AND** (g, B) is denoted as $(f, A) \wedge (g, B)$ defined as $(h, A \times B)$, where $h(a, b) = h_{a,b} = f_a \wedge g_b$, $\forall (a, b) \in A \times B$.

Definition (2.12) [10] Let $(R, +, \cdot)$ be a ring and E be a parameter set and $A \subset E$. Let $\mathcal{R}$ be a mapping given by $\mathcal{R} : A \to P(R)$. Then $(\mathcal{R}, A)$ is called a soft ring over R if and only if for each $a \in A, \mathcal{R}(a)$ is a subring of R i.e.

(i) $x, y \in \mathcal{R}(a) \Rightarrow x + y \in \mathcal{R}(a)$,

(ii) $x \in \mathcal{R}(a) \Rightarrow -x \in \mathcal{R}(a)$,

(iii) $x, y \in \mathcal{R}(a) \Rightarrow x \cdot y \in \mathcal{R}(a)$.

Definition (2.13) [10] Let $(R, +, \cdot)$ be a ring and E be a parameter set and $A \subset E$. Let $\mathcal{R}$ be a mapping given by $\mathcal{R} : A \to [0,1]^R$, where $[0,1]^R$ denotes the collection of all fuzzy subsets of R. Then $(\mathcal{R}, A)$ is called a fuzzy soft ring over R if and only if for each $a \in A$ the corresponding fuzzy subset $\mathcal{R}_a$ of R is a fuzzy subring of R i.e. $\forall x, y \in R$,

(i) $\mathcal{R}_a(x + y) \geq \mathcal{R}_a(x) * \mathcal{R}_a(y)$

(ii) $\mathcal{R}_a(-x) \geq \mathcal{R}_a(x)$

(iii) $\mathcal{R}_a(x \cdot y) \geq \mathcal{R}_a(x) * \mathcal{R}_a(y)$.

Definition (2.14) [16]**:** Let M be an $R -$ module, (F, A) be a fuzzy soft module over M, and $(\mathcal{R}, A)$ be a fuzzy soft ring over R, then $(F, \mathcal{R}, A)$ is called fuzzy soft module over fuzzy soft ring if and only if (F, A) is an $(\mathcal{R}, A) -$ module.

Remark (2.15) [16]**:** 1) We shall denote the category of fuzzy soft modules over fuzzy soft rings by *FSFS* modules.

2) For convenience we denote the fuzzy soft module over fuzzy soft ring $(F, \mathcal{R}, A)$ by (F, A) , wherever there is no risk of confusion.

Definition (2.16) [16]**:** Let $(F, \mathcal{R}, A)$ and $(H, \mathcal{R}, B)$ be two $R - FSFS$ modules over M. Then $(F, \mathcal{R}, A)$ is called $FSFS$ submodule of $(H, \mathcal{R}, B)$ if

(i) $A \subset B$

(ii) For all $a \in A$, F_a is a fuzzy submodule of H_a.

Theorem (2.17) [16]**:** Let $(F, \mathcal{R}, A)$ and $(H, \mathcal{R}, B)$ be two $R - FSFS$ modules over M. If $F_a \leq H_a$ for all $a \in A$, then $(F, \mathcal{R}, A)$ is an $FSFS$ submodule of $(H, \mathcal{R}, B)$.

Theorem (2.18) [16]**:** Let (F, A) and (H, B) be two $FSFS$ modules over M. Then their intersection $(F, A) \cap (H, B)$ is an $FSFS$ module over M .

Theorem (2.19) [16]**:** Let (F, A) and (H, B) be two $FSFS$ modules over . Then $(F, A) \wedge (H, B)$ is an $FSFS$ module over M .

Definition (2.20) [16]**:** Let $(F, \mathcal{R}_M, A)$ and $(H, \mathcal{R}_N B)$ be two $R - FSFS$ modules over M and N respectively, and let $f: M \to N$ be a homomorphism of modules, and let $g: A \to B$ be a mapping of sets. Then we say that $(f, g): (F, \mathcal{R}_M, A) \to (H, \mathcal{R}_N B)$ is an $FSFS$ homomorphism of $FSFS$ modules, if the following condition is satisfied:

$$f(F_a) = H(g(a)) = H_{g(a)} .$$

We say that (F, A) is an $FSFS$ homomorphic to (H, B).

Note that for $\forall a \in A, f: (M, F_a) \to (N, H_{g(a)})$ is a fuzzy homomorphism of fuzzy modules.

To introduce the kernel and image of $FSFS$ homomorphism of $FSFS$ modules, let $\widetilde{M} = ker\, f$. Define $\tilde{F}: A \to PF(\widetilde{M})$ by $\tilde{F}_a = F_a|_{\widetilde{M}}$. Then $(\tilde{F}, A)$ is an $FSFS$ module over $\widetilde{M}$. It is clear that this module is an $FSFS$ submodule of (F, A).

Definition (2.21) [16]: $(\tilde{F}, A)$ is said to be kernel of (f, g) and denoted by $ker(f, g)$.

And, let $\tilde{B} = g(A)$. Then for all $b \in \tilde{B}$, there exists $a \in A$ such that $g(a) = b$. Let $\tilde{N} = Im\,f < N$. We define the mapping $\tilde{H}: \tilde{B} \to PF(\tilde{N})$ as $\tilde{H}(\tilde{b}) = \tilde{B}(g(a))|_{\tilde{N}}$. Since (f, g) is an $FSFS$ homomorphism, $f(F_a) = H_{g(a)}$ is satisfied for all $a \in A$. Then the pair $(\tilde{H}, \tilde{B})$ is an $FSFS$ module over $\tilde{N}$ and $(\tilde{H}, \tilde{B})$ is an $FSFS$ submodule of (H, B).

Definition (2.22) [16]: $(\tilde{H}, \tilde{B})$ is said to be image of (f, g) and denoted by $Im(f, g)$.

Proposition (2.23): Let (F, A) be an $FSFS$ module over M and N be an R −module and $f: M \to N$ be a homomorphism of R − modules. Then $(f(F), A)$ is an $FSFS$ module over N.

Proposition (2.24) [16]: If M is an $R-$ module, (H, A) is an $FSFS$ module over N and $f: M \to N$ is a homomorphism of $R-$ modules, then $(f^{-1}(H), A)$ is an $FSFS$ module over M .

Definition (2.25) [16]: (F, A) is said to be direct sum of $\{(F_i, A_i)\}_{i \in I}$, and denoted by $\underset{i \in I}{\oplus}(F_i, A_i)$.

The mapping :

$$\varphi_j: A_j \to \prod_{i \in I} A_i$$

is defined as $\varphi_j(a_j) = \{a_i\} = \begin{cases} a_{0i} & if\ i \neq j \\ a & if\ i = j \end{cases}$.

Also for embedding mapping $q_j: M_j \to \underset{i \in I}{\oplus} M_i$, $(q_j, \varphi_j): (F_j, A_j) \to (F, A)$ is an $FSFS$ homomorphism of $FSFS$ modules.

Theorem (2.26) [16]: If $\{(F_i, A_i)\}_{i \in I}$ is a family of $FSFS$ modules over $\{M_i\}_{i \in I}$, then $\underset{i \in I}{\oplus}(F_i, A_i)$, is an $FSFS$ module over $\underset{i \in I}{\oplus} M_i$.

Corallary (2.25) [16]: If (F, A) is a $FSFS$ module over M and N is a submodule of M, $i: N \to M$ is an embedding mapping, then $(i^{-1}(F), A)$ is an $FSFS$ module over N.

Corallary (3.27) [16]: If (F, A) is an $FSFS$ module over M and $p: M \to M/\mathcal{N}$ is a canonical projection, then $(p(F), A)$ is an $FSFS$ module over quotient module $M/\mathcal{N}$.

If $\{(F_i, A_i)\}_{i \in I}$ is a family of $FSFS$ modules over the family of modules $\{M_i\}_{i \in I}$, then we can define the product and coproduct of these families by $\underset{i \in I}{\Pi}(F_i, A_i)$ and $\underset{i \in I}{\oplus}(F_i, A_i)$ respectively.

Theorem (2.28) [16]: The category of $FSFS$ modules has zero objects, sums, product, kernel and cokernel.

Theorem (2.29) [16]: Let (F, A) and (G, B) be two $FSFS$ modules over M and N, respectively, and $F \otimes G: A \times B \to M \otimes N$. Then $(F \otimes G, A \times B)$ is an $FSFS$ module over $M \otimes N$.

Noetherian* and *Artinian modules:

Definition (2.30) [18] An R-module M is said to satisfy the ascending chain condition (a.c.c.) on submodules (or to be Noetherian) if for every chain $N_1 \subseteq N_2 \subseteq N_3 \subseteq \cdots$ of submodules of M, there is an integer n such that $N_i = N_n$ for all $i \geq n$.

An R-module M is said to satisfy the descending chain condition (d.c.c.) on submodules (or to be Artinian) if for every chain $N_1 \supseteq N_2 \supseteq N_3 \supseteq \cdots$ of submodules of M, there is an integer n such that $N_i = N_n$ for all $i \geq n$.

Definition (2.31) [18] A ring R is said to be Noetherian if R is a Noetherian R- module. A ring R is said to be Artinian if R is an Artinian R-module.

Theorem (2.32). [19]

(i) Submodules and quotient modules of Noetherian are Noetherian.

(ii) Let M be an R-module and N be its submodule such that M/N and N are Noetherian. Then M is Noetherian.

Definition (2.33): A fuzzy ring X of R is called Noetherian (Artinian) if R is a Noetherian (Artinian) ring.

Lemma (2.34) [19]: Any commutative Artinian ring with identity is Noetherian .

Remark (2.35) [19]: Let X be a fuzzy ring over R .If X is Artinian, then X is Noetherian.

Theorem (2.36) [19] (i) Submodules and quotient modules of Noetherian are Noetherian.

(ii) Let M be an R-module and N be its submodule such that M/N and N are Noetherian. Then M is Noetherian.

3. *FSFS Noetherian* and *FSFS* Artinian

Definition (3.1): A fuzzy soft module $(\boldsymbol{F},\boldsymbol{A})$is said to be fuzzy soft Noetherian if for every chain $(F_1,A_1) \subseteq (F_2,A_2) \subseteq (F_3,A_3) \subseteq \cdots$ of fuzzy soft submodules of M, there is an integer n such that $(F_i,A_i) = (F_n,A_n)$ for all $i \geq n$.

Definition (3.2): A fuzzy soft module $(\boldsymbol{F},\boldsymbol{A})$ is said to be fuzzy soft Artinian if for every chain $(F_1,A_1) \supseteq (F_2,A_2) \supseteq (F_3,A_3) \supseteq \cdots$ of fuzzy soft submodules of M, there is an integer n such that $(F_i,A_i) = (F_n,A_n)$ for all $i \geq n$.

Definition (3.3): A fuzzy soft ring $(\mathcal{R},A)$ is said to be fuzzy soft Noetherian if $(\mathcal{R},A)$ is a fuzzy soft Noetherian $(\mathcal{R},A)$- module.

Definition (3.4): A fuzzy soft ring $(\mathcal{R},A)$ is said to be fuzzy soft Artinian if $(\mathcal{R},A)$ is a fuzzy soft Artinian $(\mathcal{R},A)$- module.

Definition (3.5): An $FSFS$ module $(\boldsymbol{F},\boldsymbol{A})$ is said to be $FSFS$ Noetherian if both $(\boldsymbol{F},\boldsymbol{A})$ and $(\mathcal{R},A)$ are fuzzy soft Noetherian.

An $FSFS$ module $(\boldsymbol{F},\boldsymbol{A})$ is said to be $FSFS$ Artinian $(\mathcal{R},A)$- module if both $(\boldsymbol{F},\boldsymbol{A})$ and $(\mathcal{R},A)$ is fuzzy soft Artinian.

Remark(3.6): If $(\mathcal{R},A)$ is a fuzzy soft Noetherian (Artinian) ring, then $(\mathcal{R},A)$ is $FSFS$ Noetherian (Artinian).

Theorem(3.7): The following are equivalent conditions for an $FSFS$ module $(\boldsymbol{F},\boldsymbol{A})$:

(i) $(\boldsymbol{F},\boldsymbol{A})$ is an $FSFS$ Noetherian module.

(ii) Every nonempty collection of fuzzy soft submodules of $(\boldsymbol{F},\boldsymbol{A})$ has maximal elements.

(iii) Every fuzzy soft submodule of $(\boldsymbol{F},\boldsymbol{A})$ is f.g..

Proof. $(i) \Rightarrow (ii)$ Let S be a non-empty collection of fuzzy soft submodules of $(\boldsymbol{F},\boldsymbol{A})$. If S has no maximal element, then we can construct an infinite increasing sequence of fuzzy soft submodules of $(\boldsymbol{F},\boldsymbol{A})$as follows: Suppose we have constructed(F_n,A_n). Then since (F_n,A_n) is not a maximal element in S, there is a fuzzy soft submodule of $(\boldsymbol{F},\boldsymbol{A})$, say $(F_{n+1},A_{n+1}) \in S$ such that $(F_n,A_n) \subsetneq (F_{n+1},A_{n+1})$. Therefore, S has a maximal element.

$(ii) \Rightarrow (iii)$ Let (H,B) be a fuzzy soft submodule of $(\boldsymbol{F},\boldsymbol{A})$. To show that (H,B) is f.g., let S be the set of all fuzzy soft submodule of (H,B) which is f.g.; then S is non-empty as $0 \in S$. So, by assumption, S has a maximal element say (H_0,B_0). We have to prove that $(H_0,B_0) = (H,B)$.

Notice that if $x \in (H,B) - (H_0,B_0)$ then $(H_0,B_0) + (xR,A)$ is a f.g. fuzzy soft submodule of (H,B) which is in S, so that (H,B) is not a maximal element in S, which is impossible.Therefore $(H_0,B_0) = (H,B)$.

$(iii) \Rightarrow (i)$ Let $(F_1,A_1) \subseteq (F_2,A_2) \subseteq (F_3,A_3) \subseteq \cdots$ be an ascending chain of fuzzy soft submodules of $(\boldsymbol{F},\boldsymbol{A})$. Let $(H,B) = \bigcup_{i=1}^{\infty}(F_i,A_i)$; then it is easy to show that (H,B) is a fuzzy soft submodule of $(\boldsymbol{F},\boldsymbol{A})$, so that (H,B) is f.g.. Then there exist $x_1,x_2,\dots,x_k \in (H,B)$ such that $(H,B) = (x_1R,A) + (x_2R,A) + \cdots + (x_kR,A)$; then for every j there is an integer n_j such that $x_j \in \left(F_{n_j},A_{n_j}\right)$.

Let $n = max\{n_j | j = 1, \ldots, k\}$; then $(H,B) \subseteq (F_n, A_n)$ and therefore $(F_i, A_i) = (F_n, A_n)$ for every $i \geq n$.

Corollary(3.9): Let $(\mathcal{R}, A)$ be a fuzzy soft Noetherian ring and (F,A) be a f.g. $FSFS$ module; then (F,A) is an $FSFS$ Noetherian.

Proof. Directly from theorem **(3.7).**

Corollary(3.10): Let $(\mathcal{R}, A)$ be a fuzzy soft ring. Then $(\mathcal{R}, A)$ is a fuzzy soft Noetherian ring if and only if every Fuzzy soft ideal of $(\mathcal{R}, A)$is f.g..

Theorem(3.11): For an $FSFS$ module $(\boldsymbol{F}, \boldsymbol{A})$the following are equivalent conditions:

(i) $(\boldsymbol{F}, \boldsymbol{A})$is an $FSFS$ Artinian module.

(ii) Every nonempty collection of fuzzy soft submodules of $(\boldsymbol{F}, \boldsymbol{A})$has minimal elements.

Proof. $(i) \Rightarrow (ii)$ Let S be a non-empty collection of fuzzy soft submodules of $(\boldsymbol{F}, \boldsymbol{A})$. If S has no minimal element, then we can construct an infinite decreasing sequence of fuzzy soft submodules of $(\boldsymbol{F}, \boldsymbol{A})$as follows: Suppose we have constructed (F_n, A_n). Then since (F_n, A_n) is not a minimal element in S, there is an fuzzy soft submodule of $(\boldsymbol{F}, \boldsymbol{A})$, say $(F_{n+1}, A_{n+1}) \in S$ such that $(F_n, A_n) \supsetneq (F_{n+1}, A_{n+1})$. Therefore, S has a minimal element.

$(ii) \Rightarrow (i)$ Let $(F_1, A_1) \supseteq (F_2, A_2) \supseteq (F_3, A_3) \supseteq \cdots$ be an descending chain of fuzzy soft submodules of $(\boldsymbol{F}, \boldsymbol{A})$. Let $T = \{(F_i, A_i) | i \in \mathbb{N}\}$ be a non-empty collection which has a minimal element, so that there is an integer n such that $(F_i, A_i) = (F_n, A_n), \forall i \in \mathbb{N}$.

Example (3.12): Let $M_{n \times n}(\mathbb{R})$ be the set of all $n \times n$ matrices over $\mathbb{R}$, and $R = A = M_{n \times n}(\mathbb{R})$, define the function $\mathcal{R} : A \to [0,1]^R$ by $\mathcal{R}(B) = \{C \cdot B \mid C \in M_{n \times n}(\mathbb{R})\}$ for all $B \in A$. Then $(\mathcal{R}, A)$ is a fuzzy soft ring over R, now consider $M = M_{n \times n}(\mathbb{R})$ as an $R-$ module and $F: A \to FP(M)$, defined by $F(\mathcal{M}) = \{\mathcal{N} \mid \mathcal{M} \cdot \mathcal{N} = \mathcal{N} \cdot \mathcal{M}\}$ for all $\mathcal{M} \in M$. Then (F,A) is a fuzzy soft module over M, and (F,A) is an $(\mathcal{R}, A)-$module, which means $(F, \mathcal{R}, A)$ is an FSFS module. And since $(\mathcal{R}, A)$ is a field, (F,A) is finitely generated, then (F,A) is both $FSFS$ Noetherian and $FSFS$ Artinian.

Theorem(3.13): Let $0 \longrightarrow (F, \mathcal{R}, A) \xrightarrow{f} (H, \mathcal{R}, B) \xrightarrow{g} (G, \mathcal{R}, C) \longrightarrow 0$ be a short exact sequence of $FSFS$ modules, then $(H, \mathcal{R}, B)$ is $FSFS$ Noetherian if and only if $(F, \mathcal{R}, A)$ and $(G, \mathcal{R}, C)$ are both $FSFS$ Noetherian.

Proof. $(\Rightarrow)$ Suppose $(H, \mathcal{R}, B)$ is $FSFS$ Noetherian. Since $f(F)$ is an fuzzy soft submodule of H, and every submodule of $f(F)$ is an fuzzy soft submodule of H,and since every fuzzy soft submodule of $f(F)$ is isomorphic to a fuzzy soft submodule of F, and hence every fuzzy soft submodule of F is f.g.. Therefore F is $FSFS$ Noetherian. On the other hand, if G_o is an $FSFS$ submodule of G, then the fuzzy soft submodule $g^{-1}(G_o)$ is f.g., so that G_o is f.g. as g is surjective. Therefore G is also $FSFS$ Noetherian.

$(\Leftarrow)$ Let $(H_1, B_1) \subseteq (H_2, B_2) \subseteq (H_3, B_3) \subseteq \cdots$ be an ascending chain of fuzzy soft submodules of H. Since $(F, \mathcal{R}, A)$ and $(G, \mathcal{R}, C)$ are $FSFS$ Noetherian, then there is an integer n such that $f^{-1}(H_i) = f^{-1}(H_n)$ and $g(H_i) = g(H_n)$ for every $i \geq n$, which implies that $H_{n+1} = H_n$. Let $x \in H_{n+1}$; then there is an element $y \in H_n$ such that $g(x) = g(y)$, so that $x - y \in kerg = Imf$, it follows that there is an element $z \in F$ such that $f(z) = x - y$. However, $x - y \in H_{n+1} \wedge z \in F, z \in f^{-1}(H_{n+1}) = f^{-1}(H_n)$. Therefore $x - y \in H_n$ as f is 1-1. And hence $x \in H_n$.

Theorem(3.14): Let $0 \longrightarrow (F, \mathcal{R}, A) \xrightarrow{f} (H, \mathcal{R}, B) \xrightarrow{g} (G, \mathcal{R}, C) \longrightarrow 0$ be a short exact sequence of $FSFS$ modules, then $(H, \mathcal{R}, B)$ is $FSFS$ Artinian if and only if $(F, \mathcal{R}, A)$ and $(G, \mathcal{R}, C)$ are both $FSFS$ Artinian.

Proof. $(\Rightarrow)$ Suppose $(H,\mathcal{R},B)$ is $FSFS$ Artinian. Since $f(F)$ is an fuzzy soft submodule of H, and every submodule of $f(F)$ is an fuzzy soft submodule of H,and since every fuzzy soft submodule of $f(F)$ is isomorphic to a fuzzy soft submodule of F, and hence every fuzzy soft submodule of F is f.g.. Therefore F is $FSFS$ Artinian. On the other hand, if G_o is an $FSFS$ submodule of G, then the fuzzy soft submodule $g^{-1}(G_o)$ is f.g., so that G_o is f.g. as g is surjective. Therefore G is also $FSFS$ Artinian.
$(\Leftarrow)$ Let $(H_1,B_1)\supseteq(H_2,B_2)\supseteq(H_3,B_3)\supseteq\cdots$ be an ascending chain of fuzzy soft submodules of H.
Since $(F,\mathcal{R},A)$ and $(G,\mathcal{R},C)$ are $FSFS$ Artinian, then there is an integer n such that $f^{-1}(H_i)=f^{-1}(H_n)$ and $g(H_i)=g(H_n)$ for every $i\geq n$, which implies that $H_{n+1}=H_n$. Let $x\in H_{n+1}$; then there is an element $y\in H_n$ such that $g(x)=g(y)$, so that $x-y\in kerg=Imf$, it follows that there is an element $z\in F$ such that $f(z)=x-y$. However, $x-y\in H_{n+1}\wedge z\in F, z\in f^{-1}(H_{n+1})=f^{-1}(H_n)$. Therefore $x-y\in H_n$ as f is 1-1. And hence $x\in H_n$.

Corollary(3.15): Let (F,A) and (G,B) be $FSFS$ Noetherian; then $(F\otimes G, A\times B)$ is an $FSFS$ Noetherian over $M\otimes N$.

Proof. Since (F,A) and (G,B) are isomorphic to submodules of $(F\otimes G, A\times B)$ and (F,A) and (G,B) are both $FSFS$ Noetherian, then by Theorem (2.29) $(F\otimes G, A\times B)$ is $FSFS$ Noetherian.

Corollary(3.16): Let (F,A) and (G,B) be $FSFS$ Noetherian; then $(F\oplus G, A\times B)$ is a $FSFS$ Noetherian.

Corollary(3.17): Let R be an $FSFS$ Noetherian ring; then $R^n=\left\{\begin{matrix}R\oplus R\oplus\ldots\oplus R\\ \leftarrow n\ times\rightarrow\end{matrix}\right\}$ is an $FSFS$ Noetherian module.
Proof: Using mathematical induction on **Corollary(3.16)**.

Theorem(3.18): Let (G,B) be an $FSFS$ submodule of (F,A), then (F,A) is $FSFS$ Noetherian if and only if (G,B) and $(F/_G, A)$ are both $FSFS$ Noetherian.

Proof. $\Rightarrow)$ Let (F,A) be an $FSFS$ Noetherian and let (G,B) be its $FSFS$ submodule. And since every submodule of (G,B) is a submodule of (F,A). Then (G,B) is an $FSFS$ Noetherian**.** In the other hand if
$\left(F_1/_G, A_1\right)\subseteq\left(F_2/_G, A_2\right)\subseteq\left(F_3/_G, A_3\right)\subseteq\cdots$ is an ascending chain of $FSFS$ submodules of $(F/_G, A)$, then $(F_1,A_1)\subseteq(F_2,A_2)\subseteq(F_3,A_3)\subseteq\cdots$ is an ascending chain in (F,A) which is $FSFS$ Noetherian,which means that there is $n\in N$ such that
$\left(F_i/_G, A_i\right)=\left(F_n/_G, A_n\right)$, $\forall i\geq n$.
$\Leftarrow)$ Let (G,B) and $(F/_G, A)$ are both $FSFS$ Noetherian, assume that $(F_1,A_1)\subseteq(F_2,A_2)\subseteq(F_3,A_3)\subseteq\cdots$ is an ascending chain in (F,A), then $\left(F_1/_G, A_1\right)\subseteq\left(F_2/_G, A_2\right)\subseteq\left(F_3/_G, A_3\right)\subseteq\cdots$ is an ascending chain in $(F/_G, A)$, so we get that there is $n\in N$ such that
$\left(F_i/_G, A_i\right)=\left(F_n/_G, A_n\right)$, $\forall i\geq n$. And hence $(F_i,A_i)=(F_n,A_n), \forall i\in\mathbb{N}$, which means (F,A) is an $FSFS$ Noetherian.

Theorem (3.19): Let (G,B) be an $FSFS$ submodule of (F,A), then (F,A) is $FSFS$ Artinian if and only if (G,B) and $(F/_G, A)$ are both $FSFS$ Artinian.

Proof. $\Rightarrow)$ Let (F,A) be an $FSFS$ Artinian and let (G,B) be its $FSFS$ submodule, since every submodule of (G,B) is a submodule of (F,A). Then (G,B) is an $FSFS$ Artinian**.** In the other hand if

$\left({}^{F_1}/_{G}, A_1\right) \supseteq \left({}^{F_2}/_{G}, A_2\right) \supseteq \left({}^{F_3}/_{G}, A_3\right) \supseteq \cdots$ is an ascending chain of $FSFS$ submodules of $\left({}^{F}/_{G}, A\right)$, then $(F_1, A_1) \supseteq (F_2, A_2) \supseteq (F_3, A_3) \supseteq \cdots$ is an ascending chain in (F, A) which is $FSFS$ Artinian, so that there is $n \in N$ such that

$\left({}^{F_i}/_{G}, A_i\right) = \left({}^{F_n}/_{G}, A_n\right), \ \forall i \geq n.$

$\Leftarrow$) Let (G, B) and $\left({}^{F}/_{G}, A\right)$ are both $FSFS$ Artinian, assume that $(F_1, A_1) \supseteq (F_2, A_2) \supseteq (F_3, A_3) \supseteq \cdots$ is an ascending chain in (F, A), then $\left({}^{F_1}/_{G}, A_1\right) \supseteq \left({}^{F_2}/_{G}, A_2\right) \supseteq \left({}^{F_3}/_{G}, A_3\right) \supseteq \cdots$ is an ascending chain in $\left({}^{F}/_{G}, A\right)$, so we get that there is $n \in N$ such that

$\left({}^{F_i}/_{G}, A_i\right) = \left({}^{F_n}/_{G}, A_n\right), \ \forall i \geq n$. And hence $(F_i, A_i) = (F_n, A_n), \forall i \in \mathbb{N}$, which means (F, A) is an $FSFS$ Artinian.

Conclusion:

Fuzzy soft Artinian and Fuzzy soft Noetherian modules are far more similar than they appear on the surface. While $FSFS$ Artinian satisfies a minimum condition and $FSFS$ Noetherian satisfies a maximum condition, these conditions do coincide in several cases. And Every fuzzy soft submodule of an $FSFS$ Noetherian module is f.g. We proved that tensor prodect and sum prodect of $FSFS$ Noetherian are $FSFS$ Noetherian. Finely a submodule and quotient module of an $FSFS$ Noetherian ($FSFS$ Artinian) are both $FSFS$ Noetherian ($FSFS$ Artinian).

References:

1. Molodtsov, D. **1999**. Soft set theory-First results, *Coumpt. Math. Appl.*, 37, pp:19-31.
2. Maji, P.K., Biswas, R., and Roy, A.R. **2001**. Fuzzy soft sets, *The Journal of Fuzzy Mathematics*, 9(3), pp:589-602.
3. Maji, P.K., Biswas, R. and Roy, A.R. **2001**. Intuitionistic fuzzy soft sets, *The Journal of Fuzzy Mathematics*, 9(3), pp:677-692.
4. Maji, P.K., Biswas, R. and Roy, A.R. **2004.** On intuitionistic fuzzy soft sets, *The Journal of Fuzzy Mathematics*, 12(3), pp: 669-683.
5. Zadeh L.A. **1965.** Fuzzy sets. *Information and Control*, 8, pp:338-353.
6. Aktas H. and Cagman N. **2007**. Soft sets and soft groups. *Information Science*, 177, pp: 2726-2735.
7. Aygunoglu A. and Aygun H. **2009**. Introduction to fuzzy soft group, *Coumpt. Math. Appl.*, 58:1279-1286.
8. Feng F., Jun Y.B. and Zhao X. **2008**. Soft semirings, *Comput. Math. Appl*.56:2621 -2628.
9. Acar U., Koyuncu F. and Tanay B. **2010**. Soft sets and soft rings, *Comput. Math. Appl.* 59, pp:3458-3463.
10. Ghosh J., Dinda B. and Samanta T.K. **2011**. Fuzzy Soft Rings and Fuzzy Soft Ideals. *Int. J. Pure Appl. Sci. Technol.*, 2(2), pp: 66-74.
11. Lopez-Permouth S.R. and Malik D.S. **1990**. On categories of fuzzy modules, *Information Sciences*, 52, pp: 211-220.
12. Lopez-Permouth S.R. and Morita L. **1992**. Equivalence to Categories of Fuzzy Modules, *Information Sciences*, 64, pp: 191-201.
13. Zahedi M. and Ameri R. **1995**. On Fuzzy Projective and Injective Modules, *The Journal of Fuzzy Mathematics*, 3(1), pp: 181-190.
14. Sun Q.M. and Zhang Z.L. and Liu J. **2008**. Soft sets and soft modules, *Lecture Notes in Comput. Sci*. 5009, pp: 403-409.
15. Gunduz C. and Bayramov S. **2011**. Fuzzy Soft Modules, *International Mathematical Forum*, 6(11), pp:517 – 527.
16. Alhusiny A. **2013**. Fuzzy soft modules over fuzzy soft rings. International Scientific Conference of Applied and Pure Mathematics, Iraqi Journal of Science / 8 - 9 may.

17. Schweizer B. and Sklar A. **1960**. Statistical metric space, *Pacific Journal of Mathematics*, 10, pp:314-334.
18. Abou –Dareb A. **2009**. Noetherian Fuzzy Rings, *Journal of Kerbala University*, 7(1) Scientific.
19. On Fuzzy Submodules and Their Radicals, Mukherjee T.K. Sen M.K. and Roy D.**1996**. *The Journal of Fuzzy Mathematics*, 4(3), pp:549 – 558**.**