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Occasionally Weakly Compatible Maps and Common Fixed Point Theorems in Certain New Generalized Symmetric Space

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Abstract
The major objective of the present paper is to study and investigate certain new class of generalized metric spaces namely, $\mathcal{D}^*$-symmetric spaces which is generalization of both $\mathcal{D}^*$-metric space and G-symmetric space. Moreover, employ the idea of the (E. A) property to establish the uniqueness of common fixed points of two pair of occasionally weakly compatible (OWC) maps under effect generalized contractive conditions in $\mathcal{D}^*$-symmetric spaces. Our main outcomes are generalization of various results of common fixed point theorems in the literature. In addition, we are equipping diverse convenient examples that support our major results.

Keywords: $\mathcal{D}^*$-symmetric space, $\mathcal{D}^*$-metric spaces, common fixed point, generalized contractive maps, (E. A.) property, (OWC) maps.

التطبيقات المتناسقة أحياناً بشكل ضعيف ونظريات النقطة الثابتة المشتركة في فضاء متناظر معمم جديد

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الخلاصة
الغرض الرئيس من تقديم هذا البحث هو دراسة وتقديم فضاء متري جديد من الفضاءات المترية المعممة يسمى الفضاء المتري المتناظر $\mathcal{D}^*$ الذي يُعد تعميما لفضائين متريين معممين هما الفضاء المتري $\mathcal{D}^*$ والفضاء المتري المتناظر G. اضافة الى ذلك وظفنا فكرة خاصية (E. A.) لاثبات وحدانية النقاط الثابتة المشتركة لزوجين من التطبيقات المتناسقة بشكل ضعيف حينًا تحت تاثير شروط انكماشية معممة في الفضاءات المترية المتناظرة $\mathcal{D}^*$. نتائجنا الرئيسة هي تعميم لنتائج مختلفة لنظريات النقطة الثابتة المشتركة في الادبيات. علاوة على ذلك قدمنا امثلة متنوعة ومناسبة تدعم نتائجنا الرئيسة.

1. Introduction

The study of common fixed point (CFP) theorems for (OWC) maps satisfying contractive conditions in generalized metric spaces expressed the most significant role to many fields both in pure and applied sciences. Moreover, it has a wide range of applications in other science fields such as biology, physics and computer sciences. Therefore, Jungck [1] extended the Banach's contraction principle exploiting the idea of commuting maps and established

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(CFP) theorems for commuting maps. On the other hand, Sessa in [2] initiated the tradition improvement of commutability in fixed point theorems. Jungck [3] soon enlarged this idea to compatible maps. A compatible map in a Mengar space has been presented via Mishra [4]. Further, Jungck [5] gave the idea of weakly compatible. In addition, Liu et al. [6] studied the idea of common (E. A.) property, which contains (E. A.) property introduced by M. Aamri [7]. Deferent generalizations of the notion of metric spaces have been presented via numerous researchers in literature. In particular, S. Shaban, et al. [8] have been verified the idea of $\mathcal{D}^*$-metric spaces. The concept of occasionally weakly compatible maps has been defined via Jungck and Rhoades in [9] which is more general than the concept of weakly compatible maps.

K. S. Eke and J. O. Olaleru [10] presented the idea of G-symmetric space via deleting the rectangle axiom of G-metric spaces; they also established in [11] the (CFP) of single valued and multivalued maps satisfying some contractive conditions in G-symmetric space, as well K. Eke in [12] proved various (CFP) results for contraction maps in uniform spaces.
In generalized partial ordering metric spaces; the fixed point theory has been developed quickly. In [13] A. AL-Jumaili, presented some coincidence fixed point theorems for maps satisfying contractive conditions in partially ordered complete $\mathcal{D}^*$-metric spaces. Afterwards, the authors in [14] extension the conception of $\mathcal{D}^*$-metric spaces by changing R by an ordered Banach space in $\mathcal{D}^*$-metric spaces.

Recently**,** in [15] the existence of the (CFP) theory has been verified for Hardy and Rogers-type maps and Ciric-type maps in G-symmetric space. Also, S. Q. Latif and S. S. Abed [16] studied fixed point of set-valued contractions on ordered G-metric spaces, as well A. M. Hashim and A. A. Kazem [17] introduced and prove some fixed point theorems for two maps that satisfy (ϕ, ψ)-contractive conditions. Furthermore, S. M. Ahmad and AL-Jumaili [18] introduced and established some of new (CFP) results in complete $\mathcal{D}^*$-metric spaces.

The aim of the present this paper is to define new kind of generalized metric spaces namely, $\mathcal{D}^*$-symmetric space which is generalization of both $\mathcal{D}^*$-metric and G-symmetric space. In addition, several (CFP) results for maps satisfying generalized contractive conditions in $\mathcal{D}^*$-symmetric spaces have been established. This paper is divided into two sections. In the first section, we provide some definitions, examples, and basic ideas that are important to our main argument. The uniqueness of common fixed points in $\mathcal{D}^*$-symmetric spaces has been investigated in the second section.

2. Preliminaries

In this section, recall various notions and important results which play vital role in this study and helpful for verifying our main results.

Definition 2.1: [**8**] Let $\mathcal{X}$ be a non-empty set. A generalized metric (or $\mathcal{D}^*$-metric) on $\mathcal{X}$ is a mapping $\mathcal{D}^*: \mathcal{X} \times \mathcal{X} \times \mathcal{X} \to [0, \infty)$, that satisfies the following conditions for each $x, y, z, w \in \mathcal{X}$:
$(\boldsymbol{\mathcal{D}_1^*})$ $\mathcal{D}^*(x, y, z) \geq 0, \forall\, x, y, z \in \mathcal{X}$,
$(\boldsymbol{\mathcal{D}_2^*})$ $\mathcal{D}^*(x, y, z) = 0$ if and only if $= y = z$,
$(\boldsymbol{\mathcal{D}_3^*})$ $\mathcal{D}^*(x, y, z) = \mathcal{D}^*(\mathcal{P}\{x, y, z\})$, wherever, $\mathcal{P}$ is permutation map. (symmetry),
$(\boldsymbol{\mathcal{D}_4^*})\mathcal{D}^*(x, y, z) \leq \mathcal{D}^*(x, y, w) + \mathcal{D}^*(w, z, z)$.
Then, $\mathcal{D}^*$ is called $\mathcal{D}^*$-metric and $(\mathcal{X}, \mathcal{D}^*)$ is $\mathcal{D}^*$-metric space.

Definition 2.2: A $\mathcal{D}^*$-symmetric on a set $\mathcal{X}$ is a map $\mathcal{D}_d^*: \mathcal{X} \times \mathcal{X} \times \mathcal{X} \to [0, \infty)$ such that for all $x, y, z \in \mathcal{X}$, the following conditions are satisfied:

$(\mathcal{D}^*_{d_1})$ $\mathcal{D}^*(x, y, z) \geq 0, \forall\, x, y, z \in \mathcal{X}$,
$(\mathcal{D}^*_{d_2})$ $\mathcal{D}^*(x, y, z) = 0$ if and only if $x = y = z$,
$(\mathcal{D}^*_{d_3})$ $\mathcal{D}^*(x, y, z) = \mathcal{D}^*(\mathcal{P}\{x, y, z\})$, where, $\mathcal{P}$ is a permutation map. (symmetry).

Remark 2.3: It should be observed our idea of $\mathcal{D}^*$-symmetric space is the same as that of $\mathcal{D}^*$-metric space Definition 2.1 without $(\mathcal{D}^*_4)$ (The rectangle inequality property).

Example 2.4: Let $\mathcal{X} = [0,1]$ equipped with $\mathcal{D}^*$-symmetric defined as $\mathcal{D}^*_d(x, y, z) = (x - y)^2 + (y - z)^2 + (z - x)^2$ for all $x, y, z \in \mathcal{X}$. Then $(\mathcal{D}^*_d, \mathcal{X})$ is $\mathcal{D}^*$-symmetric. This doesn't satisfy $(\mathcal{D}^*_4)$ property of $\mathcal{D}^*$-metric, for this reason it isn't $\mathcal{D}^*$-metric space.
The analogue of axioms of Wilson [19] in $\mathcal{D}^*$-symmetric space is as follows:
(W_3) Given $\{x_s\}, x, y \in \mathcal{X}$, $\mathcal{D}^*_d(x_s, x, x) \longrightarrow 0$ and $\mathcal{D}^*_d(x_s, y, y) \longrightarrow 0$ imply that $x = y$.
(W_4) Given $\{x_s\}, \{y_s\}$ $x, y \in \mathcal{X}$, $\mathcal{D}^*_d(x_s, x, x) \longrightarrow 0$ and $\mathcal{D}^*_d(x_s, y_s, y_s) \longrightarrow 0$ imply that $\mathcal{D}^*_d(y_s, x, x) \longrightarrow 0$.

Definition 2.5: Let $(\mathcal{X}, \mathcal{D}^*_d)$ be a $\mathcal{D}^*$-symmetric then:
$(\boldsymbol{i})$ $(\mathcal{X}, \mathcal{D}^*_d)$ is $\mathcal{D}^*_d$-complete if $\forall\, \mathcal{D}^*_d$-Cauchy-sequence $\{x_s\}, \exists\, x \in \mathcal{X}$ with $\lim\limits_{s\to\infty} \mathcal{D}^*_d(x_s, x, x) = 0$.
$(\boldsymbol{ii})$ $\mathcal{F}: \mathcal{X} \longrightarrow \mathcal{X}$ is $\mathcal{D}^*_d$-continuous
if $\lim\limits_{s\to\infty} \mathcal{D}^*_d(x_s, x, x) = 0 \Longrightarrow \lim\limits_{s\to\infty} \mathcal{D}^*_d(\mathcal{F}x_s, \mathcal{F}x, \mathcal{F}x) = 0$.

Definition 2.6: [20] Let $\mathcal{F}, \mathcal{G}: (\mathcal{X}, \mathcal{D}^*) \longrightarrow (\mathcal{X}^*, \mathcal{D}^{**})$ be self-maps . If $\mathcal{F}(x) = \mathcal{G}(x) = r$ for some x in $\mathcal{X}$, so x called coincidence of $\mathcal{F}$ and $\mathcal{G}$, and r the point of coincidence of $\mathcal{F}$ and $\mathcal{G}$.

Definition 2.7: A pair $(\mathcal{F}, \mathcal{G})$ of self-maps of $\mathcal{D}^*$-symmetric $(\mathcal{X}, \mathcal{D}^*_d)$ is called weakly compatible if they are commute by their coincidence points, (i. e.) if $\mathcal{F}x = \mathcal{G}x$, for some $x \in \mathcal{X}$, so $\mathcal{F}\mathcal{G}x = \mathcal{G}\mathcal{F}x$.

Definition 2.8: [21] A self-maps $\mathcal{F}$ and $\mathcal{G}$ of a set $\mathcal{X}$ are occasionally weakly compatible (briefly. OWC) if and only if $\exists\, x \in \mathcal{X}$ which is coincidence of $\mathcal{F}$ and $\mathcal{G}$ at which $\mathcal{F}$ and $\mathcal{G}$ are commute.

Definition 2.9: [5] Let $\mathcal{F}, \mathcal{G}: (\mathcal{X}, \mathcal{D}^*) \rightarrow: (\mathcal{X}, \mathcal{D}^*)$be self-maps on $\mathcal{X}$. If $\mathcal{F}(x) = \mathcal{G}(x) = x$ for some $x \in \mathcal{X}$, then x is called common fixed point (CFP) of $\mathcal{F}$ and $\mathcal{G}$.

Lemma 2.10: [21] Let $\mathcal{F}$ and $\mathcal{G}$ be (OWC) self-maps of $\mathcal{X}$. If $\mathcal{F}$ and $\mathcal{G}$ have unique point of coincidence $\mathcal{F}(x) = \mathcal{G}(x) = r$, so r is unique (CFP) of $\mathcal{F}$ and $\mathcal{G}$.

Definition 2.11: A self-maps $\mathcal{F}$ and $\mathcal{G}$ in $\mathcal{D}^*$-symmetric $(\mathcal{X}, \mathcal{D}^*_d)$ is satisfy the (E. A.) property if $\exists\, \{x_s\}$ such that $\lim\limits_{s\to\infty} \mathcal{F}\, x_s = \lim\limits_{s\to\infty} \mathcal{G}\, x_s = t$ for some $t \in \mathcal{X}$.

Definition 2.12: [22] Let $(\mathcal{X}, d)$ be a complete metric space. A map $\mathcal{F}: \mathcal{X} \longrightarrow \mathcal{X}$ is called generalized contractive map, if $\forall\, x, y \in \mathcal{X}$,

$$d(\mathcal{F}x, \mathcal{F}y) \leq \psi_1(x, y) d(x, y) + \psi_2(x, y) d(x, \mathcal{F}x) + \psi_3(x, y) d(y, \mathcal{F}y) + \psi_4(x, y)[d(x, \mathcal{F}y) + d(y, \mathcal{F}x)].$$

Wherever, ψ_1, ψ_2, ψ_3 and ψ_4 are maps from $\mathcal{X} \times \mathcal{X}$ into [0,1) such that

$$\lambda = sup\{\psi_1(x, y) + \psi_2(x, y) + \psi_3(x, y) + \psi_4(x, y); x, y \in \mathcal{X}\} < 1.$$

3. Main Results of Uniqueness of Common Fixed Points in $\mathcal{D}^*$-Symmetric Space

Our motivation in this section is to verifying the uniqueness of common fixed points of two pair of (OWC) maps satisfying the contractive conditions in $\mathcal{D}^*$-symmetric space utilizing the concept of the (E. A.) Property.

Theorem 3.1: Let $(\mathcal{X},\mathcal{D}_d^*)$ be a $\mathcal{D}^*$-symmetric space that satisfies (W_3) and $\mathcal{F},\mathcal{G}:(\mathcal{X},\mathcal{D}_d^*)\longrightarrow(\mathcal{X},\mathcal{D}_d^*)$ two self-maps of $\mathcal{X}$ such that:
$(\boldsymbol{i})$ $\mathcal{F}$ and $\mathcal{G}$ are satisfying (E. A) Property. $(\boldsymbol{ii})$ $\forall\, x\neq y\in\mathcal{X}$. If

$$\mathcal{D}_d^*(\mathcal{G}x,\mathcal{G}y,\mathcal{G}y)\le a_1\mathcal{D}_d^*(\mathcal{F}x,\mathcal{F}y,\mathcal{F}y)+a_2\mathcal{D}_d^*(\mathcal{G}x,\mathcal{G}x,\mathcal{F}x)+a_3\mathcal{D}_d^*(\mathcal{G}y,\mathcal{G}y,\mathcal{F}y)+ a_4\mathcal{D}_d^*(\mathcal{G}x,\mathcal{G}x,\mathcal{F}y)+a_5\mathcal{D}_d^*(\mathcal{G}y,\mathcal{G}y,\mathcal{F}x). \quad (1)$$

and

$$\mathcal{D}_d^*(\mathcal{G}x,\mathcal{G}x,\mathcal{G}y)\le a_1\mathcal{D}_d^*(\mathcal{F}x,\mathcal{F}x,\mathcal{F}y)+a_2\mathcal{D}_d^*(\mathcal{G}x,\mathcal{G}x,\mathcal{F}x)+a_3\mathcal{D}_d^*(\mathcal{G}y,\mathcal{G}y,\mathcal{F}y)+ a_4\mathcal{D}_d^*(\mathcal{G}x,\mathcal{G}x,\mathcal{F}y)+a_5\mathcal{D}_d^*(\mathcal{G}y,\mathcal{G}y,\mathcal{F}x). \quad (2)$$

$a_1,a_2,a_3,a_4,a_5\in[0,1)$, $a_1+a_4+a_5<1$ and $a_1+a_5<1$. Assume is $\mathcal{F}(\mathcal{X})$ $\mathcal{D}_d^*$ -closed set of $\mathcal{X}$ with $\mathcal{G}(\mathcal{X})\subset\mathcal{F}(\mathcal{X})$. If $\mathcal{F}$ and $\mathcal{G}$ are (OWC), then $\mathcal{F}$ and $\mathcal{G}$ have unique (C FP).

Proof: Since, $\mathcal{F}$ and $\mathcal{G}$ are satisfying the property of (E. A.), then there exists $\{x_s\}$ so $\lim_{s\to\infty}\mathcal{F}\,x_s=\lim_{s\to\infty}\mathcal{G}\,x_s=t$ for some $t\in\mathcal{X}$. As well, $\mathcal{F}(\mathcal{X})$ is closed implies $\exists\,p\in\mathcal{X}$ such that $\lim_{s\to\infty}\mathcal{F}\,x_s=\mathcal{F}(p)$, implies $t=\mathcal{F}(p)\in\mathcal{F}(\mathcal{X})$ by (W_3). Verify $\mathcal{F}(p)=\mathcal{G}(p)$. On the contrary and using (1) we get,

$$\mathcal{D}_d^*(\mathcal{G}x_s,\mathcal{G}p,\mathcal{G}p)\le a_1\mathcal{D}_d^*(\mathcal{F}x_s,\mathcal{F}p,\mathcal{F}p)+a_2\mathcal{D}_d^*(\mathcal{G}x_s,\mathcal{F}x_s,\mathcal{F}x_s)+a_3\mathcal{D}_d^*(\mathcal{G}p,\mathcal{F}p,\mathcal{F}p) +a_4\mathcal{D}_d^*(\mathcal{G}x_s,\mathcal{F}p,\mathcal{F}p)+a_5\mathcal{D}_d^*(\mathcal{G}p,\mathcal{F}x_s,\mathcal{F}x_s).$$

Letting $s\longrightarrow\infty$ produces

$$\mathcal{D}_d^*(\mathcal{F}p,\mathcal{G}p,\mathcal{G}p)\le a_1\mathcal{D}_d^*(\mathcal{F}p,\mathcal{F}p,\mathcal{F}p)+a_2\mathcal{D}_d^*(\mathcal{F}p,\mathcal{F}p,\mathcal{F}p)+a_3\mathcal{D}_d^*(\mathcal{G}p,\mathcal{F}p,\mathcal{F}p) +a_4\mathcal{D}_d^*(\mathcal{F}p,\mathcal{F}p,\mathcal{F}p)+a_5\mathcal{D}_d^*(\mathcal{G}p,\mathcal{F}p,\mathcal{F}p)\le(a_3+a_5)\mathcal{D}_d^*(\mathcal{G}p,\mathcal{F}p,\mathcal{F}p) \le k\mathcal{D}_d^*(\mathcal{G}p,\mathcal{F}p,\mathcal{F}p),\ \text{such that } k=a_3+a_5<1. \quad (3)$$

In the same way, utilizing (2) gives

$$\mathcal{D}_d^*(\mathcal{G}x_s,\mathcal{G}x_s,\mathcal{G}p)\le a_1\mathcal{D}_d^*(\mathcal{F}x_s,\mathcal{F}x_s,\mathcal{F}p)+a_2\mathcal{D}_d^*(\mathcal{G}x_s,\mathcal{G}x_s,\mathcal{F}x_s)+a_3\mathcal{D}_d^*(\mathcal{G}p,\mathcal{G}p,\mathcal{F}p) +a_4\mathcal{D}_d^*(\mathcal{G}x_s,\mathcal{G}x_s,\mathcal{F}p)+a_5\mathcal{D}_d^*(\mathcal{G}p,\mathcal{G}p,\mathcal{F}x_s).$$

Letting $s\longrightarrow\infty$ produces

$$\begin{aligned}\mathcal{D}_d^*(\mathcal{F}p,\mathcal{F}p,\mathcal{G}p)&\le a_1\mathcal{D}_d^*(\mathcal{F}p,\mathcal{F}p,\mathcal{F}p)+a_2\mathcal{D}_d^*(\mathcal{F}p,\mathcal{F}p,\mathcal{F}p)+a_3\mathcal{D}_d^*(\mathcal{G}p,\mathcal{G}p,\mathcal{F}p)\\ &\quad+a_4\mathcal{D}_d^*(\mathcal{F}p,\mathcal{F}p,\mathcal{F}p)+a_5\mathcal{D}_d^*(\mathcal{G}p,\mathcal{G}p,\mathcal{F}p)\\ &\le(a_3+a_5)\mathcal{D}_d^*(\mathcal{G}p,\mathcal{G}p,\mathcal{F}p)\\ &\le k\mathcal{D}_d^*(\mathcal{G}p,\mathcal{G}p,\mathcal{F}p),\ \text{such that } k=a_3+a_5<1. \quad (4)\end{aligned}$$

Merging, (6), (7) and via $(\mathcal{D}_{d_3}^*)$, we find

$$\mathcal{D}_d^*(\mathcal{F}p,\mathcal{G}p,\mathcal{G}p)\le k^2\mathcal{D}_d^*(\mathcal{F}p,\mathcal{G}p,\mathcal{G}p).$$

Since, $k<1$, then $\mathcal{F}(p)=\mathcal{G}(p)$. Therefore, $w=\mathcal{F}(p)=\mathcal{G}(p)$ where w is point of coincidence of $\mathcal{F}$ and $\mathcal{G}$, and p is coincidence point of $\mathcal{F}$ and $\mathcal{G}$. If there exists $q\in\mathcal{X}$ such that $\mathcal{F}q=\mathcal{G}q$. We verify that $\mathcal{G}p=\mathcal{G}q$. On the contrary and utilizing (1) we obtain

$$\mathcal{D}_d^*(\mathcal{G}x_s,\mathcal{G}q,\mathcal{G}q)\le a_1\mathcal{D}_d^*(\mathcal{F}x_s,\mathcal{F}q,\mathcal{F}q)+a_2\mathcal{D}_d^*(\mathcal{G}x_s,\mathcal{F}x_s,\mathcal{F}x_s)+a_3\mathcal{D}_d^*(\mathcal{G}q,\mathcal{F}q,\mathcal{F}q) +a_4\mathcal{D}_d^*(\mathcal{G}x_s,\mathcal{F}q,\mathcal{F}q)+a_5\mathcal{D}_d^*(\mathcal{G}q,\mathcal{F}x_s,\mathcal{F}x_s).$$

Letting $s\longrightarrow\infty$ produces

$$\begin{aligned}\mathcal{D}_d^*(\mathcal{G}p,\mathcal{G}q,\mathcal{G}q)&\le a_1\mathcal{D}_d^*(\mathcal{G}p,\mathcal{F}q,\mathcal{F}q)+a_2\mathcal{D}_d^*(\mathcal{G}p,\mathcal{G}p,\mathcal{G}p)+a_3\mathcal{D}_d^*(\mathcal{G}q,\mathcal{F}q,\mathcal{F}q)\\ &\quad+a_4\mathcal{D}_d^*(\mathcal{G}p,\mathcal{F}q,\mathcal{F}q)+a_5\mathcal{D}_d^*(\mathcal{G}q,\mathcal{G}p,\mathcal{G}p)\\ &\le(a_1+a_4)\mathcal{D}_d^*(\mathcal{G}p,\mathcal{G}q,\mathcal{G}q)+a_5\mathcal{D}_d^*(\mathcal{G}q,\mathcal{G}p,\mathcal{G}p).\end{aligned}$$

Thus, $\mathcal{D}_d^*(\mathcal{G}p,\mathcal{G}q,\mathcal{G}q) \leq \frac{a_5}{1-(a_1+a_4)}\mathcal{D}_d^*(\mathcal{G}q,\mathcal{G}p,\mathcal{G}p)$
$\leq k\mathcal{D}_d^*(\mathcal{G}q,\mathcal{G}p,\mathcal{G}p)$, where $k = \frac{a_5}{1-(a_1+a_4)} < 1$. (5)

In the same way, utilizing (2) we obtain,

$$\mathcal{D}_d^*(\mathcal{G}x_s,\mathcal{G}x_s,\mathcal{G}q) \leq a_1\mathcal{D}_d^*(\mathcal{F}x_s,\mathcal{F}x_s,\mathcal{F}q) + a_2\mathcal{D}_d^*(\mathcal{G}x_s,\mathcal{G}x_s,\mathcal{F}x_s) + a_3\mathcal{D}_d^*(\mathcal{G}q,\mathcal{G}q,\mathcal{F}q) + a_4\mathcal{D}_d^*(\mathcal{G}x_s,\mathcal{G}x_s,\mathcal{F}q) + a_5\mathcal{D}_d^*(\mathcal{G}q,\mathcal{G}q,\mathcal{F}x_s).$$

Letting $s \longrightarrow \infty$ produces

$$\mathcal{D}_d^*(\mathcal{G}p,\mathcal{G}p,\mathcal{G}q) \leq a_1\mathcal{D}_d^*(\mathcal{G}p,\mathcal{G}p,\mathcal{F}q) + a_2\mathcal{D}_d^*(\mathcal{G}p,\mathcal{G}p,\mathcal{G}p) + a_3\mathcal{D}_d^*(\mathcal{G}q,\mathcal{G}q,\mathcal{F}q) + a_4\mathcal{D}_d^*(\mathcal{G}p,\mathcal{G}p,\mathcal{F}q) + a_5\mathcal{D}_d^*(\mathcal{G}q,\mathcal{G}q,\mathcal{G}p)$$
$$\leq (a_1 + a_4)\mathcal{D}_d^*(\mathcal{G}p,\mathcal{G}p,\mathcal{G}q) + a_5\mathcal{D}_d^*(\mathcal{G}q,\mathcal{G}q,\mathcal{G}p).$$

Thus, $\mathcal{D}_d^*(\mathcal{G}p,\mathcal{G}p,\mathcal{G}q) \leq \frac{a_5}{1-(a_1+a_4)}\mathcal{D}_d^*(\mathcal{G}q,\mathcal{G}q,\mathcal{G}p)$
$\leq k\mathcal{D}_d^*(\mathcal{G}q,\mathcal{G}q,\mathcal{G}p)$, where $k = \frac{a_5}{1-(a_1+a_4)} < 1$. (6)

Merging (3), (4) and via $(\mathcal{D}_{d_3}^*)$, we get,

$$\mathcal{D}_d^*(\mathcal{G}p,\mathcal{G}q,\mathcal{G}q) \leq k^2\mathcal{D}_d^*(\mathcal{G}p,\mathcal{G}q,\mathcal{G}q).$$

Since, $k < 1$ then it is a contradiction. Consequently, $\mathcal{G}p = \mathcal{G}q$. So, $\mathcal{F}p = \mathcal{G}p = \mathcal{G}q = \mathcal{F}q$. Therefore, w is the unique point of the coincidence of $\mathcal{F}$ and $\mathcal{G}$, by lemma 2.10, w is unique (CFP) of $\mathcal{F}$ and $\mathcal{G}$.

Corollary 3.2: Let $(\mathcal{X},\mathcal{D}_d^*)$ be $\mathcal{D}^*$-symmetric satisfies (W_3), and $\mathcal{F},\mathcal{G}: (\mathcal{X},\mathcal{D}_d^*) \longrightarrow (\mathcal{X},\mathcal{D}_d^*)$ two self-maps of $\mathcal{X}$ such that:
$(\boldsymbol{i})$ $\mathcal{F}$ and $\mathcal{G}$ are satisfying (E.A) property. $(\boldsymbol{ii})$ $\forall\ x \neq y \in \mathcal{X}$.
If $\mathcal{D}_d^*(\mathcal{G}x,\mathcal{G}y,\mathcal{G}y) \leq k\mathcal{D}_d^*(\mathcal{F}x,\mathcal{F}y,\mathcal{F}y)$, (7)
and $\mathcal{D}_d^*(\mathcal{G}x,\mathcal{G}x,\mathcal{G}y) \leq k\mathcal{D}_d^*(\mathcal{F}x,\mathcal{F}x,\mathcal{F}y)$ (8)
$k < 1$. Assume $\mathcal{F}(\mathcal{X})$ is $\mathcal{D}_d^*$ -closed of $\mathcal{X}$ with $\mathcal{G}(\mathcal{X}) \subset \mathcal{F}(\mathcal{X})$. If $\mathcal{F}$ and $\mathcal{G}$ are (OWC), then $\mathcal{F}$ and $\mathcal{G}$ have unique (CFP).

Proof: The proof follows immediately from Theorem 3.1.

Theorem 3.3: Let $(\mathcal{X},\mathcal{D}_d^*)$ be a $\mathcal{D}^*$-symmetric space that satisfies (W_3), and $\mathcal{F},\mathcal{G}: (\mathcal{X},\mathcal{D}_d^*) \longrightarrow (\mathcal{X},\mathcal{D}_d^*)$ two self-maps of $\mathcal{X}$ such that:
$(\boldsymbol{i})$ $\mathcal{F}$ and $\mathcal{G}$ are satisfying (E. A) property.
$(\boldsymbol{ii})$ $\forall\ x \neq y \in \mathcal{X}$.
If $\mathcal{D}_d^*(\mathcal{G}x,\mathcal{G}y,\mathcal{G}y) \leq \psi_1(x,y,y)\mathcal{D}_d^*(\mathcal{F}x,\mathcal{F}y,\mathcal{F}y) + \psi_2(x,y,y)\mathcal{D}_d^*(\mathcal{G}x,\mathcal{G}x,\mathcal{F}x) + \psi_3(x,y,y)\mathcal{D}_d^*(\mathcal{G}y,\mathcal{G}y,\mathcal{F}y) + \psi_4(x,y,y)[\mathcal{D}_d^*(\mathcal{G}x,\mathcal{G}x,\mathcal{F}y) + \mathcal{D}_d^*(\mathcal{G}y,\mathcal{G}y,\mathcal{F}x)]$, (9)
and
$\mathcal{D}_d^*(\mathcal{G}x,\mathcal{G}x,\mathcal{G}y) \leq \psi_1(x,x,y)\mathcal{D}_d^*(\mathcal{F}x,\mathcal{F}x,\mathcal{F}y) + \psi_2(x,x,y)\mathcal{D}_d^*(\mathcal{G}x,\mathcal{G}x,\mathcal{F}x) + \psi_3(x,x,y)\mathcal{D}_d^*(\mathcal{G}y,\mathcal{G}y,\mathcal{F}y) + \psi_4(x,x,y)[\mathcal{D}_d^*(\mathcal{G}x,\mathcal{G}x,\mathcal{F}y) + \mathcal{D}_d^*(\mathcal{G}y,\mathcal{G}y,\mathcal{F}x)]$, (10)
where, $\psi_1,\psi_2,\psi_3,\psi_4$ are maps from $\mathcal{X} \times \mathcal{X} \times \mathcal{X}$ into [0,1), such that:

$$\lambda = sup\{\psi_1(x,y,y) + \psi_2(x,y,y) + \psi_3(x,y,y) + \psi_4(x,y,y)\} < 1.$$

Assume $\mathcal{F}(\mathcal{X})$ is $\mathcal{D}_d^*$ -closed of $\mathcal{X}$ with $\mathcal{G}(\mathcal{X}) \subset \mathcal{F}(\mathcal{X})$. If $\mathcal{F}$ and $\mathcal{G}$ are (OWC), then $\mathcal{F}$ and $\mathcal{G}$ have unique (CFP).

Proof: Let $\mathcal{F}$ and $\mathcal{G}$ are satisfying the property of (E. A.), then $\exists\ \{x_s\}$, such that $\lim_{s\to\infty} \mathcal{F}\, x_s = \lim_{s\to\infty} \mathcal{G}\, x_s = t$ for some $t \in \mathcal{X}$. As well, $\mathcal{F}(\mathcal{X})$ closed implies $\exists\ p \in \mathcal{X}$ such that, $\lim_{s\to\infty} \mathcal{F}\, x_s = \mathcal{F}(p)$. This, produces $t = \mathcal{F}(p) \in \mathcal{F}(\mathcal{X})$ via (W_3). Now, verify $\mathcal{F}(p) = \mathcal{G}(p)$. On the contrary and utilizing (12), we obtain,

$\mathcal{D}_d^*(\mathcal{G}x_s, \mathcal{G}p, \mathcal{G}p) \leq \psi_1(x_s, p, p)\mathcal{D}_d^*(\mathcal{F}x_s, \mathcal{F}p, \mathcal{F}p) + \psi_2(x_s, p, p)\mathcal{D}_d^*(\mathcal{G}x_s, \mathcal{F}x_s, \mathcal{F}x_s) + \psi_3(x_s, p, p)\mathcal{D}_d^*(\mathcal{G}p, \mathcal{F}p, \mathcal{F}p) + \psi_4(x, p, p)[\mathcal{D}_d^*(\mathcal{G}x_s, \mathcal{F}p, \mathcal{F}p) + \mathcal{D}_d^*(\mathcal{G}p, \mathcal{F}x_s, \mathcal{F}x_s)]$.

Evaluating $\psi_1, \psi_2, \psi_3, \psi_4$ at (x_s, p, p) produces,

$$\begin{aligned}\mathcal{D}_d^*(\mathcal{G}x_s, \mathcal{G}p, \mathcal{G}p) &\leq \psi_1\mathcal{D}_d^*(\mathcal{F}x_s, \mathcal{F}p, \mathcal{F}p) + \psi_2\mathcal{D}_d^*(\mathcal{G}x_s, \mathcal{F}x_s, \mathcal{F}x_s) + \psi_3\mathcal{D}_d^*(\mathcal{G}p, \mathcal{F}p, \mathcal{F}p) \\ &\quad + \psi_4[\mathcal{D}_d^*(\mathcal{G}x_s, \mathcal{F}p, \mathcal{F}p) + \mathcal{D}_d^*(\mathcal{G}p, \mathcal{F}x_s, \mathcal{F}x_s)]\end{aligned}$$

as $s \longrightarrow \infty$ produces

$$\begin{aligned}\mathcal{D}_d^*(\mathcal{F}p, \mathcal{G}p, \mathcal{G}p) &\leq \psi_1\mathcal{D}_d^*(\mathcal{F}p, \mathcal{F}p, \mathcal{F}p) + \psi_2\mathcal{D}_d^*(\mathcal{F}p, \mathcal{F}p, \mathcal{F}p) + \psi_3\mathcal{D}_d^*(\mathcal{G}p, \mathcal{F}p, \mathcal{F}p) \\ &\quad + \psi_4[\mathcal{D}_d^*(\mathcal{F}p, \mathcal{F}p, \mathcal{F}p) + \mathcal{D}_d^*(\mathcal{G}p, \mathcal{F}p, \mathcal{F}p)] \\ &\leq (\psi_3 + \psi_4)\, \mathcal{D}_d^*(\mathcal{G}p, \mathcal{F}p, \mathcal{F}p) \\ &\leq \lambda\, \mathcal{D}_d^*(\mathcal{G}p, \mathcal{F}p, \mathcal{F}p). \qquad (11)\end{aligned}$$

In the same way, utilizing (10) provides

$$\begin{aligned}\mathcal{D}_d^*(\mathcal{G}x_s, \mathcal{G}x_s, \mathcal{G}p) &\leq \psi_1(x_s, x_s, p)\mathcal{D}_d^*(\mathcal{F}x_s, \mathcal{F}x_s, \mathcal{F}p) + \psi_2(x_s, x_s, p)\mathcal{D}_d^*(\mathcal{G}x_s, \mathcal{G}x_s, \mathcal{F}x_s) \\ &\quad + \psi_3(x_s, x_s, p)\mathcal{D}_d^*(\mathcal{G}p, \mathcal{G}p, \mathcal{F}p) \\ &\quad + \psi_4(x, x_s, p)[\mathcal{D}_d^*(\mathcal{G}x_s, \mathcal{G}x_s, \mathcal{F}p) + \mathcal{D}_d^*(\mathcal{G}p, \mathcal{G}p, \mathcal{F}x_s)].\end{aligned}$$

Evaluating $\psi_1, \psi_2, \psi_3, \psi_4$ at (x_s, x_s, p) produces,

$$\begin{aligned}\mathcal{D}_d^*(\mathcal{G}x_s, \mathcal{G}x_s, \mathcal{G}p) &\leq \psi_1\mathcal{D}_d^*(\mathcal{F}x_s, \mathcal{F}x_s, \mathcal{F}p) + \psi_2\mathcal{D}_d^*(\mathcal{G}x_s, \mathcal{G}x_s, \mathcal{F}x_s) + \psi_3\mathcal{D}_d^*(\mathcal{G}p, \mathcal{G}p, \mathcal{F}p) \\ &\quad + \psi_4[\mathcal{D}_d^*(\mathcal{G}x_s, \mathcal{G}x_s, \mathcal{F}p) + \mathcal{D}_d^*(\mathcal{G}p, \mathcal{G}p, \mathcal{F}x_s)]\end{aligned}$$

as $s \longrightarrow \infty$ produces

$$\begin{aligned}\mathcal{D}_d^*(\mathcal{F}p, \mathcal{F}p, \mathcal{G}p) &\leq \psi_1\mathcal{D}_d^*(\mathcal{F}p, \mathcal{F}p, \mathcal{F}p) + \psi_2\mathcal{D}_d^*(\mathcal{F}p, \mathcal{F}p, \mathcal{F}p) + \psi_3\mathcal{D}_d^*(\mathcal{G}p, \mathcal{G}p, \mathcal{F}p) \\ &\quad + \psi_4[\mathcal{D}_d^*(\mathcal{F}p, \mathcal{F}p, \mathcal{F}p) + \mathcal{D}_d^*(\mathcal{G}p, \mathcal{G}p, \mathcal{F}p)] \\ &\leq (\psi_3 + \psi_4)\, \mathcal{D}_d^*(\mathcal{G}p, \mathcal{G}p, \mathcal{F}p) \\ &\leq \lambda\, \mathcal{D}_d^*(\mathcal{G}p, \mathcal{G}p, \mathcal{F}p). \qquad (12)\end{aligned}$$

Merging (11), (12) and via $(\mathcal{D}_{d_3}^*)$, we get

$$\mathcal{D}_d^*(\mathcal{F}p, \mathcal{G}p, \mathcal{G}p) \leq k^2\mathcal{D}_d^*(\mathcal{F}p, \mathcal{G}p, \mathcal{G}p).$$

Since, $k < 1$ then it is contradiction. Thus, $\mathcal{F}p = \mathcal{G}p$. Consequently, $w = \mathcal{F}p = \mathcal{G}p$, where w is the point of coincidence of $\mathcal{F}$ and $\mathcal{G}$ and p is coincidence point of $\mathcal{F}$ and $\mathcal{G}$. If there exists $q \in \mathcal{X}$, such that $\mathcal{F}q = \mathcal{G}q$. Assume $\mathcal{G}p \neq \mathcal{G}q$ so utilizing (9), we obtain

$$\begin{aligned}\mathcal{D}_d^*(\mathcal{G}x_s, \mathcal{G}q, \mathcal{G}q) &\leq \psi_1(x_s, q, q)\mathcal{D}_d^*(\mathcal{F}x_s, \mathcal{F}q, \mathcal{F}q) + \psi_2(x_s, q, q)\mathcal{D}_d^*(\mathcal{G}x_s, \mathcal{F}x_s, \mathcal{F}x_s) \\ &\quad + \psi_3(x_s, q, q)\mathcal{D}_d^*(\mathcal{G}q, \mathcal{F}q, \mathcal{F}q) \\ &\quad + \psi_4(x_s, q, q)[\mathcal{D}_d^*(\mathcal{G}x_s, \mathcal{F}q, \mathcal{F}q) + \mathcal{D}_d^*(\mathcal{G}q, \mathcal{F}x_s, \mathcal{F}x_s)].\end{aligned}$$

Evaluating $\psi_1, \psi_2, \psi_3, \psi_4$ at (x_s, q, q) produces,

$\mathcal{D}_d^*(\mathcal{G}x_s, \mathcal{G}q, \mathcal{G}q) \leq \psi_1\mathcal{D}_d^*(\mathcal{F}x_s, \mathcal{F}q, \mathcal{F}q) + \psi_2\mathcal{D}_d^*(\mathcal{G}x_s, \mathcal{F}x_s, \mathcal{F}x_s) + \psi_3\mathcal{D}_d^*(\mathcal{G}q, \mathcal{F}q, \mathcal{F}q) + \psi_4[\mathcal{D}_d^*(\mathcal{G}x_s, \mathcal{F}q, \mathcal{F}q) + \mathcal{D}_d^*(\mathcal{G}q, \mathcal{F}x_s, \mathcal{F}x_s)]$,

as $s \longrightarrow \infty$ produces

$$\begin{aligned}\mathcal{D}_d^*(\mathcal{G}p, \mathcal{G}q, \mathcal{G}q) &\leq \psi_1(\mathcal{D}_d^*(\mathcal{G}p, \mathcal{G}q, \mathcal{G}q) + \psi_2\mathcal{D}_d^*(\mathcal{G}p, \mathcal{G}p, \mathcal{G}p) + \psi_3\mathcal{D}_d^*(\mathcal{G}q, \mathcal{G}q, \mathcal{G}q) \\ &\quad + \psi_4[\mathcal{D}_d^*(\mathcal{G}p, \mathcal{G}q, \mathcal{G}q) + \mathcal{D}_d^*(\mathcal{G}q, \mathcal{G}p, \mathcal{G}p)] \\ &\leq (\psi_1 + \psi_4)\mathcal{D}_d^*(\mathcal{G}p, \mathcal{G}q, \mathcal{G}q) + \psi_4\mathcal{D}_d^*(\mathcal{G}q, \mathcal{G}p, \mathcal{G}p) \\ &\leq (\psi_1 + 2\psi_4) max\{\mathcal{D}_d^*(\mathcal{G}p, \mathcal{G}q, \mathcal{G}q), \mathcal{D}_d^*(\mathcal{G}q, \mathcal{G}p, \mathcal{G}p)\} \\ &\leq \lambda\, max\{\mathcal{D}_d^*(\mathcal{G}p, \mathcal{G}q, \mathcal{G}q), \mathcal{D}_d^*(\mathcal{G}q, \mathcal{G}p, \mathcal{G}p)\}.\end{aligned}$$

We have two cases shown as follows:

Case(i): If $max\{\mathcal{D}_d^*(\mathcal{G}p, \mathcal{G}q, \mathcal{G}q), \mathcal{D}_d^*(\mathcal{G}q, \mathcal{G}p, \mathcal{G}p)\} = \mathcal{D}_d^*(\mathcal{G}p, \mathcal{G}q, \mathcal{G}q)$ then

$$\mathcal{D}_d^*(\mathcal{G}p, \mathcal{G}q, \mathcal{G}q) \leq \lambda\mathcal{D}_d^*(\mathcal{G}p, \mathcal{G}q, \mathcal{G}q).$$

Case(ii): If $max\{\mathcal{D}_d^*(\mathcal{G}p, \mathcal{G}q, \mathcal{G}q), \mathcal{D}_d^*(\mathcal{G}q, \mathcal{G}p, \mathcal{G}p)\} = \mathcal{D}_d^*(\mathcal{G}p, \mathcal{G}q, \mathcal{G}q)$ then

$$\mathcal{D}_d^*(\mathcal{G}p, \mathcal{G}q, \mathcal{G}q) \leq \lambda \mathcal{D}_d^*(\mathcal{G}q, \mathcal{G}p, \mathcal{G}p). \quad (13)$$

Utilizing (10) produces

$$\begin{aligned}\mathcal{D}_d^*(\mathcal{G}x_s, \mathcal{G}x_s, \mathcal{G}q) &\leq \psi_1(x_s, x_s, q)\mathcal{D}_d^*(\mathcal{F}x_s, \mathcal{F}x_s, \mathcal{F}q) + \psi_2(x_s, x_s, q)\mathcal{D}_d^*(\mathcal{G}x_s, \mathcal{G}x_s, \mathcal{F}x_s)\\ &+ \psi_3(x_s, x_s, q)\mathcal{D}_d^*(\mathcal{G}q, \mathcal{G}q, \mathcal{F}q)\\ &+ \psi_4(x_s, x_s, q)[\mathcal{D}_d^*(\mathcal{G}x_s, \mathcal{G}x_s, \mathcal{F}q) + \mathcal{D}_d^*(\mathcal{G}q, \mathcal{G}q, \mathcal{F}x_s)].\end{aligned}$$

Evaluating $\psi_1, \psi_2, \psi_3, \psi_4$ at (x_s, x_s, q) produces,
$\mathcal{D}_d^*(\mathcal{G}x_s, \mathcal{G}x_s, \mathcal{G}q) \leq \psi_1\mathcal{D}_d^*(\mathcal{F}x_s, \mathcal{F}x_s, \mathcal{F}q) + \psi_2\mathcal{D}_d^*(\mathcal{G}x_s, \mathcal{G}x_s, \mathcal{F}x_s) + \psi_3\mathcal{D}_d^*(\mathcal{G}q, \mathcal{G}q, \mathcal{F}q) + \psi_4[\mathcal{D}_d^*(\mathcal{G}x_s, \mathcal{G}x_s, \mathcal{F}q) + \mathcal{D}_d^*(\mathcal{G}q, \mathcal{G}q, \mathcal{F}x_s)]$,
as $s \longrightarrow \infty$ produces

$$\begin{aligned}\mathcal{D}_d^*(\mathcal{G}p, \mathcal{G}p, \mathcal{G}q) &\leq \psi_1\mathcal{D}_d^*(\mathcal{G}p, \mathcal{G}p, \mathcal{G}q) + \psi_2\mathcal{D}_d^*(\mathcal{G}p, \mathcal{G}p, \mathcal{G}p) + \psi_3\mathcal{D}_d^*(\mathcal{G}q, \mathcal{G}q, \mathcal{G}q)\\ &+ \psi_4[\mathcal{D}_d^*(\mathcal{G}p, \mathcal{G}p, \mathcal{G}q) + \mathcal{D}_d^*(\mathcal{G}q, \mathcal{G}q, \mathcal{G}p)]\\ &\leq (\psi_1 + \psi_4)\mathcal{D}_d^*(\mathcal{G}p, \mathcal{G}p, \mathcal{G}q) + \psi_4\mathcal{D}_d^*(\mathcal{G}q, \mathcal{G}q, \mathcal{G}p)\\ &\leq (\psi_1 + 2\psi_4)\, max\{\mathcal{D}_d^*(\mathcal{G}p, \mathcal{G}p, \mathcal{G}q), \mathcal{D}_d^*(\mathcal{G}q, \mathcal{G}q, \mathcal{G}p)\}\\ &\leq \lambda\, max\{\mathcal{D}_d^*(\mathcal{G}p, \mathcal{G}p, \mathcal{G}q), \mathcal{D}_d^*(\mathcal{G}q, \mathcal{G}q, \mathcal{G}p)\}.\end{aligned}$$

We have two cases shown as follows:
Case(i) If $max\{\mathcal{D}_d^*(\mathcal{G}p, \mathcal{G}p, \mathcal{G}q), \mathcal{D}_d^*(\mathcal{G}q, \mathcal{G}q, \mathcal{G}p)\} = \mathcal{D}_d^*(\mathcal{G}p, \mathcal{G}p, \mathcal{G}q)$ then

$$\mathcal{D}_d^*(\mathcal{G}p, \mathcal{G}p, \mathcal{G}q) \leq \lambda\mathcal{D}_d^*(\mathcal{G}p, \mathcal{G}p, \mathcal{G}q)$$

Case(ii) If $max\{\mathcal{D}_d^*(\mathcal{G}p, \mathcal{G}p, \mathcal{G}q), \mathcal{D}_d^*(\mathcal{G}q, \mathcal{G}q, \mathcal{G}p)\} = \mathcal{D}_d^*(\mathcal{G}q, \mathcal{G}q, \mathcal{G}p)$ then

$$\mathcal{D}_d^*(\mathcal{G}p, \mathcal{G}p, \mathcal{G}q) \leq \lambda\mathcal{D}_d^*(\mathcal{G}q, \mathcal{G}q, \mathcal{G}p). \quad (14)$$

Merging (13), (14) and via ($\mathcal{D}_{d_3}^*$), we get $\mathcal{D}_d^*(\mathcal{G}p, \mathcal{G}q, \mathcal{G}q) \leq k^2\mathcal{D}_d^*(\mathcal{G}p, \mathcal{G}q, \mathcal{G}q)$.
Since, $k < 1$ then it is a contradiction. Thus, $\mathcal{G}p = \mathcal{G}q$. Consequently, $\mathcal{F}p = \mathcal{G}p = \mathcal{G}q = \mathcal{F}q$. Therefore w is the unique of the coincidence of $\mathcal{F}$ and $\mathcal{G}$, via lemma 2.10, w is unique (CFP) of $\mathcal{F}$ and $\mathcal{G}$.

Corollary 3.4: Let $(\mathcal{X}, \mathcal{D}_d^*)$ be $\mathcal{D}^*$-symmetric space satisfies (W_3), and $\mathcal{F}, \mathcal{G}: (\mathcal{X}, \mathcal{D}_d^*) \longrightarrow (\mathcal{X}, \mathcal{D}_d^*)$ two self-maps of $\mathcal{X}$ such that:
(***i***) $\mathcal{F}$ and $\mathcal{G}$ satisfying (E. A.) property. (***ii***) $\forall\, x \neq y \in \mathcal{X}$.
If

$$\mathcal{D}_d^*(\mathcal{G}x, \mathcal{G}y, \mathcal{G}y) < max\Big\{\mathcal{D}_d^*(\mathcal{F}x, \mathcal{F}y, \mathcal{F}y), \frac{k}{2}[\mathcal{D}_d^*(\mathcal{G}x, \mathcal{G}x, \mathcal{F}x) + \mathcal{D}_d^*(\mathcal{G}y, \mathcal{G}y, \mathcal{F}y)], \frac{k}{2}[\mathcal{D}_d^*(\mathcal{G}x, \mathcal{G}x, \mathcal{F}y) + \mathcal{D}_d^*(\mathcal{G}y, \mathcal{G}y, \mathcal{F}x)]\Big\},$$

(15)
and

$$\mathcal{D}_d^*(\mathcal{G}x, \mathcal{G}x, \mathcal{G}y) < max\Big\{\mathcal{D}_d^*(\mathcal{F}x, \mathcal{F}x, \mathcal{F}y), \frac{k}{2}[\mathcal{D}_d^*(\mathcal{G}x, \mathcal{G}x, \mathcal{F}x) + \mathcal{D}_d^*(\mathcal{G}y, \mathcal{G}y, \mathcal{F}y)], \frac{k}{2}[\mathcal{D}_d^*(\mathcal{G}x, \mathcal{G}x, \mathcal{F}y) + \mathcal{D}_d^*(\mathcal{G}y, \mathcal{G}y, \mathcal{F}x)]\Big\}.$$

(16)
$k \in [0,1)$. Assume $\mathcal{F}(\mathcal{X})$ is $\mathcal{D}_d^*$-closed of $\mathcal{X}$, with $\mathcal{G}(\mathcal{X}) \subset \mathcal{F}(\mathcal{X})$. If $\mathcal{F}$ and $\mathcal{G}$ are (OWC), then $\mathcal{F}$ and $\mathcal{G}$ have a unique (CFP).

Example 3.5: Assume that $\mathcal{X} = [0,4]$ and define $\mathcal{D}_d^*: \mathcal{X} \times \mathcal{X} \times \mathcal{X} \longrightarrow R$, by $\mathcal{D}_d^*(x, y, z) = max\{(x - y)^2, (y - z)^2, (z - x)^2\}$ so $(\mathcal{X}, \mathcal{D}_d^*)$ is a $\mathcal{D}^*$-symmetric space. Assume, $\mathcal{F}, \mathcal{G}: \mathcal{X} \longrightarrow \mathcal{X}$is maps, such that $\mathcal{F}(x) = \frac{1+2x}{3}$ and $\mathcal{G}(x) = \frac{1+4x}{5}$, $x \in \mathcal{X}$. Obviously, $\mathcal{F}$ and $\mathcal{G}$ are (OWC), and $\mathcal{F}$ and $\mathcal{G}$ satisfied the condition of (9) with

$$\mathcal{D}_d^*(\mathcal{G}x, \mathcal{G}y, \mathcal{G}y) \leq \frac{4}{9}\mathcal{D}_d^*(\mathcal{F}x, \mathcal{F}y, \mathcal{F}y)$$

$$\mathcal{D}_d^*(\mathcal{F}x,\mathcal{F}y,\mathcal{F}y) \leq \frac{4}{125}[\mathcal{D}_d^*(\mathcal{G}x,\mathcal{F}x,\mathcal{F}x)+\mathcal{D}_d^*(\mathcal{G}y,\mathcal{F}y,\mathcal{F}y)]$$
$$\mathcal{D}_d^*(\mathcal{F}x,\mathcal{F}y,\mathcal{F}y) \leq \frac{4}{125}[\mathcal{D}_d^*(\mathcal{G}x,\mathcal{F}y,\mathcal{F}y)+\mathcal{D}_d^*(\mathcal{G}y,\mathcal{F}x,\mathcal{F}x)].$$

The unique (CFP) of $\mathcal{F}$ and $\mathcal{G}$ equal 1.

4. Conclusions

In this work new class of generalized metric spaces namely, $\mathcal{D}^*$-symmetric metric space which is generalization of both $\mathcal{D}^*$-metric space and G-symmetric space has been introduced and investigated. Furthermore, some (CFP) results for two pair of (OWC) maps satisfying generalized contractive conditions in $\mathcal{D}^*$-symmetric spaces have been proved. Our main results which are obtained in our work give the proper generalization of some important (CFP) results and open up a wider scope for the study of (CFP) under generalization contractive conditions in generalized symmetric spaces.

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