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Solving the System of Two-Dimensional Coupled Burgers' Equations by Modified Variational Iteration Method with Genetic Algorithm

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Abstract

In this paper, a new modified variational iteration method (MVIM) with a genetic algorithm has been applied for solving nonlinear partial differential equations. Therefore, a new correction function through an auxiliary parameter w that makes sure the convergence of the standard method and improved results by using genetic techniques are introduced. The standard variational iteration method (VIM) is first applied to solve numerically the system of two-dimensional coupled Burgers' equations. Then an improvement on this method is done. Numerical experiments have been conducted to demonstrate the efficiency and high-order accuracy of this method. The algorithm converges readily which yields correct solutions. Better accuracy in comparison with other previous methods has been noticed. Moreover, the method can be easily applied to a wide number of linear and nonlinear differential equations with better accuracy.

Keywords: Coupled Burgers' equations, Genetic Algorithm, Lagrange multiplier, Modified variational iteration method, Variational iteration method.

حل نظام معادلات برجر ثنائية الأبعاد المزدوجة باستخدام طريقة التكرار المتغاير المعدلة مع الخوارزمية الجينية

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الخلاصة

في هذا البحث ، طريقة جديدة لمعادلة التكرار المتغير (MVIM) باستخدام الخوارزمية الجينية لحل المعادلات التفاضلية الجزئية غير الخطية تم تطبيقها. حيث تم تقديم تصحيح وظيفي جديد من خلال المعلمة المساعدة (w) للتأكد من تقارب الطريقة القياسية وتحسين النتائج باستخدام التقنيات الجينية. تم تطبيق الطريقة القياسية لـ (VIM) أولاً قبل تحسينها. تمت دراسة هذه الطرق لحل نظام معادلات برجر المزدوجة ثنائية الأبعاد. حيث تم إجراء تجارب عددية لإثبات كفاءة هذه الطريقة ودقتها العالية. نلاحظ أن الخوارزمية تتقارب بسهولة وتنتج حلولاً صحيحة ودقيقة أفضل مقارنة بالطرق السابقة الأخرى. علاوة على ذلك ، يمكن تطبيق الطريقة بسهولة لعدد كبير من المعادلات التفاضلية الخطية وغير الخطية وبدقة أفضل.

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1. Introduction

During the past years, nonlinear differential equations are vastly utilized as models to describe physical phenomena in different areas of science, especially in fluid mechanics, biology, solid state physics, chemical physics, optics, hydrodynamics, plasma wave, etc. [1]–[4]. One of the important nonlinear PDEs is known as Burgers' equation which is used to model a range of applications and it depicts the interaction between two fundamental physical principles in nature, convection and diffusion. This comprises turbulence, the sedimentation of two types of particles in fluid suspensions under the influence of gravity, which flows through a shock wave traveling through a viscous fluid, as well as the dispersion of contaminants in rivers [5]–[7]. Several authors solved the system of coupled Burgers' equations by different techniques such as the Laplace decomposition method [8], the finite difference and the modified cubic trigonometric B-spline differential quadrature method [9], the finite difference method based on Rubin-Graves type linearization [10], the semi-implicit characteristic Galerkin (SICG) method [11], the Bernstein differential quadrature method [12], the Sumudu decomposition method (SDM) [13], the Gaussian-based collocation meshless method [14], the efficient hybrid multistep numerical method [15] and the El-Zaki transform variational iteration method [16], etc. The variational iteration method is one of the most direct and effective ways to find approximations of PDEs. Most authors used this approach to obtain a variety of results [17]–[20] that show this method is reliable and efficient for a variety of technical and scientific applications. It presents convergent approximations of the exact results when a solution exists, otherwise, only a few approximations can be employed numerically.

The aim of this work is to solve the system of two-dimensional coupled Burgers' equations by the standard and a new modification of the variational iteration method with a genetic algorithm. We will also support the results by comparisons with other methods. Many plots in 3D graphics of the solutions are given.

The remainder of this essay is structured as follows: in section 2, the fundamental idea of the standard variational iteration method. In section 3, a novel modification for the variational iteration method is shown. Some definitions for genetic algorithms are given in section 4. Section 5 discusses a convergence analysis. In section 6, the standard and modified variational iteration method with the genetic technique are applied to solve the system of PDE and we also illustrate solutions by graphing. In the last section 7, several conclusions are made.

2. Standard Variational Iteration Method

Consider the following nonlinear PDE:

$$L[u(x)] + N[u(x)] = c(x), \quad (1)$$

where c is the source term, L[u] is a linear term, and N[u] is a nonlinear term. For a given $u_0(x)$, the solution $u_{n+1}(x)$ can be obtained as follows [21]:

$$u_{n+1}(x) = u_n(x) + \int_0^t \lambda(\tau)\left[L\{u_n(\tau)\} + N\{\widetilde{u_n(\tau)}\} - c(\tau)\right]d\tau, \quad (2)$$

where λ is the Lagrange multiplier (LM) that is gained by variational theory, where $\widetilde{u_n}$ is the constrain which results in $\delta\widetilde{u_n}$ =0 and gives the following:

$$\lambda = \frac{(-1)^m(\tau-t)^{m-1}}{(m-1)!}, \ m \geq 1 \quad (3)$$

where $m = 1$, the Lagrange multiplier is $\lambda = -1$, while if $m = 2$, then $\lambda = \tau - t$.
Substituting the Eq.(3) into Eq.(2), then the following formula will be obtained:

$$u_{n+1}(x) = u_n(x) + \int_0^t \frac{(-1)^m(\tau-t)^{m-1}}{(m-1)!}[L\{u_n(\tau)\} + N\{u_n(\tau)\} - c(\tau)]d\tau. \quad (4)$$

The successive approximations $u_n(x,t)$, $n > 0$ of the solution $u(x,t)$ will be obtained by utilizing the obtained Lagrange multiplier and any initial condition $u_0(x,t)$, consequently, the solution is

$$u(x,t) = \lim_{n\to\infty} u_n(x,t). \quad (5)$$

In the next section, we will develop the variational iteration method for nonlinear PDEs. We also improve the results by using genetic techniques.

3. Modified Variational Iteration Method

It takes a differential Eq. (1), the approximate result $u_{n+1}(x)$ of Eq.(1) for given initial $u_0(x)$ can be obtained to clarify the MVIM as follows:

$$u_{n+1}(x) = u_0(x) + w\int_0^t \lambda(\tau)\left[N\{\widetilde{u_n(\tau)}\} - c(\tau)\right]d\tau, \quad (6)$$

where w is an auxiliary term that is used to ensure the convergence to the accurate solution by limiting the norm 2 over the space of the presented problem. The optimal resolution of this w increases the accuracy of the algorithm. While the Lagrange multiplier λ can be found by equation (3). Substituting Eq. (3) into Eq. (6). Then, we have the iterative algorithm for Eq. (1) as follows:

$$\begin{cases} u_0(x) \text{ is a suitable initial approximation} \\ u_{n+1}(x) = u_0(x) + w\int_0^t \frac{(-1)^m(\tau-t)^{m-1}}{(m-1)!}\left[N\{\widetilde{u_n(\tau)}\} - c(\tau)\right]d\tau \end{cases} \quad (7)$$

The evolutionary algorithm will use to find the best value of w to solve the system of two-dimensional Burger's equation which can supply numerical solutions for linear and nonlinear problems immediately and very carefully.

4. Genetic Algorithms (GA)

Definition 1: Genetic algorithms are search methods utilized in computing to get approximate results.

Definition 2: Three different types of operators are included in the form of genetic algorithms:
selection, crossover and mutation [22].

Selection: Chromosomes in the population are selected for reproduction by this factor. If a chromosome is acceptable, it will be likely chosen for reproduction.

Crossover: To produce two children, this factor selects a point on two chromosomes and switches the subsequence before and after that location. Nearly identical to biological recombination between two single-chromosome organisms.

Mutation: The chromosome's bits are flipped by this factor. The string 00000100, for instance, might be changed to 01000100 at its second position.

Algorithm. The basics of the genetic algorithm [23].
a simple genetic algorithm works as follows:

1. Start with an nl-bit chromosomal population that was produced randomly.
2. Determine the population's total chromosomal fitness.
3. Repeat the subsequent steps until n progeny have been produced:

a. From the current population, choose a couple of parent chromosomes with an increasing probability of selection based on fitness. Selection is carried out with replacement in the sense that the same chromosome may be selected as a parent more than once.

b. Cross through the couple at a randomly selected spot to produce two offspring with probability pc. Create two offspring that are exact duplicates of each parent if there is no crossover.
4. Change each of the two offspring's loci with probability pc which is also known as the mutation probability or mutation rate, and then add the resulting chromosomes to the new population. If n is odd, a random member of the new population may be removed.
5. Move on to step 2.

5. Convergence Analysis

This section examines the convergence of our suggested modification which is known as MVIA with the genetic algorithm when applied to equation (1). When coupled Burgers' equations are used with the suggested approach. The operator O will now be defined as follows.

$$\mathrm{O}u(x,t,w) = w\int_0^t \lambda(\tau)[N\{\widetilde{u_n}(x,\tau,w)\} - c(\tau)]d\tau, \quad (8)$$

and defining $v_n, z_n, n \geq 0$ as

$$\begin{cases} v_0(x,t) = u_0(x,t), \\ z_0(x,t) = v_0(x,t), \end{cases}$$

$$\begin{cases} v_1(x,t,w) = \mathrm{O}z_0(x,t), \\ z_1(x,t,w) = z_0(x,t) + \mathrm{O}z_0(x,t), \end{cases}$$

$$\begin{cases} u_1(x,t,w) = \mathrm{O}z_0(x,t), \\ z_1(x,t,w) = z_0(x,t) + v_1(x,t,w), \end{cases}$$

$$\begin{cases} v_{n+1}(x,t,w) = \mathrm{O}z_n(x,t,w), \\ z_{n+1}(x,t,w) = z_n(x,t,w) + \mathrm{O}v_n(x,t,w), \end{cases}$$

Generally, for $n \geq 0$, we can write it as:

$$\begin{cases} u_{n+1}(x,t,w) = \mathrm{O}z_n(x,t,w), \\ z_{n+1}(x,t,w) = z_n(x,t,w) + v_{n+1}(x,t,w). \end{cases}$$

So that

$$u(x,t,w) = \lim_{n\to\infty} z_n(x,t,w) = v_0(x,t) + \sum_{n=1}^{\infty} v_n(x,t,w). \quad (9)$$

If the initial conditions and boundary requirements are satisfied by it, the $u_0(x,t)$ can be chosen without restriction. The procedure will successfully produce accurate and beneficial results if an appropriate beginning approximation is used. By using nth-order truncated series, we may roughly estimate the solution as $u_n(x,t,w) = v_0(x,t) + \sum_{n=1}^{n} v_n(x,t,w)$.
By using the 2-norm error of the residual function, the parameter w in $u_n(x,t,w)$ ensures that the hypothesis is satisfied. The theorems in [24, 25] are used to show the error analysis and convergence requirements of MVIA with an auxiliary parameter.

Theorem 1: Assume that O is an operator from a Hilbert space H to H which is defined in equation (8). The series solution (9)

$$u(x,t) = \lim_{n\to\infty} z_n(x,t,w) = v_0(x,t) + \sum_{n=1}^{\infty} v_n(x,t,w).$$

It converges if $\exists\ w \neq 0, 0 < \gamma < 1$, such that

$$\begin{cases} \|0z_0(x,t)\| \le \gamma\|z_0(x,t)\|, \\ \|0z_1(x,t,w)\| \le \gamma\|0z_0(x,t)\|, \\ \|0z_n(x,t,w)\| \le \gamma\|0z_{n-1}(x,t,w)\|, \quad n = 2,3,4,\dots \end{cases}$$

Proof. To prove that the sequence $\{z_n\}_{n=0}^{\infty}$ is a Cauchy sequence in the Hilbert space H. Therefore, we consider

$z_0(x,t) = v_0(x,t)$

$z_{0+1}(x,t,w) = z_0(x,t) + 0v_0(x,t)$

$z_{n+1}(x,t,w) = z_n(x,t,w) + 0v_0(x,t,w)$

that implies

$\|z_{n+1}(x,t,w) - z_n(x,t,w)\| = \|0v_n(x,t,w)\|$

$\le \gamma\|0v_{n-1}(x,t,w)\|$

$\le \gamma^2\|0v_{n-2}(x,t,w)\|.$

Following the same procedure, we achieve

$\|z_{n+1}(x,t,w) - z_n(x,t,w)\| \le \gamma^n\|0v_0(x,t,w)\|$

For $n \to \infty$, $\gamma^n \to 0$, thus

$\|z_{n+1}(x,t,w) - z_n(x,t,w)\| \to 0$

For every $n \ge i$

$\|z_n - z_i\| = \|(z_n - z_{n-1}) + (z_{n-1} - z_{n-2}) + \cdots + (z_{i+1} - z_i)\|,$

that implies

$\|z_n - z_i\| \le \|0v_{n-1}\| + \|0v_{n-2}\| + \cdots + \|0v_j\|$

$\le \gamma^{n-1}\|0v_0\| + \gamma^{n-2}\|0v_{n-2}\| + \cdots + \gamma^i\|0v_0\|$

$\le (\gamma^{n-1} + \gamma^{n-2} + \cdots + \gamma^i)\|0v_0\|$

$= \frac{1-\gamma^{n-i}}{1-\gamma}\gamma^{i+1}\|0v_0\|$

Since $0 < \gamma < 1$, we obtain

$\lim_{nj\to\infty}\|z_n - z_i\| = 0.$

Hence, it is proved that $\{z_n\}_{n=0}^{\infty}$ is a Cauchy sequence in the Hilbert space H, so that $u(x,t,w) = \lim_{n\to\infty} z_n(x,t,w) = v_0(x,t) + \sum_{n=1}^{\infty} v_n(x,t,w)$ converges.

Theorem 2. Let the definition of the operator L required in equation (1) be $L = \frac{\partial^i}{\partial t^i}$, $i = 1,2$. If we have the solution (9) which is defined as $u(x,t) = v_0(x,t) + \sum_{n=1}^{\infty} v_n(x,t,w)$, then $u(x,t)$ is an exact solution to equation (1).

Proof. Suppose that the series solution $u(x,t) = v_0(x,t) + \sum_{n=1}^{\infty} v_n(x,t,w)$ converges. Consequently, it means that $\lim_{n\to\infty} v_n(x,t,w)$ and

$[v_0(x,t) - v_1(x,t,w)] + \sum_{j=1}^{\infty}[v_j(x,t,w) - v_{j+1}(x,t,w)] = v_0(x,t) - v_{j+1}(x,t,w).$

Therefore,

$$[v_0(x,t) - v_1(x,t,w)] + \sum_{j=1}^{\infty}[v_j(x,t,w) - v_{j+1}(x,t,w)] = v_0(x,t) - \lim_{n\to\infty} v_{n+1}(x,t,w) = v_0(x,t).$$

Equation (9) leads to the conclusion that

$u(x,t) = v_0(x,t) + \sum_{n=1}^{\infty} v_n(x,t,w).$

By using the operator L on both sides, we obtain

$$L[v_0(x,t) - v_1(x,t,w)] + \sum_{j=1}^{\infty} L[v_j(x,t,w) - v_{j+1}(x,t,w)] = L[v_0(x,t)] = 0. \quad (10)$$

So that,

$L[v_0(x,t) - v_1(x,t,w)] = L[v_0(x,t)] - L[0z_0(x,t)].$

For simplicity, let $K = N\{\widetilde{u_n(\tau)}\} - c(\tau)$, which implies

$L[v_0(x,t) - v_1(x,t,w)] = -L\left[w\int_0^t \lambda(\tau)Kz_0(x,t)d\tau\right]$
$= wKz_0(x,t)$
$L[v_1(x,t) - v_2(x,t,w)] = L[v_1(x,t)] - L[0z_2(x,t)],$
$= L[0z_0(x,t)] - L[0z_1(x,t,w)],$
$= L\left[w\int_0^t \lambda(\tau)Kz_0(x,t)d\tau\right],$
$= -L\left[w\int_0^t \lambda(\tau)Kz_1(x,t)d\tau\right],$
$= w[0z_1(x,t,w) - 0z_0(x,t)].$
In a similar way, where $j \geq 2$,
$L\left[v_j(x,t,w) - v_{j+1}(x,t,w)\right] = w\left[0z_j(x,t,w) - 0z_{j-1}(x,t)\right].$
Therefore, it is as follows
$L[v_0(x,t) - v_1(x,t,w)] + L[v_1(x,t,w) - v_2(x,t,w)] + \sum_{j=2}^{n} L\left[v_j(x,t,w) - v_{j+1}(x,t,w)\right]$
$= wKz_0(x,t) + w[0z_1(x,t,w) - 0z_0(x,t)] + w[0z_n(x,t,w) - 0z_1(x,t)]$
$= wKz_n(x,t,w)$
$= wK\left[v_0(x,t) + \sum_{j=1}^{n} v_j(x,t,w)\right].$
Therefore,
$$L[v_0(x,t) - v_1(x,t,w)] + \sum_{j=1}^{n} L\left[v_j(x,t,w) - v_{j+1}(x,t,w)\right] = wK\left[v_0(x,t) + \sum_{j=1}^{n} v_j(x,t,w)\right], \quad (11)$$
Equations (10) and (11) can be used to derive
$wK\left[v_0(x,t) + \sum_{j=1}^{n} v_j(x,t,w)\right] = 0.$
Hence, the auxiliary parameter w is an optimal number, and it is shown that
$u(x,t) = v_0(x,t) + \sum_{n=1}^{\infty} v_n(x,t,w)$ is an exact solution of equation (1).

6. Applications and numerical results

In this section, the variational iteration method and MVIM with genetic algorithms have been applied to obtain approximate-exact solutions for nonlinear PDEs are displayed in the following Burger's equation. To show the high accuracy of the solution results compared with the exact solutions, the maximum errors are defined as follows:

$$\|\cdot\|_\infty = \underbrace{Matrix}_{0\le i\le k}\left(k, \left\|U_{Exact}(x_i,y_i,t) - U_{Appr}(x_i,y_i,t)\right\|_\infty\right), \quad (12)$$

$$\|\cdot\|_{2,\Sigma} = \sqrt{\sum_{j=0}^{k}\sum_{i=0}^{k}\left(U_{Exact}\left(x_{i,j},y_{i,j},t\right) - U_{Appr}\left(x_{i,j},y_{i,j},t\right)\right)^2} \quad (13)$$

and the mean square error (MSE) is

$$\text{MSM} = \sqrt{\frac{\sum_{j=0}^{k}\frac{\sum_{i=0}^{k}\left(U_{Exact}\left(x_{i,j},y_{i,j},t\right) - U_{Appr}\left(x_{i,j},y_{i,j},t\right)\right)^2}{(k+1)}}{(k+1)}} \quad (14)$$

where k represents the number of partitions of the interval $0 \leq x, y \leq 1$. Moreover, giving the maximum residual error (MRE). The computations associated with the problems were performed using the Maple 18 package with a precision of 20 digits.

6.1. Numerical example

Consider the following two-dimensional Burger's equation [26]:

$$\begin{cases} u_t + uu_x + vu_y = \frac{1}{Re}\left(u_{xx} + u_{yy}\right), \\ v_t + uv_x + vv_y = \frac{1}{Re}\left(v_{xx} + v_{yy}\right), \end{cases} \tag{15}$$

where R_e is the Reynolds number and is subject to the following initial conditions:

$$\begin{cases} u(x,y,0) = -\frac{4\varepsilon\pi\cos(2\pi x)\sin(2\pi y)}{2+\sin(2\pi x)\sin(\pi y)}, \\ v(x,y,0) = -\frac{2\varepsilon\pi\sin(2\pi x)\cos(2\pi y)}{2+\sin(2\pi x)\sin(\pi y)}. \end{cases} \tag{16}$$

The exact solutions of the system (15) were given by

$$\begin{cases} U_{Exact}(x,y,t) = -\frac{4\varepsilon\pi e^{-5\pi^2\varepsilon t}\cos(2\pi x)\sin(2\pi y)}{2+e^{-5\pi^2\varepsilon t}\sin(2\pi x)\sin(\pi y)}, \\ V_{Exact}(x,y,t) = -\frac{2\varepsilon\pi e^{-5\pi^2\varepsilon t}\sin(2\pi x)\cos(2\pi y)}{2+e^{-5\pi^2\varepsilon t}\sin(2\pi x)\sin(\pi y)}. \end{cases} \tag{17}$$

where $\varepsilon = \frac{1}{Re}$.

Standard variational iteration method: To solve the system (15) using the variational iteration method, the following correction functional will be constructed as follows:

$$\begin{cases} u_{n+1}(x,y,t) = u_n(x,y,t) + \int_0^t \lambda_1(\tau)\left[(u_t)_n + u_n u_{nx} + v_n v_{ny} - \varepsilon(u_{nxx} + u_{yy})\right]d\tau, \\ v_{n+1}(x,y,t) = v_n(x,y,t) + \int_0^t \lambda_2(\tau)\left[(v_t)_n + u_n v_{nx} + v_n v_{ny} - \varepsilon(v_{nxx} + v_{yy})\right]d\tau, \end{cases} \tag{18}$$

where $\lambda_1(\tau)$ and $\lambda_1(\tau)$ are the general Lagrange multiplier and $u_n u_{nx}$, $u_n v_{nx}$, $v_n u_{ny}$ and $v_n v_{ny}$ denote restricted variations i.e., $\delta u_n u_{nx} = \delta u_n v_{nx} = \delta v_n u_{ny} = \delta v_n v_{ny} = 0$. the following stationary conditions are:

$$\begin{cases} \lambda'_i(\tau) = 0, \\ 1 + \lambda_i(\tau)]_{\tau=t} = 0 \end{cases}, \quad i = 1,2 \tag{19}$$

So, the Lagrange multiplier can be identified as $\lambda_1(\tau) = \lambda_2(\tau) = -1$. Then the formulas (18) become the following:

$$\begin{cases} u_{n+1}(x,y,t) = u_n(x,y,t) - \int_0^t\left[(u_t)_n + u_n u_{nx} + v_n v_{ny} - \varepsilon(u_{nxx} + u_{yy})\right]d\tau, \\ v_{n+1}(x,y,t) = v_n(x,y,t) - \int_0^t\left[(v_t)_n + u_n v_{nx} + v_n v_{ny} - \varepsilon(v_{nxx} + v_{yy})\right]d\tau, \end{cases} \tag{20}$$

The different approximations can be taken by using the recurrence relations shown in equation (20)

$$u_0(x,y,t) = \frac{-8\varepsilon\pi\sin(\pi y)\cos(\pi x)^2 + 4\varepsilon\pi\sin(\pi y)}{2+\sin(\pi y)\sin(\pi x)\cos(\pi x)}$$

$$v_0(x,y,t) = -\frac{4\varepsilon\pi\cos(\pi y)\sin(\pi x)\cos(\pi x)}{2+\sin(\pi y)\sin(\pi x)\cos(\pi x)}$$

$$u_1(x,y,t) = \frac{4\varepsilon\pi\sin(\pi y)[1-2\cos(\pi x)^2 - \varepsilon\pi^2 t - 4\varepsilon\pi^2 t\sin(\pi x)^2]}{2+\sin(\pi y)\sin(\pi x)\cos(\pi x)}$$
$$+\frac{8\varepsilon^2 t\pi^3\sin(\pi x)[-2\sin(\pi y)^2\cos(\pi x)^3 - 3\sin(\pi y)^2\cos(\pi x) + 8\sin(\pi y)^2\cos(\pi x)]}{(2+\sin(\pi y)\sin(\pi x)\cos(\pi x))^2}$$
$$+\frac{32\varepsilon^2 t\pi^3\sin(\pi y)^3[-\cos(\pi x)^2 + 2\cos(\pi x)^6 - 3\cos(\pi x)^6 + \sin(\pi x)^4}{(2+\sin(\pi y)\sin(\pi x)\cos(\pi x))^3}$$

$$v_1(x,y,t) = \frac{4\varepsilon\pi\sin(\pi x)\cos(\pi x)\cos(\pi y)[5\varepsilon\pi^2 t - 1]}{2+\sin(\pi y)\sin(\pi y)\cos(\pi x)}$$
$$+\frac{8\varepsilon^2\pi^3 t\sin(\pi y)\cos(\pi y)[-2\cos(\pi x)^4 - 5\cos(\pi x)^2\sin(\pi x)^2 + 2\cos(\pi x)^2 - 2\sin(\pi x)^2]}{(2+\sin(\pi y)\sin(\pi x)\cos(\pi x))^2}$$

$$+\frac{32\varepsilon^2 t\pi^3 \sin(\pi y)^2 \cos(\pi y)[\cos(\pi x)^3 \sin(\pi x)+\sin(\pi x)^3 \cos(\pi x)-\sin(\pi x)^5 \cos(\pi x)]}{(2+\sin(\pi y)\sin(\pi x)\cos(\pi x))^3}$$

$\vdots$

Tables 1 and 2 display a comparison of the numerical results applying the standard variational iteration method with the exact solutions within the interval $0 \le x, y \le 1$ where $h = 0.1$. But in Tables 4 and 5, the numerical results applying the standard variational iteration method developed by the genetic algorithm have been compared with the exact solutions within the interval $0 \le x, y \le 1$ where $h = 0.1$. The best value for ε and t will be selected based on the genetic algorithm [25] and based on which the results are calculated VIM-GA.

In Table 3, the maximum errors ($\|\cdot\|_\infty$), $\|\cdot\|_{2,\Sigma}$), MSE, MRE have been presented by the standard variational iteration method on the interval $0 \le x, y \le 1$, where m represents the number of iterations, and in Table 6, the maximum errors ($\|\cdot\|_\infty$), $\|\cdot\|_{2,\Sigma}$), MSE, MRE has been shown by the standard variational iteration method developed with genetic algorithm on the interval $0 \le x, y \le 1$.

Modified variational iteration method**:** Now, to solve the system (15) with the initial conditions in equation (16) using the MVIM, the following correction functional will be constructed as

$$\begin{cases} u_{n+1}(x,y,t) = u_0(x,y,t) + w\int_0^t \lambda_1(\tau)\left[u_n u_{nx} + v_n v_{ny} - \varepsilon(u_{nxx} + u_{yy})\right]d\tau\,, \\ v_{n+1}(x,y,t) = v_0(x,y,t) + w\int_0^t \lambda_2(\tau)\left[u_n v_{nx} + v_n v_{ny} - \varepsilon(v_{nxx} + v_{yy})\right]d\tau\,, \end{cases} \tag{21}$$

In a similar way as above, the formulas of variational iteration can be obtained

$$\begin{cases} u_{n+1}(x,y,t) = u_0(x,y,t) - w\int_0^t \left[u_n u_{nx} + v_n v_{ny} - \varepsilon(u_{nxx} + u_{yy})\right]d\tau\,, \\ v_{n+1}(x,y,t) = v_0(x,y,t) - w\int_0^t \left[u_n v_{nx} + v_n v_{ny} - \varepsilon(v_{nxx} + v_{yy})\right]d\tau\,, \end{cases} \tag{22}$$

The following different approximations can be taken by utilizing the relations shown in equation (22)

$$u_0(x,y,t) = \frac{-8\varepsilon\pi\sin(\pi y)\cos(\pi x)^2 + 4\varepsilon\pi\sin(\pi y)}{2+\sin(\pi y)\sin(\pi x)\cos(\pi x)}$$

$$v_0(x,y,t) = -\frac{4\varepsilon\pi\cos(\pi y)\sin(\pi x)\cos(\pi x)}{2+2\sin(\pi y)\sin(\pi x)\cos(\pi x)}$$

$$u_1(x,y,t) = \frac{4\varepsilon\pi\sin(\pi y)[1-2\cos(\pi x)^2 - \varepsilon t w\pi^2 - 4\varepsilon t w\pi^2 sin(\pi x)^2]}{2+2\sin(\pi y)\sin(\pi x)\cos(\pi x)}$$
$$+\frac{8\varepsilon^2 t w\pi^3\sin(\pi y)^2[-2\sin(\pi x)\cos(\pi x)^3 - 3\sin(\pi x)\cos(\pi x) + 8\sin(\pi x)^3\cos(\pi x)]}{(2+2\sin(\pi y)\sin(\pi x)\cos(\pi x))^2}$$

$$+\frac{32\varepsilon^2 t w\pi^3\sin(\pi y)^3[\cos(\pi x)^2 - \sin(\pi x)^2 + 2\cos(\pi x)^6 - 3\cos(\pi x)^6 + \sin(\pi x)^4 + 2\cos(\pi x)^2\sin(\pi x)^2]}{(2+2\sin(\pi y)\sin(\pi x)\cos(\pi x))^3}$$

$$v_1(x,y,t) = \frac{4\varepsilon w\pi\sin(\pi x)\cos(\pi x)\cos(\pi y)[5\varepsilon t\pi^2 - 1]}{2+\sin(\pi y)\sin(\pi y)\cos(\pi x)}$$

$$+\frac{8\varepsilon^2 tw\pi^3 \sin(\pi y)\cos(\pi y)[-2cos(\pi x)^4-5\cos(\pi x)^2 \sin(\pi x)^2+2\cos(\pi x)^2-2\sin(\pi x)^2]}{(2+\sin(\pi y)\sin(\pi x)\cos(\pi x))^2}$$
$$+\frac{32\varepsilon^2 tw\pi^3 \sin(\pi y)^2\cos(\pi y)[\cos(\pi x)^3 \sin(\pi x)+\sin(\pi x)^3\cos(\pi x)-\sin(\pi x)^5\cos(\pi x)]}{(2+\sin(\pi y)\sin(\pi x)\cos(\pi x))^3}$$
$$\vdots$$

In Tables 7 and 8, the numerical results applying the MVIM developed by the genetic algorithm are compared with the exact solutions within the interval $0 \le x, y \le 1$ where $h = 0.1$. The best value for $w,\ \varepsilon$ and t will be selected based on the genetic algorithm [27] and based on which the results are calculated MVIM-GA.

In Table 9, the maximum errors ($\|\cdot\|_\infty$), $\|\cdot\|_{2,\Sigma}$), MSE, MRE have been found by the MVIM developed with the genetic algorithm on the interval $0 \le x, y \le 1$.

Table 1: Comparison of the numerical results applying the variational iteration method (U_{Appr}) with the exact solutions (U_{Exact}) when $\varepsilon = 0.01$ and $t = 0.01$

y	Sols	x				
		0.1	0.2	0.4	0.6	0.8
0.1	U_{Exact}	-0.0143351594	-0.0253702178	-0.0376378474	-0.0376378474	-0.0253702178
	U_{Appr}	-0.0143351595	-0.0253702177	-0.0376378468	-0.0376378468	-0.0253702177
0.2	U_{Exact}	-0.00520873591	-0.0088850905	-0.0126721126	-0.0126721126	-0.0088850904
	U_{Appr}	-0.00520873598	-0.0088850904	-0.0126721124	-0.0126721124	-0.0088850904
0.4	U_{Exact}	0.01433515941	0.02537021791	0.03763784748	0.03763784747	0.02537021790
	U_{Appr}	0.01433515959	0.02537021782	0.03763784687	0.03763784689	0.02537021781
0.6	U_{Exact}	0.01718352677	0.03590276003	0.06664126580	0.06664126577	0.03590276002
	U_{Appr}	0.01718352711	0.03590276192	0.06664127202	0.06664127199	0.03590276190
0.8	U_{Exact}	-0.00699289427	-0.0157318688	-0.0334106397	-0.0334106397	-0.0157318688
	U_{Appr}	-0.0069928944	-0.0157318707	-0.0334106565	-0.0334106565	-0.0157318707

Table 2: Comparison of the numerical results applying the variational iteration method (V_{Appr}) with the exact solutions (V_{Exact}) when $\varepsilon = 0.01$ and $t = 0.01$

y	Sols	x				
		0.1	0.2	0.4	0.6	0.8
0.1	V_{Exact}	-0.01602719544	-0.01268510894	-0.004442545252	0.004442545256	0.01268510894
	V_{Appr}	-0.01602719568	-0.01268510908	-0.004442545284	0.004442545288	0.01268510908
0.2	V_{Exact}	-0.02466892743	-0.01881892371	-0.006336056298	0.006336056305	0.018818923710
	V_{Appr}	-0.02466892746	-0.01881892364	-0.006336056259	0.006336056272	0.01881892364
0.4	V_{Exact}	-0.01602719541	-0.01268510892	-0.004442545247	0.004442545251	0.01268510892
	V_{Appr}	-0.01602719564	-0.01268510906	-0.004442545275	0.004442545283	0.01268510907
0.6	V_{Exact}	0.01921176696	0.01795137999	0.007865934394	-0.007865934410	-0.01795137999
	V_{Appr}	0.01921176730	0.01795138017	0.007865934133	-0.007865934143	-0.01795138017
0.8	V_{Exact}	0.03311882262	0.03332063289	0.01670531980	-0.016705319821	-0.03332063287
	V_{Appr}	0.03311882399	0.03332063632	0.01670532573	-0.016705325760	-0.03332063634

Table 3: Maximum error $(\|\cdot\|_\infty, \|\cdot\|_{2,\Sigma})$, MSE and MRE by the standard variational iteration method on the interval $0 \leq x, y \leq 1$ when $\varepsilon = 0.01$ and $t = 0.01$

n	i	$\|\cdot\|_\infty$	$\|\cdot\|_{2,\Sigma}$	MSE	MRE
1	1	2.402E-06	1.549E-05	1.048E-06	1.627E-04
	2	1.264E-06	8.124E-06	3.740E-07	8.048E-05
2	1	2.229E-08	1.389E-05	1.587E-06	1.435E-06
	2	6.12E-09	9.560E-06	7.791E-07	5.138E-07

where n is the number of iterations and i is the equation.

Table 4: Comparison of the numerical results applying the standard variational iteration method (U_{Appr}) developed by the genetic algorithm with the exact solutions (U_{Exact}) when $\varepsilon = 0.00325$ and $t = 0.0275$

y	Sols	x				
		0.1	0.2	0.4	0.6	0.8
0.1	U_{Exact}	-0.004661167616	-0.008249010569	-0.01223731916	-0.01223731916	-0.00824901056
	U_{Appr}	-0.004661167655	-0.008249010540	-0.01223731903	-0.01223731902	-0.00824901055
0.2	U_{Exact}	-0.001693613689	-0.002888839180	-0.00411992595	-0.00411992594	-0.00288883917
	U_{Appr}	-0.001693613705	-0.002888839167	-0.00411992590	-0.00411992599	-0.00288883916
0.4	U_{Exact}	0.004661167620	0.008249010577	0.01223731918	0.01223731917	0.008249010577
	U_{Appr}	0.004661167663	0.008249010554	0.01223731903	0.01223731904	0.008249010554
0.6	U_{Exact}	0.005587866279	0.01167578772	0.02167415011	0.02167415011	0.035902760020
	U_{Appr}	0.005587866358	0.01167578814	0.02167415155	0.02167415154	0.035902760190
0.8	U_{Exact}	-0.002274086840	-0.005116572784	-0.01086881712	-0.01086881711	-0.00511657278
	U_{Appr}	-0.002274086870	-0.005116573228	-0.01086882102	-0.01086882100	-0.00511657322

Table 5: Comparison of the numerical results applying the standard variational iteration method (V_{Appr}) developed by the genetic algorithm with the exact solutions (V_{Exact}) when $\varepsilon = 0.00325$ and $t = 0.0275$

y	Sols	x				
		0.1	0.2	0.4	0.6	0.8
0.1	V_{Exact}	-0.005211343824	-0.004124505285	-0.001444419590	0.001444419591	0.004124505286
	V_{Appr}	-0.005211343877	-0.004124505317	-0.001444419597	0.001444419598	0.004124505318
0.2	V_{Exact}	-0.008021069576	-0.006118659583	-0.002059962974	0.002059962977	0.006118659584
	V_{Appr}	-0.008021069581	-0.006118659565	-0.002059962965	0.002059962968	0.006118659565
0.4	V_{Exact}	-0.005211343815	-0.004124505278	-0.001444419588	0.001444419589	0.004124505281
	V_{Appr}	-0.005211343866	-0.004124505308	-0.001444419594	0.001444419597	0.004124505312
0.6	V_{Exact}	0.006247424422	0.005837893854	0.002558286383	-0.00255828638	-0.00583789385
	V_{Appr}	0.006247424497	0.005837893893	0.002558286322	-0.00255828632	-0.00583789389
0.8	V_{Exact}	0.01077022985	0.01083707505	0.005434408543	-0.00543440854	-0.01083707505
	V_{Appr}	0.01077023017	0.01083707584	0.005434409918	-0.00543440992	-0.01083707585

Table 6: Maximum error $(\|\cdot\|_{\infty}, \|\cdot\|_{2,\Sigma})$, MSE and MRE by the standard variational iteration method developed with the genetic algorithm on the interval $0 \le x, y \le 1$, when $\varepsilon = 0.00325$ and $t = 0.0275$

n	i	$\|\cdot\|_{\infty}$	$\|\cdot\|_{2,\Sigma}$	MSE	MRE
1	1	6.240E-07	5.477E-05	3.847E-07	1.536E-05
	2	3.284E-07	1.516E-06	2.683E-07	7.599E-06
2	1	5.18E-09	6.519E-06	7.071E-07	1.209E-07
	2	1.419E-09	1.584E-06	1.109 E-07	4.336E-08

Table 7: Comparison of the numerical results applying the MVIM (U_{Appr}) developed by genetic algorithm with the exact solutions (U_{Exact}) when $\varepsilon = 0.00125$, $w = 0.99985$ and $t = 0.01$

y	Sols	x				
		0.1	0.2	0.4	0.6	0.8
0.1	U_{Exact}	-0.001799003758	-0.003182979921	-0.004720644433	-0.004720644433	-0.003182979920
	U_{Appr}	-0.001799003910	-0.003182980171	-0.004720644773	-0.004720644774	-0.003182980170
0.2	U_{Exact}	-0.000653548674	-0.001114392982	-0.001588734871	-0.001588734874	0.00111439298-
	U_{Appr}	-0.000653548728	-0.001114393064	-0.001588734976	-0.001588734978	-0.00111439306
0.4	U_{Exact}	0.001799003760	0.003182979924	0.004720644440	0.004720644440	0.003182979924
	U_{Appr}	0.001799003912	0.003182980176	0.004720644781	0.004720644782	0.003182980174
0.6	U_{Exact}	0.002158163423	0.004511317469	0.008380177375	0.008380177368	0.004511317467
	U_{Appr}	0.002158163644	0.004511317974	0.008380178448	0.008380178448	0.004511317973
0.8	U_{Exact}	-0.000878545422	-0.00197829152	-0.004209306912	-0.004209306912	-0.001978291524
	U_{Appr}	-0.000878545518	-0.00197829178	-0.004209307633	-0.004209307633	-0.001978291779

Table 8: Comparison of the numerical results applying the MVIM (V_{Appr}) developed by genetic algorithm with the exact solutions (V_{Exact}) when $\varepsilon = 0.00125$, $w = 0.99985$ and $t = 0.01$

y	Sols	x				
		0.1	0.2	0.4	0.6	0.8
0.1	V_{Exact}	-0.00201134734	-0.001591489961	-0.000557196491	0.0005571964915	0.001591489961
	V_{Appr}	-0.00201134751	-0.001591490085	-0.000557196531	0.0005571965322	0.001591490085
0.2	V_{Exact}	-0.00309525095	-0.002360322217	-0.0007943674372	0.0007943674382	0.002360322218
	V_{Appr}	-0.00309525120	-0.002360322388	-0.0007943674874	0.0007943674889	0.002360322388
0.4	V_{Exact}	-0.00201134734	-0.001591489958	-0.0005571964903	0.0005571964910	0.001591489959
	V_{Appr}	-0.00201134751	-0.001591490082	-0.0005571965306	0.0005571965315	0.001591490084
0.6	V_{Exact}	0.002412900059	0.002255658732	0.0009891457591	-0.000989145760	-0.00225565873
	V_{Appr}	0.002412900305	0.002255658985	0.0009891458854	-0.000989145886	-0.00225565898
0.8	V_{Exact}	0.004160850839	0.004190088683	0.002104653450	-0.002104653454	-0.00419008868
	V_{Appr}	0.004160851291	0.004190089221	0.002104653807	-0.002104653810	-0.00419008922

Table 9: Maximum error $\left(\|\cdot\|_{\infty}, \|\cdot\|_{2,\Sigma}\right)$, MSE and MRE by the MVIM developed with the genetic algorithm on the interval $[0,1]$ when $\varepsilon = 0.00125$, $w = 0.99985$ and $t = 0.01$.

n	i	$\|\cdot\|_{\infty}$	$\|\cdot\|_{2,\Sigma}$	MSE	MRE
1	1	3.901E-09	1.732E-06	3.355E-07	2.754E-07
	2	1.944E-09	5.3851E-07	9.327E-08	1.361E-07
2	1	1.181E-09	9.746E-07	3.716E-07	4.597E-08
	2	5.40E-10	5.029E-07	6.774 E-08	2.295E-08

6.2. Illustrate solutions by graphing

In this section, Figures 1 and 2 present the plot of the absolute errors in 3D on the domain $(x, y) \in [0,1] \times [0,1]$, Figures 3 and 4 show the contour plot 3D on the domain $(x, y) \in [0,1] \times [0,1]$ and the Figures 5 and 6 present the contour plot 2D on the $(x, y) -$ plane for $n = 2,\ \varepsilon = 0.01$ and $t = 0.01$ by the standard variational iteration method.

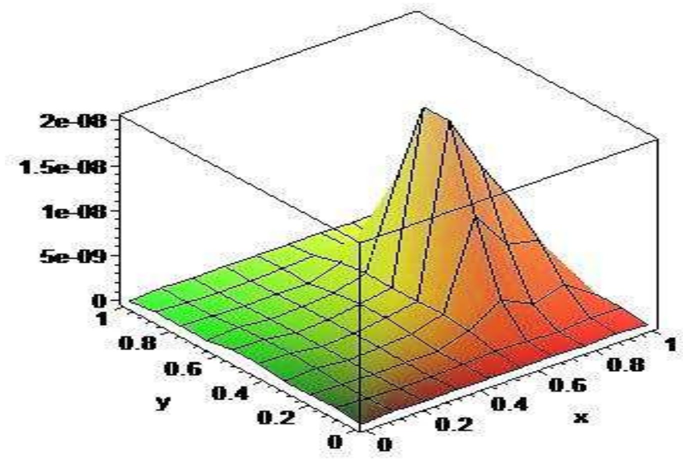

Figure 1: $\boldsymbol{U_{Exact}(x, y, t) - U_{Appr}(x, y, t)}$

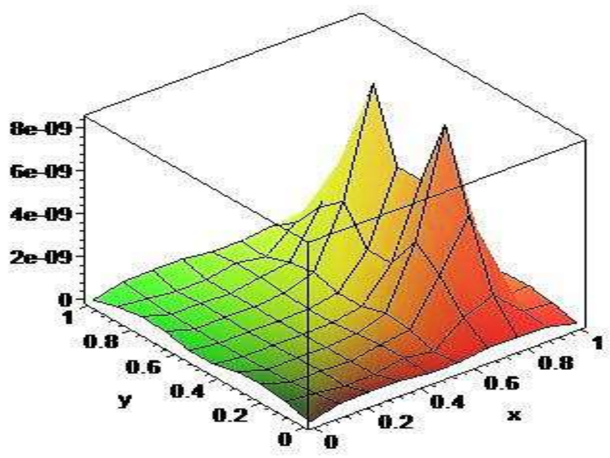

Figure 2: $\boldsymbol{V_{Exact}(x, y, t) - V_{Appr}(x, y, t)}$

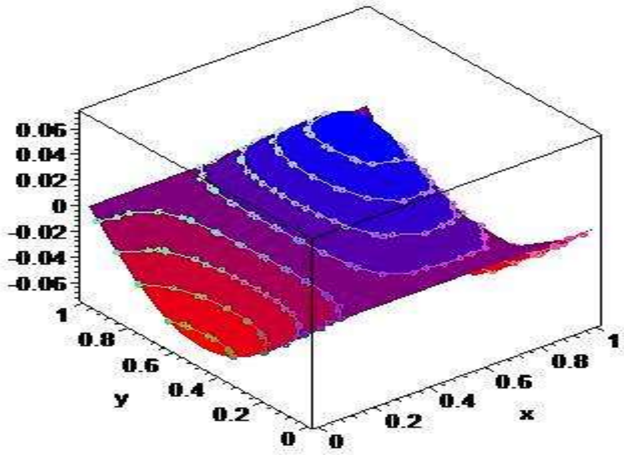

Figure 3: $U_{Exact}(x, y, t)$, $U_{Appr}(x, y, t)$

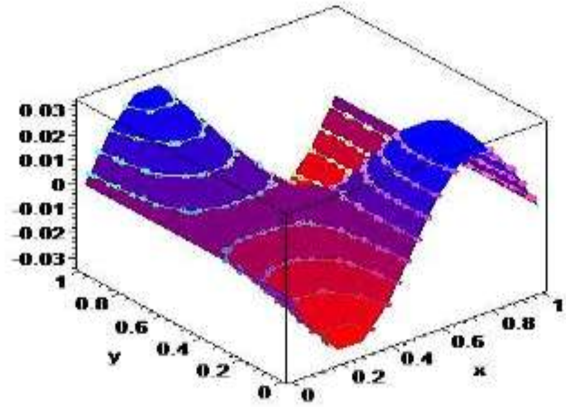

Figure 4: $V_{Exact}(x, y, t)$, $V_{Appr}(x, y, t)$

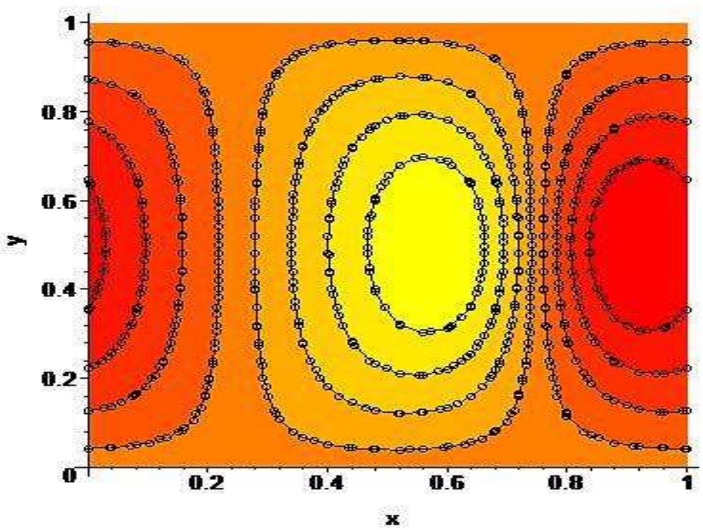

Figure 5: $U_{Exact}(x, y, t)$, $U_{Appr}(x, y, t)$

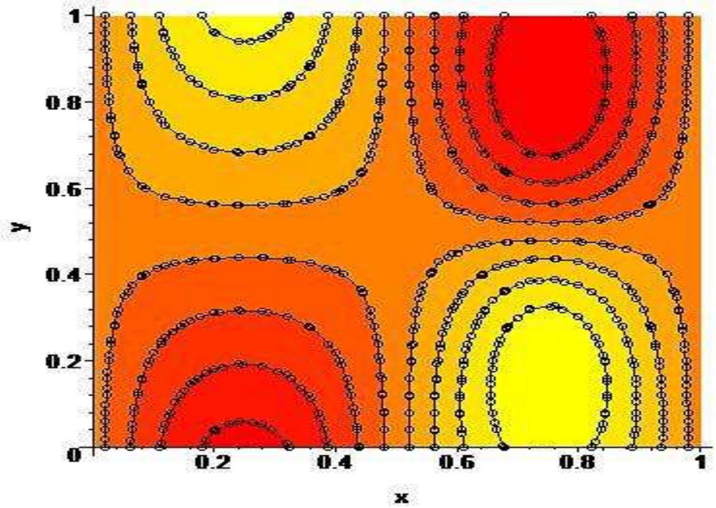

Figure 6: $V_{Exact}(x, y, t), \quad V_{Appr}(x, y, t)$

6.3. Discussions of numerical results

The two-dimensional Burgers equation system was solved by using the variational iteration method and MVIM in turn with the help of a genetic algorithm which enhanced the solutions. For accuracy and convergence, MVIM's numerical evidence was compared with the exact solution and VIM. The following findings were noted. From Table 3, where $n = 2$ the maximum residual error (MRE)=10^{-6} but in the table 6 the (MRE)= 10^{-7} where $n = 2$. so that the variational iteration method with genetic technique converges faster to the exact solution than the variational iteration method. From Table 9, where $n = 2$ the maximum residual error (MRE)= 10^{-8}. Thus, it is evident that MVIM with genetic algorithm converges better and more rapidly to exact than the standard variational iteration method and as also seen in Figures 1 and 2.

7. Conclusion:

In this research, the system of nonlinear (PDEs) that describes several physical phenomena in three ways has been solved. In the first part, the standard variational iteration method has been used. The variational iteration method with the genetic algorithm is utilized in the second part. In the last part, the modified variational iteration method with genetic techniques was successfully applied to solve the system of two-dimensional Burger's equation. By comparing the results, the variational iteration method with the genetic algorithm is more accurate than the standard variational iteration method and the modified variational iteration method with genetic techniques is more accurate than the two methods that are presented in the first and second parts. Furthermore, it has been shown that the present method is straightforward for finding approximate solutions in many other fields. Afterward, this notable method could be more efficiently used to find linear and nonlinear partial differential equations which orderly emerge in engineering, physics, and other technological areas.

References

[1] M. Alobaidi, W. Taha, A. Hazza, and P. Oguntunde, "Stability Analysis and Assortment of Exact Traveling Wave Solutions for the (2 + 1)-Dimensional Boiti-Leon-Pempinelli System", *Iraqi Journal of Science,* vol. 62, no. 7, pp. 2393–2400, 2021, doi: 10.24996/ijs.2021.62.7.29.

[2] M. Rasheed, R. Hameed, S. Obeid, and A. Jameel, "On Numerical Blow-Up Solutions of Semilinear Heat Equations", *Iraqi Journal of Science,* vol. 61, no. 8, pp. 2077–2086, 2020, doi: 10.24996/ijs.2020.61.8.

[3] M. G. S. AL-Safi, W. R. A. AL-Hussein, and R. M. Fawzi, " Numerical and Analytical Solutions of Space-Time Fractional Partial Differential Equations by Using a New Double Integral

Transform Method", *Iraqi Journal of Science*, vol. 64, No. 4, pp: 1935-1947, 2023, doi: 10.24996/ijs.2023.64.4.31.

[4] R. Redi, Y. Obsie, and A. Shiferaw, "The Improved (G'/G)-Expansion Method to the Generalized Burgers-Fisher Equation", *Mathematical Modelling and Applications*, vol. 3, no. 1, pp. 16–30, 2018, doi: 10.11648/j.mma.20180301.13.

[5] N. A. Mohamed, "Solving one and two-dimensional unsteady Burgers'equation using fullyimplicit finite difference schemes", *Arab Journal of Basic and Applied Sciences*, vol. 260, no. 1, pp. 254–268, 2019, doi:10.1080/25765299.2019.1613746.

[6] E. R. Fernandez, and R. Cuerno, "Gaussian statistics as an emergent symmetry of the stochastic scalar Burgers equation", *Physical Review*, vol. 99, no. 4, 2019, doi: 10.1103/PhysRevE.99.042108.

[7] N. A. Mohamed, "Fully Implicit Scheme for Solving Burgers' Equation based on Finite Difference Method", *The Egyptian International Journal of Engineering Sciences and Technology*, vol. 26, pp. 38–44, 2018.

[8] H. Eltayeb, and I. Bachar, "A note on singular two-dimensional fractional coupled Burgers' equation and triple Laplace Adomian decomposition method", *boundary value problems*, vol. 129, 2020, doi: 10.1186/s13661-020-01426-0.

[9] V. Kumar, S. Singh, and M. E. Koksal, "A Composite Algorithm for Numerical Solutions of Two-Dimensional Coupled Burgers Equations", *Journal of Mathematics*,vol. 2021, no. 1012, pp. 13, 2021, doi: 10.1155/2021/7240300.

[10] N. Murat, and A. Gagir, "Numerical Simulation of Two Dimensional Coupled Burgers Equations by Rubin-Graves Type Linearization", *Mathematical Sciences and Applications E-Notes*, vol. 9, no. 4, pp. 158–169, 2021.

[11] Y. Chai, and J. Ouyang, "Appropriate stabilized Galerkin approaches for solving two-dimensional coupled Burgers equations at high Reynolds numbers", *Computers and Mathmatics with Applications*, vol. 79, no. 5, 2019, doi: 10.1016/j.camwa.2019.08.036.

[12] R. Rohila, and R. C. Mittal, "A numerical study of two-dimensional coupled systems and higher order partial differential equations", *Asian-European Journal of Mathematics*, vol. 12, no. 5, pp. 1-12, 2019, doi: 10.1142/S1793557119500712.

[13] S. A. Ahmed, M. Elbadri, and M. Z. Mohamed, "A New Efficient Method for Solving Two-Dimensional Nonlinear System of Burger's Differential Equations", *Abstract and Applied Analysis*, PP. 1–7, 2020, doi: 10.1155/2020/7413859.

[14] N. Chuathong, and S. Kaennakham, "Numerical solution to coupled burgers' equations by Gaussian-based hermite collocation scheme", *Journal of Applied Mathematics*, pp. 1-18, 2018, doi: 10.1155/2018/3416860.

[15] V. Mukundan, A. Awasthi, and V. S. Aswin, "Multistep Methods for the Numerical Simulation of Two-Dimensional Burgers' Equation", *Differential Equations and Dynamical Systems*, vol. 30, no. 4, pp. 1-24, 2019, doi: 10.1007/s12591-019-00468-w.

[16] M. H. Al-Tai, and A. Al-Fayadh, "Solving Two Dimensional Coupled Burger's Equations and Sine-Gordon Equation Using El-Zaki Transform-Variational Iteration Method", *Al-NahrainJournal of Science*, vol. 24, no. 2, pp. 41–47, 2021, doi: 10.22401/anjs.24.2.07.

[17] S. A. Ojobor, and A. Obihia, "Modified Variational Iteration Method for Solving Nonlinear Partial Differential Equation Using Adomian Polynomials", *Mathematics and Statistics*, vol. 9, no. 4, pp. 456-464, 2021 doi: 10.13189/ms.2021.090406.

[18] A. Wazwaz, "Optical bright and dark soliton solutions for coupled nonlinear Schrodinger (CNLS) equations by the variational iteration method", *International Journal for Light and Electron Optics*, vol. 207, no. 2, 2020, doi: 10.1016/j.ijleo.2020.164457.

[19] M. Nadeem, and J. H. He, "He–Laplace variational iteration method for solving the nonlinear equations arising in chemical kinetics and population dynamics", *Journal of Mathematical Chemistry*, vol. 59, no. 5, pp. 1234–1245, 2021, doi: 10.1007/s10910-021-01236-4.

[20] H. Ahmad, and T. A. Khan, "Variational iteration algorithm-I with an auxiliary parameter for the solution of differential equations of motion for simple and damped mass – spring systems", *Noise and Vibration Worldwide*, vol. 51, no. 2, pp. 12–20, 2020, doi: 10.1177/0957456519889958.

[21] A. Aljuhani, D. Maturi, and H. Alshehri, "Variational Iteration Method for Solving Boussinesq Equations Using Maple", *Applied Mathematics*, vol. 13, no. 12, pp. 960-967, 2022, doi:

10.4236/am.2022.1312060.

[22] K. Deb, "An introduction to genetic algorithms", *Academy Proceedings in Engineering Sciences*, vol. 24, no. 4, pp. 293–315, 1999, doi: 10.1007/BF02823145.

[23] M. Sari, and C. Tuna, "Prediction of Pathological Subjects Using Genetic Algorithms", *Computational and Mathematical Methods in Medicine,* 2018, doi: 10.1155/2018/6154025.

[24] H. Ghaneai, and M. Hosseini, "Solving differential-algebraicequations through variational iteration method with an auxiliary parameter", *Applied Mathematical Modelling*, vol. 40, pp. 3991–4001, 2016, doi: 10.1016/j.apm.2015.10.002.

[25] Z. M. Odibat, "A study on the convergence of variational iteration method", *Mathematical and Computer Modelling*, vol. 51, pp. 1181–1192, 2010, doi: 10.1016/j.mcm.2009.12.034.

[26] V. Joshi, and M. Kapoor, "A Novel Technique for Numerical Approximation of 2 Dimensional Non-Linear Coupled Burgers Equations using Uniform Algebraic Hyperbolic (UAH) Tension B-Spline based Differential Quadrature Method", *Applied Mathematics and Information Sciences*, vol. 15, no. 2, pp. 217–239, 2021, doi: 10.18576/amis/150215.

[27] S. Katoch, S. S. Chauhan, and V. Kumar, "A review on genetic algorithm: past, present, and future", *Multimedia Tools and Applications*, vol. 80, no. 5, pp. 8091-8126, 2021, doi: 10.1007/s11042-020-10139-6.