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Majorization and Quasi-Subordination Problems for Certain Subclasses of Holomorphic Functions Utilizing Differential Operator

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Abstract
In the current paper, we define some subclasses of holomorphic functions in the open disk U that are defined by the differential operator. These subclasses are defined in the open disk U. We make some estimates for the bounds of the Fekete-Szego inequality as well as the coefficients $|a_2|$ and $|a_3|$ for functions that belong to these subclasses. Also, some of the results of these subclasses have been generalized, particularly those that involve the majorization and quasi-subordination concepts.

Keywords: Differential Operator, Majorization, Quasi-Subordination, Holomorphic Functions, and Fekete-Szego inequality.

مسائل الاخصية وشبه التابعية لفئات معينة من الدوال الهولومورفية باستخدام مؤثر تفاضلي

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الخلاصة
في هذه الورقة البحثية ، عرفنا بعض الفئات لدوال هولومورفية في قرص الوحدة المفتوح U والمعرفة بواسطة مؤثر تفاضلي ، لتقدير حدود متراجحة فيكيتي− سيزيجو ومعاملات $|a_2|$ و $|a_3|$ لهذه الدوال في هذه الفئات ، وكذلك عممنا بعض النتائج لهذه الفئات من ضمنها مفهومي الاخصية وشبه التابعية .

Introduction

Many authors examine the bounds of Fekete-Szego coefficient for various classes see [1] [2] The class of holomorphic functions $f(z)$ in an open unit disk $U = \{z: |z| < 1\}$, standardized by $f(0) = 0$ and $f'(0) = 1$ is represented by $\mathcal{A}$ and given by

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n. \quad (1)$$

Assume f and g be two holomorphic functions in U. If there exists a Schwarz function $w(z)$ that is univalent in U with $w(0) = 0$, and $|w(z)| < |z|, z \in U$ such that $f(z) = g(w(z))$, then we say that f is a subordinate to g in U and represented as $f(z) \prec g(z), (z \in U)$. Moreover, if $g(z)$ is holomorphic in U, then $f(z) \prec g(z)$ is equivalent to $f(0) = g(0)$ and $f(U) \subset g(U)$. In 1970, Robertson [3], introduced the notion of quasi-subordination of two holomorphic functions, f is called quasi-subordinate to g in U, written as $f(z) \prec_q g(z)$, if

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there exists holomorphic functions φ and w, with $|\varphi(z)| \leq 1$, $w(0)=0$ and $|w(z)|<1$ such that $f(z)=\varphi(z)g\big(w(z)\big)$, ($z \in U$). Observe that if $\varphi(z)=1$, then $f(z) \prec_q g(z)$ becomes $f(z) \prec g(z)$, also if $w(z)=z$, then $f(z) \prec_q g(z)$ becomes majoraization $' \prec\prec '$, written as $f(z) \prec\prec g(z)$ in U, the notion of majoraization is due to MacGeqr [4]. Therefore, quasi-subordination generalizes of both usual subordination and majoraization.
Throughout this paper, let $f(z)=z+a_2z^2+a_3z^3+\cdots$. Also, let the function $\varphi(z)$ holomorphic in U given by:

$$\varphi(z)=C_0+C_1z+C_2z^2+\cdots \; . \tag{2}$$

Where $C_0, C_1, C_2, \ldots \in \mathbb{C}$, and ϕ be of the form:

$$\phi(z)=1+B_1z+B_2z^2+\cdots \tag{3}$$

Where $B_1, B_2, \ldots \in \mathbb{R}$ and $B_1>0$.
The purpose of this present paper is to integrate a new results by applying the Frasin differential operator on some subclasses.

1. Preliminary Results

Definition 1. [5]. A function $f \in \mathfrak{R}_q(\tau,\xi,\phi)$, $\xi \in \mathbb{C}-\{0\}$, $\tau \geq 0$, if

$$\frac{1}{\xi}\left(f'(z)+\tau z f''(z)-1\right) \prec_q \left(\phi(z)-1\right), z \in U$$

Definition 2. [5]. A function $f \in \mathfrak{L}_q(\tau,\xi,\phi)$, $\xi \in \mathbb{C}-\{0\}$, $\tau \geq 0$, if

$$1+\frac{\tau}{\xi}\left(\frac{zf''(z)}{f'(z)}\right) \prec_q \phi(z), z \in U$$

Definition 3. [5]. A function $f \in M_q(\tau,\zeta,\xi,\phi)$, $\xi \in \mathbb{C}-\{0\}$, $0 \leq \zeta \leq 1$ $\tau \geq 0$, if

$$\frac{1}{\xi}\left(\frac{zf'(z)+\tau z^2 f''(z)}{(1-\zeta)z+\zeta z f'(z)}-1\right) \prec_q \left(\phi(z)-1\right), z \in U$$

A function f is in $M_q(\tau,\zeta,\xi,\phi)$ if and only if there exists a holomorphic function $\phi(z)$ with $|\phi(z)| \leq 1$, $z \in U$ such that

$$\frac{\frac{1}{\xi}\left(\frac{zf'(z)+\tau z^2 f''(z)}{(1-\zeta)z+\zeta z f'(z)}-1\right)}{\phi(z)} \prec \left(\phi(z)-1\right), z \in U \, .$$

Note that
$i)$ $M_q(\tau,0,\xi,\phi)=\mathfrak{R}_q(\tau,\xi,\phi)$.
$ii)$ $M_q(\tau,1,\xi,\phi)=\mathfrak{L}_q(\tau,\xi,\phi)$.
$iii)$ $M_q(\zeta,\zeta,\xi,\phi)$ was investigated by Kant and Vyas [6].
Frasin [7] introduced the differential operator $\mathcal{D}^{\alpha}_{k,\mu}f(z)$ which defined as

$$\mathcal{D}^0 f(z)=f(z).$$
$$\mathcal{D}^1_{k,\mu}f(z)=(1-\mu)^k f(z)+\left(1-(1-\mu)^k\right)zf'(z).$$
$$\mathcal{D}^{\alpha}_{k,\mu}f(z)=\mathcal{D}_{k,\mu}\left(\mathcal{D}^{\alpha-1}_{k,\mu}f(z)\right).$$

Where $\alpha \in \mathbb{N} \cup \{0\}$, $z \in \mathbb{C}$, $\mu \in \mathbb{R}$, $f \in \mathcal{A}$ and $k \in \mathbb{N}$, then we have

$$\mathcal{D}^{\alpha}_{k,\mu}f(z)=z+\sum_{n=2}^{\infty}\left(1+(n-1)\sum_{d=1}^{m}\binom{m}{d}(-1)^{d+1}\mu^d\right)^{k} a_n z^n.$$

Where $\mu>0$, $m \in \mathbb{N}$ and $C^m_d(\mu) := \sum_{d=1}^{m}\binom{m}{d}(-1)^{d+1}\mu^d$.
Note that, when $m=1$, we get the Al-Oboudi Differential Operator [8]. Also, when $m=\mu=1$, we get the Salagean Operator [9].
In our work, we require the next two lemmas to prove our major results.

Lemma 4. [10]. The Schwarz function $w(z)$ are given via

$$w(z) = w_1 z + w_2 z^2 + w_3 z^3 + \cdots, \tag{4}$$

then for each $t, z, w, w_2, w_3, \ldots \in \mathbb{C}$ we have

$|w| \leq 1$, and $|w_2 - t w_1^2| \leq 1 + (|t| - 1)|w_1|^2 \leq \max\{1, |t|\}$.

For the functions $w(z) = z^2$ or $w(z) = z$ the result is sharp.

Lemma 5. [10]. Assume $\varphi(z)$ be analytic function in U, with $|\varphi(z)| < 1$ and let $\varphi(z) = C_0 + C_1 z + C_2 z^2 + \cdots$.

Then $|C_0| \leq 1$ and for $n > 0$, $|C_n| \leq 1 - |C_0|^2 \leq 1$.

2. Main Results

Theorem 1. If $\in M_q(\tau, \zeta, \xi, \phi)$, then

$$|a_2| \leq \frac{|\xi| B_1}{2(1+\tau-\zeta)\left(1+C_d^m(\mu)\right)^{k}}. \tag{5}$$

$$|a_3| \leq \frac{|\xi| B_1}{3(1+2\tau-\zeta)\left(1+2C_d^m(\mu)\right)^{k}} max\left\{1, \left|-\frac{\zeta\xi B_1}{(1+\tau-\zeta)} - \frac{B_2}{B_1}\right|\right\},$$

and for any complex number $t \in \mathbb{C}$,

$$|a_3 - t a_2^2| \leq \frac{|\xi| B_1}{3(1+2\tau-\zeta)\left(1+2C_d^m(\mu)\right)^{k}} max\left\{1, \left|\mathcal{J} B_1 - \frac{B_2}{B_1}\right|\right\}, \tag{6}$$

where

$$\mathcal{J} = \xi\left(\frac{3t(1+2\tau-\zeta)\left(1+2C_d^m(\mu)\right)^{k}}{4(1+\tau-\zeta)^2\left(1+C_d^m(\mu)\right)^{2k}} - \frac{\zeta}{(1+\tau-\zeta)}\right). \tag{7}$$

Proof: If $\mathcal{A} \in M_q(\tau, \zeta, \xi, \phi)$, then there is a holomorphic functions φ and w with $|\varphi(z)| \leq 1$, $w(0) = 0$ and $|w(z)| < 1$ such that

$$\frac{1}{\xi}\left(\frac{z\left(\mathcal{D}_{k,\mu}^{\alpha} f(z)\right)' + \tau z^2\left(\mathcal{D}_{k,\mu}^{\alpha} f(z)\right)''}{(1-\zeta)z + \zeta z\left(\mathcal{D}_{k,\mu}^{\alpha} f(z)\right)'} - 1\right) = \varphi(z)\big(\phi(w(z)) - 1\big), z \in U. \tag{8}$$

Series expansions of $\mathcal{D}_{k,\mu}^{\alpha} f(z)$ and its sequential derivatives as of (1) gives

$$\frac{1}{\xi}\left(\frac{z\left(\mathcal{D}_{k,\mu}^{\alpha} f(z)\right)' + \tau z^2\left(\mathcal{D}_{k,\mu}^{\alpha} f(z)\right)''}{(1-\zeta)z + \zeta z\left(\mathcal{D}_{k,\mu}^{\alpha} f(z)\right)'} - 1\right) =$$

$$\frac{1}{\xi}\left(\frac{z\left(1+\sum_{n=2}^{\infty} n\left(1+(n-1)C_d^m(\mu)\right)^{k} a_n z^{n-1}\right) + \tau z^2\left(\sum_{n=2}^{\infty}(n-1)n\left(1+(n-1)C_d^m(\mu)\right)^{k} a_n z^{n-2}\right)}{(1-\zeta)z + \zeta z\left(1+\sum_{n=2}^{\infty} n\left(1+(n-1)C_d^m(\mu)\right)^{k} a_n z^{n-1}\right)} - 1\right) =$$

$$\frac{1}{\xi}\left(\frac{z + 2(1+\tau)\left(1+C_d^m(\mu)\right)^{k} a_2 z^2 + 3(1+2\tau)\left(1+2C_d^m(\mu)\right)^{k} a_3 z^3 + \cdots}{z + 2\zeta\left(1+C_d^m(\mu)\right)^{k} a_2 z^2 + 3\zeta\left(1+2C_d^m(\mu)\right)^{k} a_3 z^3 + \cdots} - 1\right) = \frac{1}{\xi}\Big(2(1+\tau-\zeta)\big(1 + C_d^m(\mu)\big)^{k} a_2 z + 3(1+2\tau-\zeta)\big(1+2C_d^m(\mu)\big)^{k} a_3 z^2 + \cdots\Big). \tag{9}$$

In the same way as of (2), (3) and (4), we get

$$\varphi(z)\big(\phi(w(z)) - 1\big) = B_1 c_0 w_1 z + \big(B_1 c_1 w_1 + c_0(B_1 w_2 + B_2 w_1^2)\big) z^2 + \cdots. \tag{10}$$

Making use of (9) and (10) in (8), we get

$$a_2 = \frac{\xi c_0 B_1 w_1}{2(1+\tau-\zeta)\left(1+C_d^m(\mu)\right)^{k}}, \tag{11}$$

and

$$a_3 = \frac{\xi B_1}{3(1+2\tau-\zeta)\left(1+2C_d^m(\mu)\right)^{k}}\left[c_1 w_1 + c_0\left\{w_2 + \left(\frac{\zeta\xi c_0 B_1}{1+\tau-\zeta} + \frac{B_2}{B_1}\right)\right\} w_1^2\right]. \tag{12}$$

Thus, for any $t \in \mathbb{C}$, we get

$$a_3 - t a_2^2 = \frac{\xi B_1}{3(1+2\tau-\zeta)\left(1+2C_d^m(\mu)\right)^{k}}\left[c_1 w_1 + c_0\left(w_2 + \frac{B_2}{B_1}w_1^2\right) - \mathcal{J} B_1 c_0^2 w_1^2\right]. \quad (13)$$

Where $\mathcal{J}$ is as stated in (7).

Since $\phi(z)$ holomorphic and bounded via one in U. We have

$$|c_0| \le 1 \quad \text{and} \quad c_1 = (1-c_0^2)x \qquad x \le 1. \quad (14)$$

The statement (5) shadows as of (6) utilizing (7) and Lemma 5. As of (13) and (14), we get

$$a_3 - t a_2^2 = \frac{\xi B_1}{3(1+2\tau-\zeta)\left(1+2C_d^m(\mu)\right)^{k}}\left[x w_1 + c_0\left(w_2 + \frac{B_2}{B_1}w_1^2\right) - c_0^2(\mathcal{J} B_1 w_1^2 + x w_1)\right]. \quad (15)$$

If $c_0 = 0$, then (15) yields

$$|a_3 - t a_2^2| = \frac{|\xi| B_1}{3(1+2\tau-\zeta)\left(1+2C_d^m(\mu)\right)^{k}}. \quad (16)$$

In additional, if $c_0 \ne 0$, we define function

$$L(c_0) = x w_1 + c_0\left(w_2 + \frac{B_2}{B_1}w_1^2\right) - c_0^2(\mathcal{J} B_1 w_1^2 + x w_1). \quad (17)$$

The equation (17) is quadratic in c_0 and thus holomorphic in $|c_0| \le 1$. Obviously, $|L(c_0)|$ makes its maximum value at $c_0 = e^{i\theta}$, $0 \le \theta \le 2\pi$. Thus

$$max\,|L(c_0)| = \max_{0\le\theta\le 2\pi} \left|L\left(e^{i\theta}\right)\right| = |L(1)| = \left|w_2 - \left(\mathcal{J} B_1 - \frac{B_2}{B_1}\right)w_1^2\right|.$$

It follows from (15) that

$$|a_3 - t a_2^2| \le \frac{|\xi| B_1}{3(1+2\tau-\zeta)\left(1+2C_d^m(\mu)\right)^{k}}\left|w_2 - \left(\mathcal{J} B_1 - \frac{B_2}{B_1}\right)w_1^2\right|. \quad (18)$$

By virtue of Lemma 5, we obtain

$$|a_3 - t a_2^2| \le \frac{|\xi| B_1}{3(1+2\tau-\zeta)\left(1+2C_d^m(\mu)\right)^{k}} max\left\{1, \left|\mathcal{J} B_1 - \frac{B_2}{B_1}\right|\right\}. \quad (19)$$

This ends the proof.

We determine the below Corollary for the class $\mathfrak{R}_q(\tau, \xi, \phi)$ by putting $\zeta = 0$ in Theorem 1

Corollary 2. Assume $\xi \in \mathbb{C} - \{0\}$, $\tau \ge 0$. If $f \in \mathfrak{R}_q(\tau, \xi, \phi)$, then

$$|a_2| \le \frac{B_1|\xi|}{2(1+\tau)(1+C_d^m(\mu))^{k}}.$$

$$|a_3| \le \frac{B_1|\xi|}{3(1+2\tau)(1+2C_d^m(\mu))^{k}}\left(max\left\{1, \frac{|B_2|}{B_1}\right\}\right).$$

And for some $t \in \mathbb{C}$

$$|a_3 - t a_2^2| \le \frac{B_1|\xi|}{3(1+2\tau)(1+2C_d^m(\mu))^{k}} max\left\{1, \left|\frac{3t|\xi|(1+2\tau)\left(1+2C_d^m(\mu)\right)^{k} B_1}{4(1+\tau)^2\left(1+C_d^m(\mu)\right)^{2k}} - \frac{B_2}{B_1}\right|\right\}.$$

Remark 3. For $k = 0$, Corollary 2 reduces to Corollary 2.3 of [5].

We determine the below Corollary for the class $\mathfrak{R}_q(\tau, \xi, \phi)$ by setting $\zeta = 1$ in Theorem 1.

Corollary 4. Assume $\xi \in \mathbb{C} - \{0\}$, $\tau \ge 0$. If $f \in \mathfrak{L}_q(\tau, \xi, \phi)$, then

$$|a_2| \le \frac{B_1|\xi|}{2\tau(1+C_d^m(\mu))^{k}}.$$

$$|a_3| \le \frac{B_1|\xi|}{6\tau(1+2C_d^m(\mu))^{k}} max\left\{1, \left|-\frac{\xi B_1}{\tau} - \frac{B_2}{B_1}\right|\right\}.$$

And for some $t \in \mathbb{C}$

$$|a_3 - t a_2^2| \le \frac{B_1|\xi|}{6\tau(1+2C_d^m(\mu))^{k}} max\left\{1, \left|\left(\frac{6t\xi\left(1+2C_d^m(\mu)\right)^{k}}{4\tau\left(1+C_d^m(\mu)\right)^{2k}} - \frac{\xi}{\tau}\right)B_1 - \frac{B_2}{B_1}\right|\right\}.$$

Remark 5. For $\mathscr{k} = 0$, Corollary 4 reduces to Corollary2.5 of [5].
Theorem 6. If $f \in M_q(\tau, \zeta, \xi, \phi)$, then
$|a_2| \leq \frac{|\xi|B_1}{2(1+\tau-\zeta)\left(1+C_d^m(\mu)\right)^{\mathscr{k}}}$.

$$|a_3| \leq \frac{|\xi|B_1}{3(1+2\tau-\zeta)\left(1+2C_d^m(\mu)\right)^{\mathscr{k}}}\left(1+\frac{|B_2|}{B_1}+\frac{\zeta|\xi|B_1}{1+\tau-\zeta}\right).$$

And for any complex number $t \in \mathbb{C}$,

$$|a_3 - ta_2^2| \leq \frac{|\xi|B_1}{3(1+2\tau-\zeta)\left(1+2C_d^m(\mu)\right)^{\mathscr{k}}}\left(1+\frac{|B_2|}{B_1}+\frac{\zeta|\xi|B_1}{1+\tau-\zeta}+|t|\frac{3|\xi|(1+2\tau-\zeta)\left(1+2C_d^m(\mu)\right)^{\mathscr{k}}B_1}{4(1+\tau-\zeta)^2\left(1+C_d^m(\mu)\right)^{2\mathscr{k}}}\right).$$

Proof. By taking $w(U) = z$ in the proof of Theorem 1 we get this result.
We determine the below Corollary for the class $\mathfrak{R}_q(\tau, \xi, \phi)$ by setting $\zeta = 0$ in Theorem 6.

Corollary 7. If $f \in \mathcal{A}$ satisfies
$\frac{1}{\xi}\left(\left(\mathcal{D}_{\mathscr{k},\mu}^{\alpha} f(z)\right)' + \tau z\left(\mathcal{D}_{\mathscr{k},\mu}^{\alpha} f(z)\right)'' - 1\right) \ll (\phi(z) - 1).$
Then the next inequalities hold:
$|a_2| \leq \frac{|\xi|B_1}{2(1+\tau)(1+C_d^m(\mu))^{\mathscr{k}}}$.
$|a_3| \leq \frac{|\xi|(B_1+|B_2|)}{3(1+2\tau)(1+2C_d^m(\mu))^{\mathscr{k}}}$.
And for any $t \in \mathbb{C}$
$|a_3 - ta_2^2| \leq \frac{|\xi|(B_1+|B_2|)}{3(1+2\tau)\left(1+2C_d^m(\mu)\right)^{\mathscr{k}}} + \frac{{B_1}^2|t||\xi|^2}{(2+2\tau)^2(1+C_d^m(\mu))^{2\mathscr{k}}}$.
We determine the below Corollary for the class $\mathfrak{R}_q(\tau, \xi, \phi)$ by setting $\zeta = 1$ in Theorem 6.

Corollary 8. If $f \in \mathcal{A}$ satisfies

$$1 + \frac{\tau}{\xi}\left(\frac{z\left(\mathcal{D}_{\mathscr{k},\mu}^{\alpha} f(z)\right)''}{\left(\mathcal{D}_{\mathscr{k},\mu}^{\alpha} f(z)\right)'}\right) \ll \phi(z).$$

Then the following inequalities hold:
$|a_2| \leq \frac{|\xi|B_1}{2\tau(1+C_d^m(\mu))^{\mathscr{k}}}$.
$|a_3| \leq \frac{|\xi|B_1}{6\tau(1+2C_d^m(\mu))^{\mathscr{k}}}\left(1+\frac{|B_2|}{B_1}+\frac{|\xi|B_1}{\tau}\right)$.
And for any $t \in \mathbb{C}$

$$|a_3 - ta_2^2| \leq \frac{|\xi|B_1}{6\tau\left(1+2C_d^m(\mu)\right)^{\mathscr{k}}}\left(1+\frac{|B_2|}{B_1}+\frac{|\xi|B_1}{\tau}+|t|\frac{6\tau|\xi|\left(1+2C_d^m(\mu)\right)^{\mathscr{k}}B_1}{4\tau^2\left(1+C_d^m(\mu)\right)^{2\mathscr{k}}}\right).$$

3. Conclusions

The study of quasi-subordination of holomorphic functions has very interesting and useful applications in geometric functions. Therefore, in this work, we have estimate majorization and quasi-subordination problems for some subclasses which is defined by differential operator. Furthermore, many new consequences of these problems are mentioned. We wish our results will be useful for the future studies in complex analysis. As a future studies, several results can be obtained by applying the convolution (or Hadamard product) of our holomorphic functions.

References

[1] AS. Juma and M.H. Saloomi, "Coefficient Bounds for Certain Subclass of Analytic Functions Defined By Quasi-Subordination," *Iraqi Journal of Science*, vol.59, no. 2C, pp.1115-1121, (2018).

[2] A.S. Juma and N.S. Shehab, "Application of Quasi Subordination Associated with Generalized Sakaguchi Type Functions," *Iraqi Journal of Science*, vol.62, no .12, pp. 4885-4891,(2021).

[3] M. S. Robertson, " Quasi-subordination and coefficient conjectures, " *Bulletin of the American Mathematical Society*, vol. 76, pp. 1-9,(1970).

[4] T.H. MacGregor Majorization by univalent functions," *Duke Mathematical Journa,* vol. 34, pp. 95-102, (1967).

[5] S.R. Swamy and Y.Sailaja," On the Fekete-Szegö coefficient functional for quasi-subordination class," *Palestine Journal of Mathematics,* vol. 10, no.2, pp. 666–672,(2021).

[6] S. Kant and P. P. Vyas," Sharp bounds of Fekete-Szegö functional for quasi-subordination class," *Acta Universitatis Sapientiae, Informatica, Math,* vol. 11, no. 1, pp. 87–98, (2019).

[7] B. A. Frasin,"A new differential operator of analytic functions involving binomial series," *Boletim da Sociedade Paranaense de Matemática*. vol. 38, no. 5, pp. 205-213, (2020).

[8] F. M. Al-Oboudi, "On univalent functions defined by a generalized Salagean operator," *International Journal of Mathematics and Mathematical Sciences,* pp.1429-1436, (2004).

[9] G. S. Salagean, "Subclasses of Univalent functions," *Lecture Notes in Math., Spinger Verlag, Berlin,* vol. 1013, pp. 362–372,(1983).

[10] F. R. Keogh and E. P. Merkes, "A coefficient inequality for certain classes of analytic functions," *Proceedings of the American Mathematical Society,* vol. 20, pp. 8–12,(1969).