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Towards Solving Fractional Order Delay Variational Problems Using Euler Polynomial Operational Matrices

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Abstract

In this paper, we introduce an approximate method for solving fractional order delay variational problems using fractional Euler polynomials operational matrices. For this purpose, the operational matrices of fractional integrals and derivatives are designed for Euler polynomials. Furthermore, the delay term in the considered functional is also decomposed in terms of the operational matrix of the fractional Euler polynomials. It is applied and substituted together with the other matrices of the fractional integral and derivative into the suggested functional. The main equations are then reduced to a system of algebraic equations. Therefore, the desired solution to the original variational problem is obtained by solving the resulting system. Error analysis has been discussed. An illustrative example is given in order to illustrate that the proposed method is very accurate and efficient for solving such kinds of problems.

Keywords: Calculus of variation, fractional order derivatives, operational matrix, Euler polynomials, fractional order Euler functions.

نحو حل مسائل التغاير التباطؤية ذات الرتب الكسرية بأستخدام مصفوفات العمليات لمتعددة حدود أويلر

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الخلاصة

في هذا البحث نقدم طريقة تقريبية لحل مسائل التغاير التباطؤية ذات الرتب الكسرية بأستخدام مصفوفات العمليات لمتعددات حدود أويلر الكسرية، ولهذا الغرض تم تصميم مصفوفات العمليات للتكامل الكسري والاشتقاق لمتعددات حدود أويلر . علاوة على ذلك، فأن حد التباطؤ في الدالي المفترض كذلك يجزء لحدود من مصفوفة العمليات لمتعددات حدود أويلر الكسرية ليتم تطبيقها وتعويضها مع بقية مصفوفات التكامل الكسري والاشتقاق في الدالي المقترح. يتم بعد ذلك تحويل المعادلات الرئيسية الى نظام من المعادلات الجبرية، وبالتالي يتم الحصول على الحل المطلوب لمسألة التغاير الاصلية عن طريق حل النظام الناتج. تمت مناقشة تحليل الخطأ وتم تقديم مثال توضيحي لتوضيح ان الطريقة المقترحة دقيقة للغاية وفعالة لحل مثل هذا النوع من المسائل.

1. Introduction

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Fractional calculus is one of the most interdisciplinary fields of applied mathematics which deals with the derivative and integrals of any order. Nowadays, it is used to advance mathematical models of real-world phenomena in various areas of science and engineering [1-3]. Fractional order derivatives are naturally related to the systems with memory that dominate most of the scientific system models. Models and applications containing fractional derivatives can be found in chemical physics, probability physics, astrophysics, and various fields of engineering [4-6]. There are many definitions of a fractional derivative, The commonly known fractional derivatives are the classical Riemann-Liouville and Caputo derivative. Fractional derivatives and integrals of these Riemann-Liouville and Caputo types have a huge number of applications in many fields of science and engineering [7-11].

The calculus of variation has a long history of communications with other fields of mathematics such as differential equations, geometry and with physics. However, the calculus of variation has found applications in economics and some branches of engineering [12-14]. A fractional calculus of variations problem is a problem in which either the objective functional or the constraint equations or both consist of at least one fractional derivative term. In recent years, many numerical and approximate methods have been used to solve fractional order problems such as the homotopy analysis method, variational iteration method, homotopy perturbation method, wavelet method, collocation method, spectral tau method, finite element method and other methods, see [15-21]. Recently, many researchers used different functions and polynomials. For some orthogonal polynomials, the operational matrices of fractional integrals and derivatives have been derived such as Bernstein polynomials, the Legendre polynomials, Jacobi polynomials, Chebyshev polynomials and Laguerre polynomials [22-26]. Inclusion of delay in the fractional order variational problems seems to be opening new vistas, especially in the field of bioengineering [27].

In this paper, the fractional order Euler functions based on Euler polynomials are used for solving fractional order delay variational problems. The operational matrix is derived for the fractional integration. By using the operational matrix of fractional integration and the fractional order Euler functions, we convert the varational problem into a system of linear algebraic equations. Numerical solutions are obtained by solving this linear system. By comparing the exact solution with the numerical solution using the proposed method, we exhibit the precision and efficiency of the proposed technique for various values of α.

2. Preliminaries and notations

In this section, we introduce some basic definitions of fractional calculus, namely the definition of Riemann-Liouville fractional order integral and Caputo fractional order derivative [28].

Definition 1: The Riemann-Liouville fractional integral operator I^{α} of order α of a function $f \in C_{\mu}$ and $\mu \geq -1$ is defined as follows:

$$I^{\alpha} f(t) = \begin{cases} \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f(\tau) d\tau, & \alpha > 0, \tau > 0 \\ f(t), & \alpha = 0 \end{cases} \tag{1}$$

where $\Gamma(\alpha)$ is the Gamma function.

For the Riemann-Liouville fractional integral, we have:

1- $I^{\alpha}\big(\lambda_1 f(t) + \lambda_2 g(t)\big) = \lambda_1 I^{\alpha} f(t) + \lambda_2 I^{\alpha} g(t)$, λ_1 and λ_2 are constants.

2- $I^{\alpha 1} I^{\alpha 2} f(t) = I^{\alpha 1+\alpha 2} f(t)$.
3- $I^{\alpha 1} I^{\alpha 2} f(t) = I^{\alpha 2} I^{\alpha 1} f(t)$.
4- $I^{\alpha} t^{\beta} = \frac{\Gamma(\beta+1)}{\Gamma(\beta+\alpha+1)} t^{\alpha+\beta}$, $\beta > -1$.

Definition 2: The Caputo fractional derivative of order α of a function $f \in C_\mu^m$ and $\mu \geq -1$, is defined as follows:

$$ {}_0^cD_t^\alpha f(t) = \begin{cases} \frac{1}{\Gamma(m-\alpha)} \int_0^t \frac{f^{(m)}(\tau)}{(t-\tau)^{\alpha+1-m}} d\tau, & m-1 < \alpha < m, \tau > 0, m \in N \\ \frac{d^m}{dt^m} f(t) & \alpha = m. \end{cases} \quad (2)$$

For the Caputo fractional derivative and $\alpha > 0$, we have:
1- ${}_0^cD_t^\alpha \left(I^\alpha f(t)\right) = f(t)$, $t > 0$
2- $I^\alpha \left({}_0^cD_t^\alpha f(t)\right) = f(t) - \sum_{k=0}^{n-1} f^{(k)}(0_+) \frac{t^k}{k!}$,
3- ${}_0^cD_t^\alpha (c) = 0$, $c \in R$
4- ${}_0^cD_t^\alpha \left(t^\beta\right) = \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)} t^{\beta-\alpha}$, $\beta > -1$.

3. Fractional order Euler functions

In this section, we give some definitions and basic properties of the Euler polynomials that are used in this paper [29].

3.1 Euler polynomials

The Euler polynomials basis of degree n which is denoted by $E_n(t)$ are introduced by using the following relation:

$$\sum_{k=0}^{n} \binom{n}{k} E_k(t) + E_n(t) = 2t^n, \quad (3)$$

where $\binom{n}{k}$ is a binomial coefficient, and the Euler vector $E(t)$ is defined as follows:

$$E(t) = [E_0(t), E_1(t), \ldots, E_n(t)]^T. \quad (4)$$

Then, using Eq. (3), we can write $E(t)$ as follows:

$$E(t) = A\, T(t) \quad (5)$$

where $T(t) = [1, t, t^2, \ldots, t^n]$ and A is an upper triangular matrix with non-zero diagonal elements, thus it is a non-singular matrix and then A^{-1} exists.

3.2 Euler polynomials Properties

The Euler polynomials have the following interesting properties for all $n = 1,2, \ldots$
1- $E_n'(t) = nE_{n-1}(t)$.
2- $E_{n+1}(t+1) + E_n(t) = 2t^n$.
3- $\int_0^1 E_n(t)dt = -\frac{2E_{n+1}(t)}{n+1}$.
4- $E_n(t) = \frac{1}{n+1} \sum_{k=1}^{n+1} (2 - 2^{k+1}) \binom{n+1}{k} B_k(0)\, t^{n+1-k}$,

where $B_k(t)$ are the Bernoulli polynomials of order k for , $k = 0,1, \ldots$, which are defined as follows:

$$\sum_{k=0}^{n} \binom{n+1}{k} B_k(t) = (n+1)t^n . \quad (6)$$

Using the last property, the Euler polynomials can be expressed in the following matrix form:

$$E(t) = A\, T(t)$$

where

$$A=\begin{bmatrix} \frac{(2-2^2)}{1}\binom{1}{1}B_1(0) & 0 & \cdots & 0 \\ \frac{(2-2^3)}{2}\binom{2}{2}B_2(0) & \frac{(2-2^2)}{2}\binom{2}{1}B_1(0) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \frac{(2-2^{n+2})}{n+1}\binom{n+1}{n+1}B_{n+1}(0) & \frac{(2-2^{n+1})}{n+1}\binom{n+1}{n}B_n(0) & \cdots & \frac{(2-2^2)}{n+1}\binom{n+1}{1}B_1(0) \end{bmatrix}$$

We can write the following relation by means of property (1):

$$E'(t)^T = ME(t)^T ,$$

where

$$M=\begin{bmatrix} 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 2 & 0 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & n-1 & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & n & 0 \end{bmatrix}$$

Such that M is the operational matrix of differentiation. Hence, the k^{th} derivative of $E(t)$ can be constructed as [30]:

$$E^{(k)}(t)^T = (M^T)^k E(t)^T \tag{7}$$

Also, the Euler polynomials satisfy the following formula:

$$\int_0^1 E_m(t)E_n(t)dt = (-1)^{n-1}\frac{m!(n+1)!}{(m+n+1)!}E_{m+n+1}(0), \qquad m,n \geq 1 \tag{8}$$

Hence, they are complete basis over the interval [0,1].

3.3 Formulation of fractional order Euler functions

The fractional order Euler functions are constructed by replacing the variable t by $x^\alpha, (\alpha > 0)$ in the Euler polynomials. Let the fractional order Euler functions $E_m(x^\alpha)$ be the basis of degree m and denoted by $E_m^\alpha(x)$, then by using Eq. (3) we get:

$$\sum_{k=0}^{m}\binom{m}{k}E_k^\alpha(x) + E_m^\alpha(x) = 2x^{m\alpha} \tag{9}$$

and,

$$E^\alpha(x) = A_\alpha\, T^\alpha(x) \tag{10}$$

where $E^\alpha(x) = [E_0^\alpha(x), E_1^\alpha(x), \dots, E_m^\alpha(x)]$, $T^\alpha(x) = [1, x^\alpha, \dots, x^{m\alpha}]$,

$$A_\alpha=\begin{bmatrix} \frac{(2-2^2)}{1}\binom{1}{1}B_1(0) & 0 & \cdots & 0 \\ \frac{(2-2^3)}{2}\binom{2}{2}B_2(0) & \frac{(2-2^2)}{2}\binom{2}{1}B_1(0) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \frac{(2-2^{m+2})}{m+1}\binom{m+1}{m+1}B_{m+1}(0) & \frac{(2-2^{m+1})}{m+1}\binom{m+1}{m}B_m(0) & \cdots & \frac{(2-2^2)}{m+1}\binom{m+1}{1}B_1(0) \end{bmatrix}$$

And the first fractional order Euler functions are given by:

$E_0^\alpha(x) = 1,$

$E_1^\alpha(x) = x^{\boldsymbol{\alpha}} - \frac{\mathbf{1}}{2},$

$E_2^\alpha(x) = x^{\boldsymbol{2\alpha}} - x^\alpha,$

$E_3^\alpha(x) = x^{\boldsymbol{3\alpha}} - \frac{\mathbf{3}}{\mathbf{2}}x^{\boldsymbol{2\alpha}} + \frac{\mathbf{1}}{\mathbf{4}},$

and so on.

Moreover, the fractional order Euler functions satisfy the following formula:

$$\int_0^1 E_m^\alpha(x)\, E_n^\alpha(x)\, x^{1-\alpha}dx = (-1)^{n-1}\frac{m!(n+1)!}{(m+n+1)!}E_{m+n+1}(0), \quad m,n \geq 1. \tag{11}$$

Therefore, the fractional order Euler functions are complete basis over the interval [0,1].

3.4 Fractional order Euler functions approximation

A function $\gamma(x)$ which is square integrable in [0,1] can be expanded as :

$$\gamma(x) = \sum_{i=0}^{\infty} c_i E_i^{\alpha}(x) \simeq \sum_{i=0}^{m} c_i E_i^{\alpha}(x) = C^T E^{\alpha}(x) \quad (12)$$

where

$$E^{\alpha}(x) = [E_0^{\alpha}(x), E_1^{\alpha}(x), \dots, E_m^{\alpha}(x)]^T, \; C = [c_0, c_1, \dots, c_m]^T$$

Now, to evaluate c_i, we take

$$\int_0^1 E_j^{\alpha}(x)\,\gamma(x) x^{1-\alpha} dx = \sum_{i=0}^{m} c_i \int_0^1 E_i^{\alpha}(x)\, E_j^{\alpha}(t)\, x^{1-\alpha} dx\,,\; i = j = 0,1,2,\dots,m$$

then $a_j = \sum_{i=0}^{m} c_i b_{ij}^{\alpha}$,

where $a_j = \int_0^1 E_j^{\alpha}(x)\,\gamma(x) x^{1-\alpha} dx$ and $b_{ij}^{\alpha} = \int_0^1 E_i^{\alpha}(x)\, E_j^{\alpha}(t)\, x^{1-\alpha} dx$

Therefore,

$$A^T = C^T B \quad (13)$$

with $A^T = [a_0, a_1, \dots, a_m]^T$, $B = [b_{ij}^{\alpha}]$ is $(m+1) \times (m+1)$ matrix,

such that $B = \int_0^1 E^{\alpha}(x)\, E^{\alpha}(x)^T\, x^{\alpha-1} dx$. Then we can compute the matrix B by using Eq. (11).

4. The operational matrix of the fractional integration

The fractional Riemann-Liouville integration of order α of the vector $E^{\boldsymbol{\alpha}}(x)$ that is defined in (12) can be expressed by the following [31]:

$$I^{\beta} E^{\alpha}(x) = P_{\alpha}\, E^{\alpha}(x) \quad (14)$$

where P_{α} is the Riemann-Liouville fractional operational matrix of integration. We apply the properties of the operator I^{β} and by Eq. (10), we have:

$$I^{\beta} E^{\boldsymbol{\alpha}}(x) = I^{\beta} A_{\alpha} T^{\boldsymbol{\alpha}}(x) = A_{\alpha} I^{\beta}\; T^{\boldsymbol{\alpha}}(x) \quad (15)$$

then $I^{\beta} T^{\boldsymbol{\alpha}}(x) = I^{\beta}[1, x^{\alpha}, x^{2\alpha}, \dots, x^{m\alpha}]^T$

$$= [\frac{\Gamma(1)}{\Gamma(\beta+1)} x^{\beta}, \frac{\Gamma(\alpha+1)}{\Gamma(\alpha+\beta+1)} x^{\alpha+\beta}, \dots, \frac{\Gamma(m\alpha+1)}{\Gamma(m\alpha+\beta+1)} x^{m\alpha+\beta}]^T$$

Now, we expand $\frac{\Gamma(i\alpha+1)}{\Gamma(i\alpha+\beta+1)} x^{i\alpha+\beta}$ for $= 0,1,\dots,m$. By using FEFs,

$$\frac{\Gamma(i\alpha+1)}{\Gamma(i\alpha+\beta+1)} x^{i\alpha+\beta} = \sum_{j=0}^{m} u_j^i E_j^{\alpha}(x) = (U^i)^T E^{\alpha}(x)$$

where $U^i = [u_0^i, u_1^i, \dots, u_m^i]^T$, for $i = 0,1,\dots,m$. Also, we can obtain U^i in the same way of Eq.(12). Therefore,

$$I^{\beta} T^{\boldsymbol{\alpha}}(x) = \Psi E^{\alpha}(x) \quad (16)$$

where $\Psi = [U_0^T U_0^T, \dots, U_0^T]^T$.

Clearly,

$$P_{\alpha} = A_{\alpha} \Psi. \quad (17)$$

5. The operational matrix of fractional order Euler functions including delay.

Let $y(x) \in L^2[0,1]$, we can use the fractional order Euler functions to expand $y(x)$ as follows:

$$y(x) = \sum_{j=0}^{\infty} y_j E_j^{\alpha}(x) \;\approx\; \sum_{j=0}^{m} y_j E_j^{\alpha}(x) \;= Y^T E^{\alpha}(x)\,.$$

Let $d : [0,1] \to [0,1]$ be a delay function. Hence, we expand $y(d(x))$ in the same way

$$y\big(d(x)\big) = Y^T A_{\alpha}\, T^{\boldsymbol{\alpha}}(d(x))\,,$$

where $T^{\boldsymbol{\alpha}}(d(x)) = [1, (d(x))^{\alpha}, \dots, (d(x))^{m\alpha}]^T$. Furthermore, we can use the fractional order Euler functions to expand $(d(x))^{i\alpha}$ as follows:

$$(d(x))^{m\alpha} = \sum_{j=0}^{\infty} d_j^i E_j^{\alpha}(x) \approx \sum_{j=0}^{m} d_j^i E_j^{\alpha}(x) = (D^i)^T E^{\alpha}(x)$$

where $D^i = [d_0^i, d_1^i, \dots, d_m^i]^T$. So,

$$y\big(d(x)\big) = Y^T A_{\alpha}\, \Omega\, E^{\alpha}(x) = Y^T D_{\alpha}\, E^{\alpha}(x) \quad (18)$$

where $\Omega = \begin{bmatrix} (D^0)^T \\ (D^1)^T \\ \vdots \\ (D^m)^T \end{bmatrix}$ and $D_\alpha = A_\alpha\, \Omega$.

6. The approximate solution of fractional order delay variational problems

In this section, we shall consider the problem of the extermination of a functional J of the form:

$$J[y(x)] = \int_a^b F[x, y(x), {}_a^cD_x^\alpha y(x)\,, I^\beta y(x), z(x), y\big(d(x)\big), y'\big(d(x)\big)]dx \qquad (19)$$

where $z(x)$ is defined by:

$$z(x) = \int_a^x L(t, y(t), {}_a^cD_x^\alpha y(t), I^\beta y(t))dt \qquad (20)$$

and the admissible functions are $y(b) = y_b \in R$ and $y(x) = \varphi(x)$ for any $x \in [a-\tau, a]$. The function φ is given.

Now, we use the operational matrix of the fractional order Euler functions to approximate ${}_a^cD_x^\alpha$. This is done as follows:

$${}_a^cD_x^\alpha y(x) \simeq C^T E^\alpha(x) \qquad (21)$$

where $= [c_0, c_1, c_2, \dots, c_n]^T$, $E^\alpha(x) = [E_0^\alpha, E_1^\alpha, E_2^\alpha, \dots, E_n^\alpha]^T$.

We apply I^α on Eq. (21), hence $y(x)$ will be:

$y(x) \simeq C^T P_\alpha E^\alpha(x) + y(0)$ that means

$$y(x) \simeq C^T P_\alpha E^\alpha(x) + y_0. \qquad (22)$$

Then Eq. (19) can be expanded into fractional order Euler functions as follows:

$$J[y(x)] = \int_a^b F[x, (C^T P_\alpha E^\alpha(x) + y_0), (C^T E^\alpha(x)), (C^T P_\alpha E^\alpha(x)), \Big(\int_a^x L(t, (C^T P_\alpha E^\alpha(t) + y_0\Big)$$
$$, (C^T E^\alpha(t)), (C^T P_\alpha E^\alpha(t)))dt), (C^T D_\alpha\, E^\alpha(x)), (C^T D_\alpha\, M E^\alpha(x))]dx$$

Now, we define:

$$\mathcal{L}(x, \lambda) = J[y(x)] - \lambda g(x) \qquad (23)$$

where λ is the Lagrange multiplier and $g(x)$ is the constraint, Finally we have:

$$\mathcal{L} = \mathcal{L}(c_0, c_1, c_2, \dots, c_{n,}\lambda)$$

Taking $\frac{\partial \mathcal{L}}{\partial c_{i,}\lambda}$ and setting them equal to zero. Solving for c_i and λ and substituting the resulting values of $c_i, i = 0,1,2, \dots, n$ into Eq. (22), we can attain the coveted approximate solution for problem (19)-(20).

7. Error analysis

Consider $y(x) \in L^2[0,1]$ which is expanded as:

$$y(x) \simeq \sum_{i=0}^m c_i E_i^\alpha(x) = C^T E^\alpha(x) = y_m(x)$$

Then the following theorem gives the error bound of the fractional order Euler functions approximation [29].

Theorem: Let ${}_a^cD_x^\alpha y(x) \in C(0,1]$, and $E^\alpha = \{E_0^\alpha(x), E_1^\alpha(x), \dots, E_m^\alpha(x)\}$ form a vector space. If $y_m(x)$ is the approximation solution to $y(x)$, then the error bound is given as follows:

$$\|y - y_m\|_2 \le \frac{M_\alpha}{\Gamma\big((m+1)\alpha+1\big)\sqrt{(2m+2)\alpha+1}} \ ,$$

where $M_\alpha \ge sup_{x\in(0,1]}\left|{}_a^cD_x^{(m+1)\alpha} y(x)\right|$.

8. Illustrative example

Consider the fractional order delay variational problems

$$J[y(x)] = \int_0^2 [({}_0^cD_x^\alpha y(x) - \Gamma(\alpha+2)x)^2 + z(x) + (y'(x-1) - y'_\alpha(x-1))^2]dx, \tag{24}$$

where

$$z(x) = \int_0^x (y(t) - t^{\alpha+1})^2 dt \tag{25}$$

is defined on the set $C^1[-1,2]$ under the constraints,

$$\begin{cases} y(2) = 2^{\alpha+1}, \\ y(x) = 0, \quad \text{for all} \quad x \in [-1,0]. \end{cases} \tag{26}$$

First, we approximate ${}_0^cD_x^\alpha y(x)$ in terms of the fractional order Euler functions as,

$${}_0^cD_x^\alpha y(x) = C^T E^\alpha(x) . \tag{27}$$

Now, we operate I^α on both sides of Eq. (27), this yields,

$$\begin{aligned} y(x) &= C^T P_\alpha E^\alpha(x) + y(0) \\ &= C^T P_\alpha E^\alpha(x) . \end{aligned} \tag{28}$$

Here, we shall consider n=3. So that from the conditions in Eq. (26) we have,

$$g_1(x) = C^T P_\alpha E^\alpha(0) = 0 \tag{29}$$

$$g_2(x) = C^T P_\alpha E^\alpha(2) - 2^{\alpha+1} = 0 \tag{30}$$

The terms $\Gamma(\alpha+2)x$, $y'_\alpha(x-1)$ that appeared in Eq. (24) can be decomposed using the fractional order Euler functions as

$$\Gamma(\alpha+2)x = R^T E^\alpha(x)$$

$$y'_\alpha(x-1) = W^T E^\alpha(x)$$

and the term $t^{\alpha+1}$ in Eq. (25) is decomposed as

$$t^{\alpha+1} = S^T E^\alpha(x)$$

where,

$$R = \begin{bmatrix} 1 \\ 2 \\ 0 \\ -7.105 \times 10^{-15} \end{bmatrix}, \quad W = \begin{bmatrix} -1 \\ 2 \\ -3.197 \times 10^{-14} \\ 1.421 \times 10^{-14} \end{bmatrix} \quad \text{and} \quad S = \begin{bmatrix} 0.5 \\ 1 \\ 1 \\ -2.132 \times 10^{-14} \end{bmatrix}$$

Define $\mathcal{L}(x, \lambda_1, \lambda_2)$ as follows:

$$\mathcal{L}(x, \lambda_1, \lambda_2) = f_1(x) + f_2(x) + f_3(x) + \lambda_1 g_1(x) + \lambda_2 g_2(x) \tag{31}$$

where,

$$f_1(x) = \int_0^2 [C^T E^\alpha(x) E^\alpha(x)^T C - 2C^T E^\alpha(x) R^T E^\alpha(x) + R^T E^\alpha(x) E^\alpha(x)^T R]dx$$

$$f_2(x) = \int_0^2 [\int_0^x [C^T P_\alpha E^\alpha(t) E^\alpha(t)^T {P_\alpha}^T C - 2C^T P_\alpha E^\alpha(t) S^T E^\alpha(t) + S^T E^\alpha(t) E^\alpha(t)^T S]dt]dx$$

and,

$$\begin{aligned} f_3(x) = \int_0^2 [&C^T D_\alpha M E^\alpha(x) E^\alpha(x)^T M^T {D_\alpha}^T C - 2C^T D_\alpha M E^\alpha(x) W^T E^\alpha(x) \\ &+ W^T E^\alpha(x) E^\alpha(x)^T W]dx \end{aligned}$$

such that f_1 and f_2 are the first term and the second term of Eq. (24), respectively. f_3 is the last term that includes delay in Eq. (24), and λ_1, λ_2 are Lagrange multipliers.

Now, by taking the partial derivative of $\mathcal{L}$ with respect to c_i , $i = 0,1,2,3$, we set λ_1 and λ_2 equal to zero and solve for c_i, λ_1 and λ_2, then we substitute them into Eq. (28), we obtain the vector values of C.

Table 1 shows the comparison between the approximate solution to Eq. (24) for different values of α and the exact solution when $\alpha = 1$ as in [32]:

$$y_\alpha(x) = \begin{cases} 0, & x \in [-1,0] \\ x^{\alpha+1}, & x \in [0,2] \end{cases} \tag{32}$$

Table 1: Comparison between the approximate solution of Eq. (22) for different values of α and the exact solution

x	**Exact solution when** $\alpha = 1$	$\alpha = 1$	$\alpha = 0.9$	$\alpha = 0.85$
0.1	0.01	0.016	0.017	0.017
0.2	0.04	0.048	0.046	0.046
0.4	0.16	0.159	0.147	0.142
0.5	0.25	0.239	0.22	0.211
0.7	0.49	0.455	0.415	0.397
0.8	0.64	0.592	0.539	0.514
1	1	0.926	0.844	0.806
1.1	1.21	1.124	1.027	0.98
1.3	1.69	1.589	1.457	1.394
1.4	1.96	1.857	1.706	1.634
1.6	2.56	2.466	2.277	2.187
1.7	2.89	2.809	2.6	2.501
1.9	3.61	3.575	3.327	3.209
2	4	4	3.732	3.605

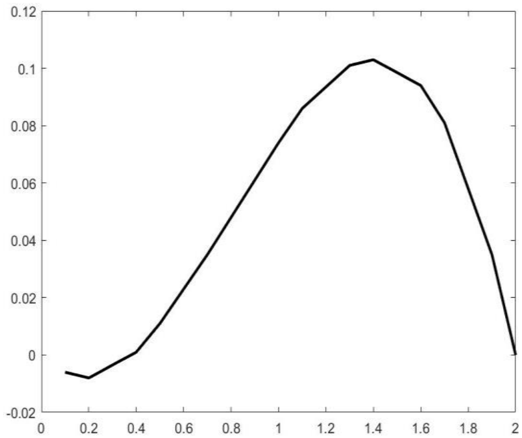

Figure 1: Represent the Error curve between the for App. solution and the exact solution.

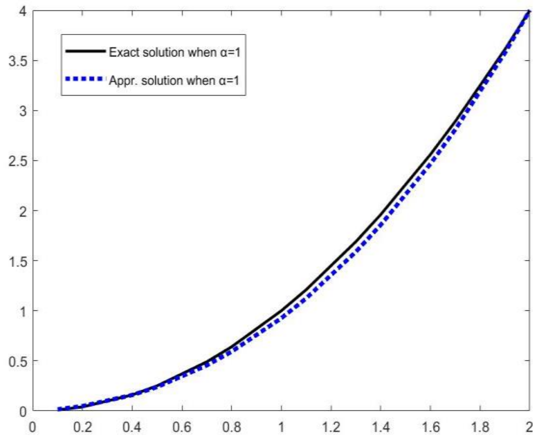

Figure 2: Represent the approximate solution $\alpha = 1$ and the exact solution at same α.

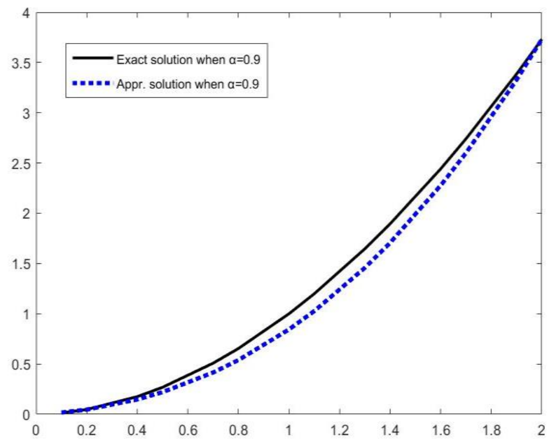

Figure 3: Represent the approximate solution for $\alpha = 0.9$ and the exact solution at same α

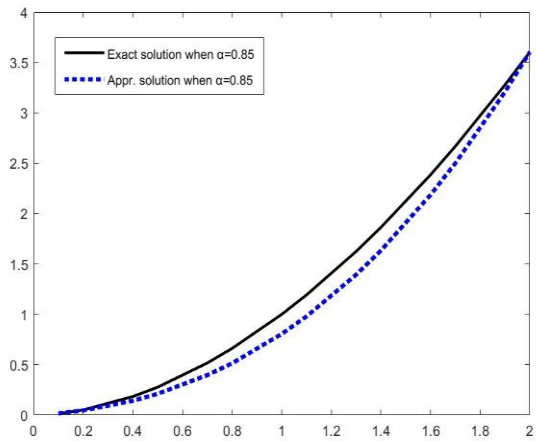

Figure 4: Represent the approximate $\alpha = 0.85$ and the exact solution at same α

9. Conclusion

This paper introduces an efficient technique for approximating the solution to fractional order delay variational problems using the operational matrices of the fractional Euler polynomials. The fractional derivative in the present paper is defined in the Caputo sense. The unknown function was decomposed in terms of the fractional Euler polynomials operational matrices which contain the unknown vector of coefficients. The proposed technique converted the original variational problem into a system of algebraic equations. Solving the resulting system gives us the unknown coefficients vector and then the desired solution. The numerical results approve that the proposed method is accurate and relatively simple to implement and has good accuracy.

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