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Z-Essential Submodules

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Abstract:

We define and investigate Z- essential submodules as a generalization of essential submodules. Various characterizations and properties of Z-essential submodules are given. Moreover we introduce the concepts of Z-singular submodule and Z-closed submodules.

Keywords : Z- small submodule , Z-essential submodule, Z-singular submodule, Z-closed submodules , Z - nonsingular submodule .

المقاسات الجزئية الواسعة من النوع – Z

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الخلاصة :

عرفنا ودرسنا مفهوم المقاسات الجزئية الواسعة النمط Z- كتعميم لمفهوم المقاسات الجزئية الواسعة وقد اعطيت تميزات وخواص عديد ة عن المقاســ ـبـريـة الواسعة Z – النمط و – Z . اضافة الى هذا قدمنا مفهوم المقاسات الجزئية المنفرد ة من النمط – Z اضافة الى هذا قدمنا المقاسات الجزئية المغلقة من النمط - Z

1- Introduction:

Throughout this paper all modules are unitary right R-modules, where R is commutative ring with unity. It is known that a submodule N of an R- module M is said to be small (superfluous), (nationally "$N << M$"), if whenever $W \leq M$, $N + W = M$, then $W = M$. A submodule N of an R-module M is called essential (large) (notationally $N \underset{e}{\leq} M$) if whenever $N \cap W = (0)$, $W \leq M$ then $W = (0)$ [1], [2]. Some authors used the notation $N \overset{0}{\hookrightarrow} M$ for small submodule and $N \overset{*}{\hookrightarrow} M$ or $N \trianglelefteq M$ for essential submodule. We shall use $N << M$ for (N is small submodule of M) and $N \underset{ess}{\leq} M$ for (N is essential submodule of M).

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Many generalizations of small submodules and essential submodules were introduced by researchers. Some of these generalizations are δ-small submodules [3], semismall submodules [4], p-small submodules [5], e-small submodules [6], J-small submodules [7], e*-essential submodules [8], t-essential submodules [9], [10], small essential submodules[10] , P-essential submodulles [11].

K.R.Goodearl in [1] introduce the concept $Z_2(M)$ (the second singular submodule of M) by $\frac{Z_2(M)}{Z(M)}=Z(\frac{M}{Z(M)})$, where Z(M) is the singular submodule of M and Z(M) ={m ∈ M : mI = 0 for some I $\leq_{ess}$ R} = {m∈ M : ann(m) $\leq_{ess}$ R}, where ann(m) = {r ∈ R:mr = 0}.

If Z(M) = M (Z(M) = (0)) M is called singular . Asgari and Haghany in [12] used the notion of $Z_2(M)$ and presented the concept "t-essential submodules", where a submodule A of M is called t-essential (briefly, $A \leq_{tes} M$), if whenever B ≤ M, A ∩ B ⊆ $Z_2(M)$ implies B ⊆ $Z_2(M)$. Equivalently A $\leq_{tes}$M if A + $Z_2(M)$ $\leq_{ess}$ M. Hence it is clear that every essential submodule is t-essential, but not conversely, see [12]. However the two concepts are equivalent in class of nonsingular modules. Also Asgari in [12] proved that $Z_2(M)$ ={m ∈ M : ann(m) $\leq_{tes}$ R} = {m ∈ M : mI = 0 for some I $\leq_{tes}$ R}. For more information about $Z_2(M)$, you can see [12], [13] , [14] .

At 2021, A mina in [15] introduced and studied Z-small submodules, where a submodule N of M is called Z-small (denoted by $N \ll_Z M$) if whenever' N + W = M', W ≤ M, W ⊇ $Z_2(M)$, then" W = M ". Note that W ⊇ $Z_2(M)$ implies $Z_2(W) = Z_2(M)$.

In this paper, we present and study the concept Z- essential submodule (as a dual of notion of Z-small submodule), where a submodule N of M is called Z - essential (briefly $N \leq_{zes} M$) if whenever N ∩ W = 0, W ≤ M, W ⊆ $Z_2(M)$ then W = (0).
In S. 2, we study Z-essential submodules and present many properties related with this concept .

In S. 3, we introduce the concept of Z-singular submodules, where for any R - module M, the set { m∈M : $ann(m) \leq_{zes}$ R } is denoted by ZS(M). It is clear that ZS(M) is submodule of M . M is called Z-singular (Z-nonsingular) if ZS (M) = M(ZS (M) = 0). Many properties related with this concept are given.

In S. 4, we define Z-closed submodule, where a submodule N of an R-module M is called Z-closed (brifely $N \leq_{ZC} M$) if N has no proper Z-essential extension in M, that is if N $\leq_{zes}$ W ≤ M , then N = W. It is clear that every Z-closed submodule is closed but the converse is not true, see Remark 4.3(1). Several other results are introduced.

2-Z- essential submodules:
2.1 Definition: A submodule N of an R- module M is called Z-essential (briefly N $\leq_{zes}$M) if whenever W≤ $Z_2(M)$, N ∩ W=(0) , then W = (0) .Note that W ≤ $Z_2(M)$ is equivalent to $Z_2(W)$ = W, that is N $\leq_{zes}$M if whenever N ∩ W=(0), $Z_2(W)$ = W (W is Z_2-torsion submodule) ,then W = (0) .

It clear that every essential submodule is Z-essential, but the converse may be not true, (see Rem. 2.2.(2)).

2.2 Remarks and Examples:

1- If Z_2 (M)=M, N≤ M , then $N \leq_{ess} M$ if and only $N \overset{\leq}{_{zes}} M$; that is essential and Z-essential submodules are coincident .

2- If Z_2 (M)=(0), then it is clear that every submodule of M is Z- essential.
In particular, if M = Z_6 as Z_6 – module; every submodule of Z_6 is Z -essential, but $(\bar{2})$ $(\bar{3})$, $(\bar{0})$ are not essential in Z_6.
3- If $(0) \overset{\leq}{_{zes}} M$, then Z_2 (M) = 0, and the converse hold by part (2)
Proof: Since (0) ∩ Z_2 (M) = (0), so Z_2 (M) =(0) since $(0) \overset{\leq}{_{zes}} M$.

4- Consider Z_4 as Z_4-module , Z_2 (Z_4) = $(\bar{2})$, Z_4 and $(\bar{2})$ are essential in Z_4 , so they are Z− essential. But (o) is not Z−essential, since (0) ∩ Z_2 (Z_4) = $(\bar{0}) \cap (\bar{2}) = (\bar{0})$ and $(\bar{2}) \neq (\bar{0})$

5- Z-essential submodules and t-essential submodules are independent concepts.
For examples: in the Z-module Z_{12} .The submodule A=$(\bar{4}) \overset{\leq}{_{tes}} Z_{12}$, see [14, Ex.1.1.16)]. However by (1) essential and Z-essential are coincide in Z_{12}, hence $(\bar{4}) \overset{\nleq}{_{tes}} Z_{12}$. In Z_6 as Z_6 – module, every submodule of Z_6 is Z−essential by part (2). But $(\bar{2}) \leq Z_6$ is not t−essential in Z_6 since $(\bar{2}) \cap (\bar{3}) = (\bar{0}) \subseteq Z_2(Z_6)$, but $(\bar{3}) \not\subset (\bar{0}) = Z_2(Z_6)$.

6- For every module M , $M \overset{\leq}{_{zes}} M$, $Z_2(M) \overset{\leq}{_{zes}} M$

2.3 Proposition: Let M be an R-module .Then
1- If N ≤ W ≤ M and $N \overset{\leq}{_{zes}} M$, then $W \overset{\leq}{_{zes}} M$.
proof: Let B ⊆ Z_2 ,(M) and W ∩ B = (0). Since N⊆W, N∩B = (0) .But $N \overset{\leq}{_{zes}} M$, so B=0. Thus $W \overset{\leq}{_{zes}} M$.
2- If N_1 and N_2 are Z-essential of an R-module M , then $N_1 \cap N_2$ is Z-essential in M.
Proof: Let B ⊆ Z_2 (M) and ($N_1 \cap N_2$) ∩ B = (0). Then $N_1 \cap (N_2 \cap B) = (0)$. But $N_2 \cap B \subseteq N_2 \cap Z_2(M) \subseteq Z_2(M)$, so that $N_2 \cap B = (0)$, since $N_1 \overset{\leq}{_{zes}} M$. Also $N_2 \overset{\leq}{_{zes}} M$, B ⊆ Z_2 (M), hence B= (0)
3- Let f :M →M' be an R-homomorphism, $N \overset{\leq}{_{zes}} M'$. Then $f^{-1}(N) \overset{\leq}{_{zes}} M$.
Proof: let $f^{-1}(N) \cap B = (0)$, B ⊆ Z_2, (M). Then $f(f^{-1}(N) \cap B) = (0)$ and so N ∩ f (B)=(0) . As B ⊆ Z_2 (M) , f(B) ⊆ f (Z_2 (M)) ⊆ Z_2 (f (M)) ⊆ Z_2(M').Thus f (B) = (o), since $N \overset{\leq}{_{zes}} M'$. This implies B⊆ Ker f = $f^{-1}(0) \subseteq f^{-1}(N)$ and so $B \cap f^{-1}(N) = B$, that is (0) = B.
4- Let f : $M_1 \rightarrow M_2$ be a monomorphism , $N \overset{\leq}{_{zes}} M$. Then $f(N) \overset{\leq}{_{zes}} f(M_1)$.
Proof: Assume f(N) ∩ B=(0), B ⊆Z_2 (f (M_1)) ⊆Z_2 (M_2). Then $f^{-1}(f(N) \cap B) = f^{-1}$ (0 = (0). This implies $f^{-1} f(N) \cap f^{-1}(B) = (0)$. But $f^{-1} f(N) = N$, since f is monomorphism , Hence $N \cap f^{-1}(B) = (0)$, but we can show that $f^{-1}(B) \subseteq Z_2(M_1)$ as follows :-
Let $x \in f^{-1}(B)$, then f (x) ∈ B ⊆ Z_2 (M_2). Hence $ann_R f(x) \overset{\leq}{_{tes}} R$. But $ann_R f(x) \subseteq ann_R(x)$ since f is 1-1 and hence $ann(x) \overset{\leq}{_{tes}} R$ which implies that x ∈Z_2(M_1). Therefore $f^{-1}(B) \subseteq Z_2(M_1)$.This implies $f^{-1}(B) = (0)$ since $N \overset{\leq}{_{zes}} M_1$. Then $(0) = f f^{-1}(B) = B \cap f(M_1) = B$. Thus $f(N) \overset{\leq}{_{zes}} f(M_1)$.

5- 1f " A ≤ B ≤ M ', then $A \overset{\leq}{_{zes}} B$ and $B \overset{\leq}{_{zes}} M$ if and only if $A \overset{\leq}{_{zes}} M$.
Proof : ⟹ Assume A∩K = (0) and K ⊆ Z_2 (M). Then A∩ (K∩B) =(0). As K⊆ Z_2 (M), K∩B ⊆ Z_2 (M)∩ B=Z_2(B). But $A \overset{\subseteq}{_{zes}} B$. So that K∩B =(0). Also $B \overset{\leq}{_{zes}} M$, and K⊆$Z_2$(M), so that K =(0).
⟸ To prove $A \overset{\leq}{_{zes}} B$ and $B \overset{\leq}{_{zes}} M$. Assume A∩K = (0), K ⊆ Z_2 (B). But Z_2 (B) ⊆ Z_2(M). So that A∩K=(0), K ⊆ Z_2 (M), hence K=0, since $A \overset{\leq}{_{zes}} M$. Now let B∩W = (0) and W ⊆ Z_2 (M). Then A∩W =(0) since A⊆ B .But $A \overset{\leq}{_{zes}} B$, so W = (0).

The following is a characterization of Z-essential submodules.

2.4 Proposition: Let N be a submodule of an R- module M. Then $N \leq_{zes} M$ if and only if for each $U \leq Z_2(M)$, $U \neq 0$, $N \cap U \neq 0$.

Proof: $\Rightarrow$ It is clear.

$\Leftarrow$ Let $U \subseteq Z_2(M)$ and $N \cap U = 0$. Suppose $U \neq 0$ then $N \cap U \neq 0$ which is a contradiction. Thus $U = 0$ and $N \leq_{zes} M$.

2.5 Corollary: Let $N \leq M$, $N \leq_{zes} M$ if and only if for each $x \in Z_2(M)$, $x \neq 0$, $\exists\, r \in R\text{-}\{0\}$ such that $0 \neq x r \in N$.

Proof: $\Rightarrow$ By Proposition 2.4, $N \cap (x) \neq (0)$, Hence there exists $0 \neq r \in R$ such that $0 \neq x r \in N$.

$\Leftarrow$ Let $0 \neq U \subseteq Z_2(M)$. Then for each $0 \neq x \in U$, $\exists\, r \in R\text{-}\{0\}$ such that $0 \neq x r \in N$, so that $xr \in N \cap U$. Therefore $N \cap U \neq 0$ for each $0 \neq U \subseteq Z_2(M)$. Therefore $N \leq_{zes} M$ by Proposition 2.4 .

2.6 Definition: A monomorphism $f: M \to M'$ is called Z-essential monomorphism if $\mathrm{Im} f \leq_{zes} M$.

2.7 Proposition : An R- module monomorphism "$f: L \to M$" is Z-essential if and only if for each homomorphism $h: M \to N$ such that $\mathrm{Ker}\, h \subseteq Z_2(M)$, hof is monomorphism implies h is monomorphism.

proof: $\Rightarrow$ Since $f: L \to M$ is Z-essential, $\mathrm{Im}\, f \leq_{zes} M$. As hof is monomorphism, $0 = \mathrm{Ker}\,(h \circ f) = f^{-1}(\mathrm{Ker}\, h)$. So $\mathrm{Ker}\, h \cap \mathrm{Im}\, f = 0$ (because if $x \in \ker h \cap \mathrm{Im}\, f$, then $h(x) = 0$, $x = f(\ell)$ f or some $\ell \in L$. So $(h \circ f)(\ell) = 0$, and hence $\ell = 0$ and $x=0$). But $\mathrm{Im}\, f \leq_{zes} M$ and $\mathrm{Ker}\, h \subseteq Z_2(M)$, so that $\mathrm{Ker}\, h = 0$.Thus h is monomorphism.

$\Leftarrow$ Assume K is monomorphism. To prove $\mathrm{Im} f \leq_{zes} M$. Let $\mathrm{Im}\, f \cap K = 0$, $K \subseteq Z_2(M)$. Consider $L \xrightarrow{f} M \xrightarrow{\pi} M/K$, where π is the natural epimorphism. Hence $\pi \circ f$ is monomorphism. To see this Let $x \in \ker(\pi \circ f)$, $(\pi \circ f)(x) = 0_{M/K}$, hence $f(x) \in K$; that is $f(x) \in \mathrm{Im}\, f \cap K = 0$. Thus $f(x) = 0$ and $x \in \mathrm{Ker}\, f = \{0\}$. Then" $\mathrm{Ker}\,(\pi \circ f) = 0$ " ; that is $\pi \circ f$ is monomorphism. Hence by assumption, π is monomorphism and as $\mathrm{Ker}\, \pi = K$, so that $K = 0$.

2.8 Proposition: Let $f : K \to L$ be a monomorphism $g: L \to M$ be a mononomorphism. Then f , g are Z-essential monomorphism if and only if $g \circ f$ is Z-essential monomorphism.

Proof: $\Longrightarrow$ Let $(g\, of)(K) \cap U = 0$, $U \subseteq Z_2(M)$. To prove

$U = (0)$, $g^{-1}((g \circ f)(K)) \cap U) = g^{-1}(0) = \mathrm{Ker}\, g = \{0\}$.

This implies $f(K) \cap g^{-1}(U) = (0)$.

We claim that $g^{-1}(U) \subseteq Z_2(L)$. Assume $x \in g^{-1}(U)$, hence $x \in L$ and $g(x) \in U \subseteq Z_2(M)$, so that $\mathrm{ann}\, g(x) \leq_{tes} R$. But $\mathrm{ann}(x) \supseteq \mathrm{ann}\, g(x)$, since g is 1-1. Hence $\mathrm{ann}(x) \leq_{zes} R$; that is $x \in Z_2(L)$. Thus $g^{-1}(U) \subseteq Z_2(L)$, and so $f(K) \cap g^{-1}(U) = 0$, $g^{-1}(U) \subseteq Z_2(L)$ which implies $g^{-1}(U) = 0$, since $f(K) \leq_{zes} L$.

It follows that $gg^{-1}(U) = g(0) = 0$. But $gg^{-1}(U) = U \cap \mathrm{Im}\, g$. Hence $U=0$, since $\mathrm{Im}\, g \leq_{zes} M$.

$\Longrightarrow$ Let $f(K) \cap B = (0)$, $B \subseteq Z_2(L)$. To prove $B = (0)$. Then $(g \circ f)(K) \cap g(B) = (0)$, But $g(B) \subseteq g(Z_2(L)) \subseteq Z_2(g(L) \subseteq Z_2(M)$. Hence $g(B)=(0)$. Since $(g \circ f)(K) \leq_{zes} M$.

It follows that $B = (0)$ since g is 1-1. Thus $f(K) \leq_{zes} L$.

Now since $(g \circ f)(K) \leq_{zes} M$ and $(g \circ f)(K) = g(f(K)) \subseteq g(L) \subseteq M$ .Hence $g(L) \leq_{zes} M$.

Recall that an R - module M is called a multiplication R-module if for any $N \leq M$, there exists an ideal I of R such that N = MI . Equivalently M is a generation R-module if for each N $\leq M$, $N = M (N :_R M)$, where $(N :_R M) = \{ r \in R : Mr \subseteq N \}$ [16] .

2.9 Proposition: Let M be a faithful finitely generated multiplication R-module, let N $\leq M$. Then $N \leq_{zes} M$ if and only if $(N :_R M) \leq_{zes} R$.

proof: $\Longrightarrow$ Assume $(N:M) \cap I = (0)$, $I \subseteq Z_2 (R)$. Then $M [(N:M) \cap I] = M (0) =(0)$ Hence by [2, Th.1.6], $M [(N:M) \cap I] = M (N:M) \cap MI = (0)$, that is $N \cap MI = (0)$. But $MI \subseteq MZ_2(R) = Z_2(M)$. Hence $MI = (0)$, since $N \leq_{zes} M$. So that $I \subseteq \text{ann } M =(0)$, thus $I =(0)$ and $(N:M) \leq R$.
$\Longleftarrow$ Assume $N \cap W = (0)$, $W \subseteq Z_2, (M)$. Then $(N \cap W:M) = (0:M) = \text{ann } M = (0)$, so $(N:M) \cap (W:M) = (0)$. But $W \subseteq Z_2 (M)$, hence $M (W: M) = W \subseteq Z_2 (M) = M Z_2 (R)$.
As M is a faithful finitely generated multiplication R-module, then by [2, Th 3.1], $(W:M) \subseteq Z_2 (R)$. Hence $(N:M) \cap (W:M) = (0)$, $(W:M) \subseteq Z_2 (R)$ and $(N:M) \leq_{zes} R$, so that $(W:M) =(0)$. It follows that $W = M (W:M) = M.(0) = (0)$.

2.10 Proposition: Let M be a finitely generated faithful multiplication R - module, let I $\leq R$. Then $I \leq_{zes} R$ if and only if $MI \leq_{zes} M$.

proof: $\Longrightarrow$ Let $MI \cap W=(0)$ and $W \subseteq Z_2 (M)$.Since M is a multiplication R module, W=MJ for some $J \leq R$. But $W=MJ \subseteq Z_2(M) = M Z_2(R)$, So that $J \subseteq Z_2 (R)$.
Now $M I \cap W=MI \cap MJ = (0)$ and so " $M (I \cap J) = 0$ " ; that is $I \cap J \subseteq \text{ann } M = (0)$. Thus J = (0) and W = MJ = (0).
$\Longleftarrow$ The proof is similarly.

2.11 Corollary: Let M be a finitely generated faithful multiplication R-module, $N=MI \leq M$. Then $N \leq_{zes} M$ if and only if $I \leq_{zes} R$.

2.12 Proposition: Let M be a finitely generated faithful multiplication R-module. and I, J ideals of R. Then $I \leq_{zes} J$ if and only if $MI \leq_{zes} MJ$.

Measure: $\Longrightarrow$ Assume $MI \cap W=0$, $W \subseteq MJ$ and $W \subseteq Z_2 (MJ)$. Since $W \leq M$, W=MK for some $K \leq R$. Hence $MK \subseteq M J$ and so $K \subseteq J$ by [2,Th.3.1].
Also $Z_2 (MJ) = Z_2 (M) \cap MJ$. But $Z_2(M)=MZ_2 (R)$, since M is a finitely generated faithful multiplication R-module. Hence $Z_2 (MJ) = MJ \cap MZ_2(R)$. Also by [2; Th.2.1] $Z_2 (MJ) = M (J \cap Z_2 (R)) = MZ_2 (J)$.
Now $MI \cap MK =0$ implies $M(I \cap K) =0$ and so $I \cap K \subseteq \text{ann } M=(0)$, that is $I \cap K=(0)$. As $W \subseteq Z_2(MJ)$, $MK \subseteq MZ_2(J)$ and since M is a finitely generated faithful multiplication R-module, $K \subseteq Z_2 (J)$. Thus $I \cap K = (0)$ and $K \subseteq Z_2(J)$, so K = (0). It follow that W=MK=(0).
$\Longleftarrow$ Assume $I \cap K= (0)$, $K \subseteq Z_2 (J)$. To prove K = (0), since $I \cap K = 0$, then $M(I \cap K) = 0$ and so $MI \cap MK = (0)$ and $MK \subseteq MZ_2 (J)$. But $Z_2(MJ) = MJ \cap Z_2 (M) = MJ \cap M Z(R) = M (J \cap Z_2 (R)) = M Z_2 (J)$. Thus $MI \cap MK = 0$ and $MK \subseteq Z_2 (MJ)$, so that MK = (0), since $MI \leq MJ$. It follows that $K \subseteq \text{ann } M = (0)$. That is K=0.

2.13 Theorem: If $\{K_\lambda: \lambda \in \Lambda\}$ and $\{L_\lambda \subseteq Z_2(M): \lambda \in \Lambda\}$ be families of submodules of an R-module M.If $\{K_\lambda: \lambda \in \Lambda\}$ is an independent family of submodules of M and $K_\lambda \leq_{zes} L_\lambda$ for all $\lambda \in A$, then $\{L_\lambda: \lambda \in \Lambda\}$ also independent family and $\bigoplus_{\lambda \in \Lambda} K_\lambda \leq_{zes} \bigoplus L_\lambda$

Proof: If $K_1 \leq_{zes} L_1$ and $K_2 \leq_{zes} L_2$ are submodules of M with $K_1 \cap K_2 = (0)$, then $K_1 \cap K_2= (0) \leq_{zes} L_1 \cap L_2$, since $Z_2(L_1 \cap L_2) = (0)$ by Remarks 2.2(3). But $Z_2(L_1 \cap L_2) =Z_2(M) \cap (L_1 \cap L_2)$ and $Z_2(M) \leq_{zes} M$ (Remarks 2.2 (6)), so $L_1 \cap L_2= 0$

Let ρ_1:$L_1 \oplus L_2 \to L_1$ and ρ_2: $L_1 \oplus L_2 \to L_2$, where ρ_1, ρ_2 are natural projections. We obtain $\rho_1^{-1}(K_1) = K_1 \oplus L_2 \leq_{zes} L_1 \oplus L_2$, $\rho_2^{-1}(K_2) = L_1 \oplus K_2 \leq_{zes} L_2 \oplus L_1$ and then $K_1 \oplus K_2 = (K_1 \oplus L_2) \cap (L_1 \oplus K_2) \leq_{zes} (L_1 \oplus L_2)$ by Proposition 2.3(2).
Thus the assertion of the theorem for families with two elements is shown, and by induction, we get it for families with finitely many elements for arbitrary index set Λ, a family $\{ L_\lambda \subseteq Z_2(M) : \lambda \in \Lambda\}$ is independent if every finite subfamily is independent and thus what we have just proved.
For any $m \in \oplus_{\lambda \in E} L_\lambda$ for some finite subset, $E \subseteq \Lambda$ and since $\oplus_{\lambda \in E} k_\lambda \leq_{zes} \oplus_{\lambda \in E} L_\lambda$, then by Proposition 2.4 $mR \cap \oplus_{\lambda \in E} k_\lambda \neq 0$. But $mR \cap \oplus_{\lambda \in E} k_\lambda \subseteq mR \cap \oplus_{\lambda \in \Lambda} K_\lambda$. So $mR \cap \oplus_{\lambda \in \Lambda} k_\lambda \neq 0$.
Hence the intersection of a nonzero submodule of $\oplus_{\lambda \in \Lambda} L_\lambda$ with $\oplus_{\lambda \in \Lambda} K_\lambda$ is again nonzero. Thus $\oplus_{\lambda \in \Lambda} k_\lambda \leq_{zes} \oplus_{\lambda \in \Lambda} L_\lambda$.
2.14 Remark: If $\{k_\lambda : \lambda \in \Lambda\}$ and $\{L_\lambda : \lambda \in \Lambda\}$ are families of R-modules with $k_\lambda \leq_{zes} L_\lambda$ for all $\lambda \in \Lambda$, then we have the external direct sum $\oplus_{\lambda \in \Lambda} k_\lambda \leq_{zes} \oplus_{\lambda \in \Lambda} L_\lambda$.

3-Z–singular submodules:
3.1 Definition: Let M be an R-module. The set $\{m \in M : \text{ann}(m) \leq_{zes} R\}$ is denoted by ZS(M).
It is easy to check ZS (M) is a submoduel of M. This submodule is called Z-singular submodule of M. It is clear that Z (M) ⊆ ZS (M).

3.2 Proposition: For any R- module M. Then ZS (M) = $\{m \in M : mI=(0)$ for any $I \leq_{zes} R\}$.
Proof: Let K = $\{m \in M: mI=(0)$ for some $I \leq_{zes} R\}$. Assume m ∈ ZS(M), so that ann (m) $\leq_{zes}$ R and so m ann (m) = 0; that is mI = (0), where I = ann(m) $\leq_{zes}$ R. Thus m ∈ K. Conversely, if m ∈ K, then mI=0 for some I $\leq_{zes}$ R, hence I ≤ ann(m) and so ann(m) $\leq_{zes}$R. Thus m ∈ ZS (K).

3.2 Proposition: Let f : M → N be any R- homomorphism then f (ZS(M)) ⊆ ZS(N).
Proof: Let y∈ f (ZS(M)) Then y = f(x) for some, x ∈ ZS(M)
Hence ann (x) $\leq_{zes}$ R. But ann f (x) ⊇ ann (x), so ann f (x) $\leq_{zes}$ R and this implies y = f(x) ∈ ZS (N).

3.3 Proposition: For N ≤ M, ZS(N) = ZS(M) ∩ N.
Proof: It is clear that ZS(N) ⊇ ZS(M) ∩ N. For any m ∈ ZS(N) , m ∈N and ann (m) $\leq_{zes}$R, so that m ∈ ZS(M) ∩ N.

3.4 Definition: An R- module M is called to be Z-singular (respectively Z- nonsingular) module if ZS(M) = M (resp. ZS(M) =(0)).
In particular, $\forall n \in Z_+$, M = Z_n as Z-module. Z_n = Z(M) = ZS(M).
For the Z-module Z, Z(Z) = (0), but for each N ≤ Z, ann (N) = (0) $\leq_{zes}$ Z; ie ZS(Z) = Z.

3.5 Remarks: Let N be a submodule of an R-module M. Then
1. M is Z-singular, implies N is Z-singular.
2. M is Z-nonsingular, implies N is Z-nonsingular.
3. Any simple faithful module is Z-singular.
Proof: (1) and (2) are easy.
(3) Since any simple module M is either nonsingular or singular. If M is singular, then Z(M) = M, and since Z(M) ⊆ ZS(M) we get ZS(M) = M, Thus M is Z-singular. If Z(M) = 0, then $Z_2(M)$ = 0. As $MZ_2(R) \subseteq Z_2(M)$, so $M Z_2(R) = (0)$. This implies $Z_2(R) = 0$, since M is faithful. Thus every ideal of R is Z-essential by Rem. and Exs. 2.2(2), hence for each m ∈ M, ann (m) $\leq_{zes}$ R. It is follows that ZS(M) = M.

3.6 Proposition: An R-module M is Z-nonsingular of and only if Hom (A,M) = 0 for all Z-singular module.

Proof: $\Longrightarrow$ If M is Z-nonsingular, then ZS(M) = (0). Let A be Z-singular module, that is ZS(A) = A. Let f :A → M be an R- homomorphism . Then f (ZS(A)) ⊆ ZS(M) and hence f(A) ⊆ 0 . Thus f = 0 .

$\Longleftarrow$ To Prove M is Z-nonsingular. Since ZS(ZS(M)) = ZS(M) ∩ M = ZS(M), Thus ZS(ZS(M)) = ZS(M), that is ZS(M) is a Z-singular module. Hence Hom(ZS(M),M) = 0. But ZS(M) ≤ M, so the inclusion mapping i ∈ Hom (ZS(M), M) = 0. This implies i = 0 and ZS(M) = 0 and so M is Z-nonsingular.

3.7 Proposition: A module M is Z-singular if and only if there exists a short exact sequence $(0) \longrightarrow A \xrightarrow{f} B \xrightarrow{g} M \longrightarrow 0$ such that f is an essential monomorphism.

Proof :

$\Longrightarrow$ Assume M is Z-singular. Choose an exact sequence $0 \longrightarrow A \xrightarrow{inc} B \xrightarrow{g} M \longrightarrow 0$ with A⊆B and B is a free module. Let $\{b_\alpha\}$ be a basis of B, then for each $\alpha \in \Lambda$, $g(b_\alpha)I_\alpha = 0$ for some Z-essential ideal, since M is Z-singular. Hence g $(b_\alpha I_\alpha) = 0$, that is $b_\alpha I_\alpha \subseteq \ker g$, $\forall \alpha \in \Lambda$. But Ker g = Im (i) = A, so $b_\alpha I_\alpha \leq A$, $\forall\alpha \in\Lambda$. Since $I_\alpha \underset{zes}{\leq} R$, we get $b_\alpha I_\alpha \underset{zes}{\leq} b_\alpha R$, $\forall \alpha\in \Lambda$. Hence $\bigoplus_{\alpha\in\Lambda} (b_\alpha I_\alpha) \underset{zes}{\leq} \bigoplus_{\alpha\in\Lambda} b_\alpha R = B$ by Theorem 2.13. But $\bigoplus_{\alpha\in\Lambda} (b_\alpha I_\alpha) \subseteq A \subseteq B$, so that $A \underset{zes}{\leq} B$ (by tansitivity of Z-essential submodules). Thus the inclusion mapping i : A ⟶ B is Z-essential monomorphism .

$\Longleftarrow$ Suppose we have exact sequence $0 \to A \xrightarrow{f} B \xrightarrow{g} M \to 0$ such that f is mononomorphism. Given b ∈ B, define k :R → B by k(r) = br, ∀r ∈ R. Since f(A) ≤ B, we get $k^{-1} f(A) \underset{zes}{\leq} R$ by Proposition 2.2 (3). But $k^{-1}f(A)=\{r\in R: k(r) \in f (A)\}=\{r\in R:br\in f(A)\}$. Put $I=k^{-1} (f(A))$ so $I \underset{zes}{\leq} R$ and bI ≤ f(A) = ker g. Hence g(bI) = g(b) I = 0 . It follows that g(b) ∈ ZS(M) . But g is an epimorphism, so for each m ∈ M, ∃ b ∈ R with g(b)=m, so that ZS(M) = M; that is M is Z-singular.

3.8 Corollary: If $A \underset{zes}{\leq} M$, where M is an R-module. Then $\frac{M}{A}$ is Z-singular.

Proof: Consider the sequence $0 \to A \xrightarrow{i} B \xrightarrow{\pi} \frac{M}{A} \to 0$ where I is the inclusion mapping and π is the natural epimorphism. Since i is monomorphism and i (A) = $A \underset{zes}{\leq} M$, i is Z-essential monomorphism. Hence by Proposition 3.7, $\frac{M}{A}$ is Z-singular.

3.9 Remarks:

(1) The following example shows that the converse of corollary 3.8 is not true in general .

The Z-module Z_2, if A = (0), then $A \underset{zes}{\nleq} Z_2$ but $\frac{Z_2}{A} \simeq Z_2$ (as Z-module) is singular, so $\frac{Z_2}{A}$ is Z-singular.

(2) Let I be an ideal of a commutative ring with identity R. Then $I \underset{zes}{\leq} R$ if and only if $\frac{R}{I}$ is Z-singular .

Proof: $\Longrightarrow$ It follows by corollary 3.8.

$\Longleftarrow$ Since R/I is Z-singular, ZS(R/I) = R/I. Hence 1+I ∈ ZS(R/I) and so ann $(1+I) \underset{zes}{\leq} R$. But ann (1+I) = {r ∈ R : (1+I) r = I} = {r ∈ R : r ∈ I} = $I \underset{zes}{\leq} R$.

3.10 Proposition: Let $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ be a short exact sequence. If A and C are Z-nonsingular. Then B is Z-nonsingular.

Proof: Let m ∈ ZS(B). Then ann (m) $\underset{zes}{\leq}$ R. Since the sequence exact , Imf = kerg, also g is an epimorphism which implies $\frac{B}{\ker g} \cong C$ which is Z-nonsingular .Hence $\frac{B}{f(A)}$ is Z-nonsingular.

But ann (m) ⊆ ann (m + f(A)), so ann (m + f(A)) $\leq_{zes}$ R; that is m + f(A) ∈ ZS($\frac{M}{f(A)}$) = 0,It is clear that m ∈ f(A). Thus m ∈ ZS(B) ∩ f(A) = ZS (f(A)). But f(A) is Z-nonsingular since f(A) ≅ A which is Z-nonsingular, it follows that m = 0 and ZS(B) = 0.

3.11 Corollary: If N and $\frac{M}{N}$ are Z-nonsingular, then M is Z-nonsingular.

Proof: The sequence $0 \to N \xrightarrow{i} M \xrightarrow{\pi} \frac{M}{N} \to 0$, where i is the inclusion mapping and π is the natural projection, is a short exact sequence. Hence by Proposition 3.10, M is Z-nonsingular.

3.12 Proposition: Let $\{M_\alpha\}_{\alpha\in\wedge}$ be a family of R-modules and M = $\oplus_{\alpha\in\Lambda} M_\alpha$. Then ZS(M) = $\oplus_{\alpha\in\Lambda}$(ZS($M_\alpha$)).

Proof: ∀α ∈Λ, M_α ⊆ M, so ZS(M_α) ⊆ ZS(M); that is ZS(M_α) ⊆ ZS($\oplus_{\alpha\in\Lambda} M_\alpha$). Thus $\oplus_{\alpha\in\Lambda}$ ZS(M_α) ⊆ ZS($\oplus_{\alpha\in\Lambda} M_\alpha$) …(1)

Let $\sum_{\alpha\in\Lambda} x_\alpha \in ZS$ ($\oplus_{\alpha\in\Lambda} M_\alpha$), where $x_\alpha \in M_\alpha$, ∀α ∈Λ and x_α = 0 for all except a finite number of α ∈Λ. Hence ann ($\sum_{\alpha\in\Lambda} x_\alpha$) $\leq_{zes}$ R and ann (x_α) $\leq_{zes}$ R; that is x_α ∈ ZS(M_α) and $\sum_{\alpha\in\Lambda} x_\alpha \in \oplus_{\alpha\in\Lambda}$ ZS(M_α). Thus ZS ($\oplus_{\alpha\in\Lambda} M_\alpha$) ⊆ $\oplus_{\alpha\in\Lambda}$ ZS(M_α) … (2)

Then by (1) and (2), ZS($\oplus_{\alpha\in\Lambda} M_\alpha$) = $\oplus_{\alpha\in\Lambda}$ (ZS(M_α)).

3.13 Theorem: The class of Z-singular R-modules is closed under (1) submodules (2) factor modules (3) direct sum.

Proof: M is Z-singular, so ZS(M) = M

1- For any A ≤ M. Since ZS(A) = ZS(M) ∩ A = M∩A = A.

2- Let A ≤ M, Let : M → M/A be the natural epimorphism π(ZS(M)) ⊆ ZS ($\frac{M}{A}$), hence π (M) = $\frac{M}{A}$ ⊆ ZS ($\frac{M}{A}$) . Thus $\frac{M}{A}$ = ZS ($\frac{M}{A}$).

3- If $\{M_\alpha\}_{\alpha\in\wedge}$ be a family of Z-singular modules By Proposition 3.12, $\oplus_{\alpha\in\Lambda}$ (ZS(M_α)) = ZS($\oplus_{\alpha\in\Lambda} M_\alpha$). Hence $\oplus_{\alpha\in\Lambda}$($M_\alpha$) = ZS ($\oplus_{\alpha\in\Lambda} M_\alpha$); ie $\oplus_{\alpha\in\Lambda} M_\alpha$ is Z-singular.

3.14 Theorem: The class of Z-nonsingular R-modules is closed under (1) submodules, (2) essential extension (3) direct product (4) module extension.

Proof: (1) and (2) are easy

3- Let $\{C_\alpha\}_{\alpha\in\wedge}$ be a collection of Z-nonsingular R-modules. Let A be Z-singular R-module , hence Hom (A,C_α) = 0 , ∀α ∈Λ . It follows that Hom (A, $\pi_{\alpha\in\Lambda} C_\alpha$) = 0, and so $\pi_{\alpha\in\Lambda} C_\alpha$ is Z-nonsingular.

4- Suppose that 0 → C → B → A → 0 is an exact sequence with A and C are Z-nonsingular. Then by Proposition 3.10, B is Z- nonsingular.

4- Z - Closed submodule:

Recall that a submodue A of an R-module M is called closed (A $\leq_c$ M) if whenever B ≤ M such that A $\leq_{ess}$ B, then A = B; ie A has no proper essential extension in M [3]. In this section, we introduce (Z-closed submodule) as a generalization of closed submodule.

4.1 Definition: A submodule C of an R-module M is called Z-closed (briefly C $\leq_{zc}$M) if whenever C $\leq_{zes}$ W and where C≤M implies C = W; ie C $\leq_{zc}$ M if C has no proper Z-essential extension in M.

4.2 Proposition: For each A ≤ M, there exists B ⊇ A such that A $\leq_{zes}$ B and B is Z-closed.

Proof: Let T = {K≤M : A $\leq_{zes}$K}. T ≠ 0 since A ∈ T. By Zorn's Lemma, T has a maximal element expressed K_0, We claim that K_0 is Z-closed. Suppose ∃$K^{/}$ ≤ M such that K_0 $\leq_{zes}$ $K^{/}$. As A ≤ K_0; so A $\leq_{zes}$ $K^{/}$ and this implies $K^{/}$ ∈ T which is a contradiction , since K_0 is a maximal element of T. Thus K_0 is Z-closed .

4.3 Remarks:

1- It is clear that **e**very Z-closed submodule of an R-module M is closed.

2- A closed submodule need not be Z-closed submodule, as for example: In Z_6 as Z_6-module since $Z_2(Z_6) = 0$, every submodule of Z_6 is Z-essential, hence $N = (\bar{3})$ is not Z-closed submodule of Z_6. But N is closed. Also by the same example: a direct summand of a module may not be Z-closed.

3- If $Z_2(M) = M$, then a submodule A of M is closed if and only if it is Z-closed.

4.4 Proposition: Let $A \leq M$, $K \leq M$. if $A \underset{zc}{\leq} M$, then $\frac{A}{K} \underset{zc}{\leq} \frac{M}{K}$.

Proof: Suppose $\frac{A}{K} \underset{zes}{\leq} \frac{W}{K}$ for some $\frac{W}{K} \leq \frac{M}{K}$. Then by Proposition 2.3 (3). $A \underset{zes}{\leq} W$. Hence $A = W$, since $A \underset{zc}{\leq} M$. Thus $\frac{A}{K} = \frac{W}{K}$.

4.5 Proposition: Let $B \leq K \leq M$, if $B \underset{zc}{\leq} M$, $K \underset{zes}{\leq} M$ then $\frac{K}{B} \underset{zes}{\leq} \frac{M}{B}$.

Proof: Assume $\frac{C}{B} \leq \frac{M}{B}$, $\frac{C}{B} \leq Z_2(\frac{M}{B})$ and $\frac{K}{B} \cap \frac{C}{B} = 0$. Hence $K \cap C = B$. Since $K \underset{zes}{\leq} M$ and $C \underset{zes}{\leq} C$, so $B = (K \cap C) \underset{zes}{\leq} (M \cap C) = C$. But $B \underset{zc}{\leq} M$, so $B = C$. Thus $\frac{C}{B} = 0$ and $\frac{K}{B} \underset{zes}{\leq} \frac{M}{B}$.

4.6 Proposition: If $A \underset{zc}{\leq} M$ and $A \leq B \leq M$, then $A \underset{zc}{\leq} B$.

Proof: It is easy, so is omitted.

The converse of Proposition 4.6 may not be true in general for example:

Let M be the Z-module Z_{12}, $A = \{\bar{0}, \bar{6}\}$, $B = \{\bar{0}, \bar{2}, \bar{4}, \ldots, \overline{10}\}$. Then $A \underset{zc}{\leq} B$, but $A \underset{zc}{\nleq} M$. However we have the following:

4.7 Proposition : Let A and B be submodules of a module M. Then the following assertions are equivalent .

(1) $B \underset{zc}{\leq} M$,

(2) for each submodule C of M such that $B \leq C$, then $B \underset{zc}{\leq} C$.

Proof: (2) $\Longrightarrow$ (1) It is clear.

(1) $\Longrightarrow$ (2) Follows by Prop.4.6.

4.8 Proposition : Let $N \underset{zc}{\leq} M$ and $K \underset{zes}{\leq} M$. Then $N \cap K \underset{zc}{\leq} K$. Provided $Z_2(A+B) = Z_2(A) + Z_2(B)$ for each $A\ B \leq M$.

Proof: To Prove $N \cap K \underset{zc}{\leq} K$. Suppose $N \cap K \underset{zes}{\leq} L \leq K$. So we must prove $N \cap K = L$. First we shall prove $N \underset{zes}{\leq} N + L$. Let $x \in Z_2(N+L)$ and $x \neq 0$, so $x \in Z_2(N) + Z_2(L)$ and hence $x = n + l$ for some $n \in Z_2(N)$, $l \in Z_2(L)$.

As $Z_2(N+L) \subseteq Z_2(M)$, hence $x \in Z_2(M)$. But $K \underset{zes}{\leq} M$, so there exists $r_1 \in R - \{0\}$ such that $0 \neq xr \in K$. Thus $0 \neq (n+l)\, r \in K$, so it follows that $nr = - R + k$ for some $k \in K$, and then $nr \in N \cap K$. Since $l \in Z_2(L)$ and $0 \neq lr \in Z_2(L)$, there exists $r_1 \in R-\{0\}$ such that $0 \neq lr\, r_1 \in N \cap K$.

This implies $0 \neq nrr_1 + l_r\, r_1 \in N \cap K \subseteq K$, thus $N \underset{zes}{\leq} N+L$ by corollary 2.5 and hence $N=N+L$ since $N \underset{zc}{\leq} M$.

Now $N \cap K = (N+L) \cap K = L + (N \cap K)$, so that $L \subseteq N \cap K$. But $N \cap K \subseteq L$, hence $N \cap K = L$ and $N \cap K \underset{zc}{\leq} K$.

5.Conclusions :

1. Many properties of Z-essential submodule anologous to that of essential submodules . However we have :

i) (0) $\leq_{zes}$ M if and only if $Z_2(M) = 0$.

ii) $Z_2(M)$ $\leq_{zes}$ M .

Iii) For a submodule N of a module M , N $\leq_{zes}$ M if and only if for each $U \subseteq Z_2(M)$, $U \neq 0$, $N \cap U \neq 0$.

Iv) For a submodule N of a module M , N $\leq_{zes}$ M if and only if for each $x \in Z_2(M)$, $x \neq 0$, $\exists$ $r \in R-\{0\}$ such that $0 \neq xr \in N$.

2. Many properties of Z - singular (Z - non singular) of submodules are anologous to that of Z - singular (Z - non singular) of submodules . However we have : Any simple faithful module is Z - singular .

3. The class of Z - closed submodules which contained the class of closed submodule . Many properties of closed submodules transfer to Z - closed submodules (may be with certain condition) , for example :

If N $\leq_{zc}$ M , K $\leq_{zes}$ M , $N \cap K$ $\leq_{zc}$ N , provided $Z_2(A+B) = Z_2(A)+Z_2(B)$, for each submodules A , B of M .

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