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Bayesian Estimation of Power Law Function in Non-homogeneous Poisson Process Applied in Mosul Gas Power Plant – Iraq

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Abstract

Non-homogeneous Poisson process with power law intensity function has often been used as a model for describing the failure pattern of repairable systems. Maximum likelihood and Bayesian estimation are used to estimate model parameters. Simulation and realistic application are used and represented by shutting down the gas power plant in Mosul. Stops in hours are designed with the power law random process model in order to obtain a model that represents the average stop time of the units throughout the study period in the best way. The results of the application on the data of the three concerned stations show that the Bayes estimate is better than the maximum likelihood estimate. This proves that the Bayes methods are very accurate and effective in estimating the rate of occurrence parameters.

Keywords: Non-homogeneous Poisson process, Power law process, Maximum likelihood estimator, Bayesian estimator.

التقدير البيزي لدالة قانون القوة في عملية بواسون غير المتجانسة المطبقة في محطة توليد كهرباء الموصل الغازية–العراق

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الخلاصة

غالبًا ما يتم استخدام عملية بواسون غير المتجانسة مع دالة الكثافة لقانون القوة كنموذج لوصف نمط فشل الأنظمة القابلة للإصلاح. تم استخدام مقدري الإمكان الاعظم وبيز لتقدير معلمات النموذج ، واستخدمت المحاكاة والتطبيق الواقعي ، وتمثلت في توقفات محطة توليد كهرباء الموصل الغازية. تم تصميم التوقف في ساعات باستخدام نموذج العملية العشوائية لقانون القوة من أجل الحصول على نموذج يمثل متوسط وقت التوقف للوحدات طوال فترة الدراسة بأفضل طريقة. أظهرت نتائج التطبيق على بيانات المحطات الثلاث المعنية أن تقدير بايز أفضل من تقدير الاحتمالية القصوى. هذا يثبت أن طرق بايز دقيقة للغاية وفعالة في تقدير معدل معلمات الحدوث.

1. Introduction

A repairable system is often modelled as a counting failure process. Repairable system reliability analysis must consider the effects of successive repair actions. When there is no trend in the system failure data, the failure process can often be modelled as a renewal process

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in which successive repair actions bring the system to be good as a new state [1]. This paper deals with the estimation of power law process (PLP) parameters using two estimation methods, namely the maximum likelihood estimation (MLE) and the Bayes method (Bay). For systems undergoing reliability improvement testing, it is critically important to identify whether significant improvement is occurring. System reliability improvement can be detected by observing a significant trend of increasing successive time between failures, i.e., system failure inter-arrival times. System reliability deterioration can be detected by observing a significant trend of decreasing successive time between failures, as the non-homogeneous Poisson process (NHPP) is capable of modelling these models. If the failure intensity function is decreasing over time, the times between failures tend to be longer, and if it is increasing, the times between failures tend to be shorter. For renewal processes, the times between failures are independent and identically distributed. A homogeneous Poisson process (HPP) is a special case of the renewal process when the times between failures are independent and exponential [2].

2. Power Law Process (PLP)

The power law process is one of the most common functions in the study of reliability growth models. The first ideas of this process go back to Duane (1964), who published a paper in which he presented data on the failures of different systems during their development programs and showed that this process is equivalent to the non-homogeneous Poisson process, in which the time rate of the occurrence follows the Weibull distribution [3]. Several authors have extensively researched the use of the power law process for evaluating hardware reliability growth and identifying software failures. For instance, Pulcini (2001) delved into Bayesian prediction methods for anticipating future failure times and the number of failures in a specific time interval for a repairable system that undergoes minimal repairs and periodic overhauls. Sen (2002) explored the Bayesian prediction of the Weibull intensity, while Pfefferman and Cernuschi-Frias (2002) introduced a nonparametric prediction approach. In addition, Pievatolo et al. (2003) provided an example of how the power law process can be applied to forecast the anticipated number of failures in underground trains during a given period.

Assume that the process $\{X(t), t \geq 0\}$ represents an NHPP, if the time rate of occurrence is described by the following formula:

$$\lambda(t) = \beta\lambda t^{\beta-1}; t \geq 0\,; \lambda, \beta > 0, \tag{1}$$

then the process $\{X(t), t \geq 0\}$ is called the power law process. One of the specifications of this process is that the distribution of the periods between the occurrence of events follows the Weibull distribution with the following probability density function [8] [9]:

$$\mathrm{f}(t|\beta, \lambda) = \beta\lambda t^{\beta-1} e^{-\lambda t_0^{\beta}}\; ; t \geq 0; \lambda, \beta > 0\,, \tag{2}$$

where λ and β are parameters of the Weibull distribution, β can be seen as a measure of the non-homogeneity of the failure rate: if $\beta > 1$ ($\beta < 1$), then the failure rate is increasing (decreasing) and this indicates a deterioration (resp. growth) in the system reliability. If $\beta = 1$, then the failure rate is considered to be constant [10]. The cumulative function of the rate of occurrence in the period $(0, t_0)$ is defined as follows:

$$m(t) = \int_0^t \lambda(u)du$$
$$= \int_0^t \beta\lambda u^{\beta-1}du$$

$$= \lambda t^{\beta} \; ; \; 0 \leq t \leq t_0 \, , \tag{3}$$

which is called the mean rate of occurrence (power law function). The number of occurrences follows the Poisson distribution with parameter $\lambda(t)$ in the time period $(0,t_0]$ [11]:

$$p[N(t) = n] = \frac{[m(t_0)]^n e^{-m(t_0)}}{n!}; n = 0,1,2, \ldots \tag{4}$$

3. Parameters Estimation

There are several methods for estimating the parameters of the power function process. In this research, the maximum likelihood estimation and the Bayes estimation are used and a comparison between them is conducted.

3.1 Maximum Likelihood Estimation

The maximum likelihood estimation is one of the most widely used methods for estimating stochastic model parameters due to its efficient properties including stability and minimum variance unbiased estimators. The parameter estimations of this method are characterized by making the likelihood function at its maximum. The probability distribution of the intervals between the occurrence of failures in NHPP follows the exponential distribution with parameter $\lambda(t)$ in the period $(0,t_0]$ with the following probability density function [12]:

$$f(t) = \lambda(t) e^{-\int_0^{t_0} \lambda(t) dt} \; , \; 0 \leq t \leq t_0 \, , \tag{5}$$

The following likelihood function for the intervals between the occurrence of events is [13][14]:

$$L(t_1, t_2, \ldots, t_n; \beta, \lambda) = \left(\prod_{i=1}^{n} \lambda(t_i)\right) exp\left(-\int_0^{t_n} \lambda(x) dx\right), \tag{6}$$

where $(t_1, t_2, \ldots, t_n)$ represent the periods between failures, then we have:

$$L(\beta, \lambda) = \left(\prod_{i=1}^{n} \beta \lambda t_i^{\beta - 1}\right) exp\left(-\lambda t_n^{\beta}\right) \tag{7}$$

$$= \beta^n \lambda^n \left(\prod_{i=1}^{n} t_i^{\beta - 1}\right) exp\left(-\lambda t_n^{\beta}\right) . \tag{8}$$

By taking natural logarithm for (8):

$$\ell(\beta, \lambda) = ln[L(\beta, \lambda)] = n ln(\beta) + n ln(\lambda) - \lambda t_n^{\beta} + (\beta - 1) \sum_{i=1}^{n} ln(t_i). \tag{9}$$

To find the maximum likelihood estimators for two parameters (β, λ), the first derivative of the equation (9) is taken to each parameter as follows:

$$\frac{\partial \ell(\beta, \lambda)}{\partial \beta} = \frac{n}{\beta} - \lambda t_n^{\beta} ln(t_n) + \sum_{i=1}^{n} ln(t_i) \tag{10}$$

$$\frac{\partial \ell(\beta, \lambda)}{\partial \lambda} = \frac{n}{\lambda} - t_n^{\beta}. \tag{11}$$

After equalizing the two equations to zero, we get:

$$\frac{n}{\beta} - \lambda t_n^{\beta} ln(t_n) + \sum_{i=1}^{n} ln(t_i) = 0 \tag{12}$$

$$\frac{n}{\lambda} - t_n^{\beta} = 0 \tag{13}$$

$$\lambda t_n^{\beta} = n, \tag{14}$$

substituting (14) into (12), we get:

$$\frac{n}{\beta} = n ln(t_n) - \sum_{i=1}^{n} ln(t_i) \tag{15}$$

$$\hat{\beta}_{MLE} = \frac{n}{n ln(t_n) - \sum_{i=1}^{n} ln(t_i)} \tag{16}$$

Where $\hat{\beta}_{MLE}$ is the MLE for β, substituting (16) into (14), we get the MLE for λ:

$$\hat{\lambda}_{MLE} = \frac{n}{t_n^{\hat{\beta}_{MLE}}} . \tag{17}$$

3.2 Bayes Estimation (Bay)

The Bayes estimation is one of the best methods for estimating the parameters of stochastic processes due to its accuracy. This method depends on the prior distribution of parameters and the information resulting from the sample about the parameter that is obtained from the maximum likelihood function in order to get the posterior distribution of the function [15]. From the likelihood function (7), we assume that the prior distribution for each parameter follows the gamma distribution [16]:

$\beta \sim Gamma(a,b),\ \lambda \sim Gamma(c,d),$

The probability density function for each parameter is given as follows:

$$p(\beta) = \frac{b^a}{\Gamma(a)}\beta^{a-1}e^{-b\beta}\ , \quad (18)$$

$$p(\lambda) = \frac{d^c}{\Gamma(c)}\lambda^{c-1}e^{-d\lambda}\ . \quad (19)$$

The joint prior distribution function for (β,λ) is [5][17]:

$$p(\beta,\lambda) = \frac{b^a d^c}{\Gamma(a)\Gamma(c)}\beta^{a-1}\lambda^{c-1}e^{-(b\beta+d\lambda)}\ , \quad (20)$$

then the posterior distribution function is:

$$\begin{aligned} p(\beta,\lambda|Data) &\propto p(\beta,\lambda)L(\beta,\lambda) \\ &\propto \frac{b^a d^c}{\Gamma(a)\Gamma(c)}\beta^{a-1}\lambda^{c-1}e^{-(b\beta+d\lambda)}\beta^n\lambda^n e^{-\lambda t_n^\beta}\prod_{i=1}^n t_i^{\beta-1} \\ &\propto \frac{b^a d^c}{\Gamma(a)\Gamma(c)}\beta^{a+n-1}\lambda^{c+n-1}e^{-(b\beta+d\lambda)-\lambda t_n^\beta}\prod_{i=1}^n t_i^{\beta-1} \\ &\propto \frac{\beta^{a+n-1}\lambda^{c+n-1}e^{-(b\beta+d\lambda)-\lambda t_n^\beta}\prod_{i=1}^n t_i^{\beta-1}}{\int\int \beta^{a+n-1}\lambda^{c+n-1}e^{-(b\beta+d\lambda)-\lambda t_n^\beta}\prod_{i=1}^n t_i^{\beta-1}d\beta d\lambda}\ . \end{aligned} \quad (21)$$

The Bayes estimator for power law process parameters can be obtained as follows:

$$\beta_{Bay} = E[\beta|Data] = \int p(\beta,\lambda|Data)d\lambda = \frac{\int \beta^{a+n}\lambda^{c+n-1}e^{-(b\beta+d\lambda)-\lambda t_n^\beta}\prod_{i=1}^n t_i^{\beta-1}d\lambda}{\int\int \beta^{a+n-1}\lambda^{c+n-1}e^{-(b\beta+d\lambda)-\lambda t_n^\beta}\prod_{i=1}^n t_i^{\beta-1}d\beta d\lambda}. \quad (22)$$

$$\lambda_{Bay} = E[\lambda|Data] = \int p(\beta,\lambda|Data)d\beta = \frac{\int \beta^{a+n-1}\lambda^{c+n}e^{-(b\beta+d\lambda)-\lambda t_n^\beta}\prod_{i=1}^n t_i^{\beta-1}d\beta}{\int\int \beta^{a+n-1}\lambda^{c+n-1}e^{-(b\beta+d\lambda)-\lambda t_n^\beta}\prod_{i=1}^n t_i^{\beta-1}d\beta d\lambda}. \quad (23)$$

It is difficult to obtain results for the integrals in equations (22) and (23), so we use the Laplace approximation as an approximation method as follows [18][19][20]:
If we have the following formula:

$$I(Data) = E[u(\beta,\lambda|Data)] = \frac{\int e^{(ln[u(\beta,\lambda)]+\ell+\rho)}d\beta d\lambda}{\int e^{(\ell+\rho)}d\beta d\lambda}, \quad (24)$$

where $u(\beta,\lambda)$ is a function of parameters, in equation (22) it is equal to β, and in equation (23) it is equal to λ, ρ is the natural logarithm of the previous common distribution of parameters, and it is defined as follows:

$$\rho = ln\left[\frac{b^a d^c}{\Gamma(a)\Gamma(c)}\right] + (a-1)ln(\beta) + (c-1)ln(\lambda) - (b\beta + d\lambda), \quad (25)$$

ℓ is the natural logarithm of the likelihood function which is defined as follows:

$$\ell = nln(\beta) + nln(\lambda) - \lambda t_0^\beta + (\beta-1)\sum_{i=1}^n ln(t_i)\ . \quad (26)$$

Let:

$$h(\beta,\lambda) = \frac{1}{n}(\ell+\rho). \quad (27)$$

$$h^*(\beta,\lambda) = \frac{1}{n}ln\big(u(\beta,\lambda)\big) + h(\beta,\lambda). \quad (28)$$

Then the equation (24) becomes:

$$I(Data) = E[u(\beta,\lambda|Data)] = \frac{\int e^{nh^*(\beta,\lambda)}d\beta d\lambda}{\int e^{nh(\beta,\lambda)}d\beta d\lambda}. \tag{29}$$

Thus, Laplace's estimate for this equation is as follows:

$$I(Data) = E[u(\beta,\lambda|Data)]$$
$$= \left[\frac{|\Sigma^*|}{|\Sigma|}\right]^{1/2} exp\left\{n\left(h^*(\beta^*,\lambda^*) - h(\beta,\lambda)\right)\right\}, \tag{30}$$

where the symbolic |.| denotes the determinant of the matrix, and (β^*,λ^*) are the values that maximize the function $h^*(\beta^*,\lambda^*)$. (β,λ) are the values that maximize the function $h(\beta,\lambda)$, Σ^* and Σ are the negative inverse of the Hessian Matrix for h^* (β^*,λ^*) and $h(\beta,\lambda)$ at (β^*,λ^*), and (β,λ), respectively:

$$\Sigma = [-H_h]^{-1} = \begin{bmatrix} -\frac{\partial^2 h}{\partial\beta^2} & -\frac{\partial^2 h}{\partial\beta\partial\lambda} \\ -\frac{\partial^2 h}{\partial\beta\partial\lambda} & -\frac{\partial^2 h}{\partial\lambda^2} \end{bmatrix}^{-1} \tag{31}$$

$$\Sigma^* = [-H_{h^*})]^{-1} = \begin{bmatrix} -\frac{\partial^2 h^*}{\partial\beta^2} & -\frac{\partial^2 h^*}{\partial\beta\partial\lambda} \\ -\frac{\partial^2 h^*}{\partial\beta\partial\lambda} & -\frac{\partial^2 h^*}{\partial\lambda^2} \end{bmatrix} \tag{32}$$

Note that h is a constant, while h^* changes with u, whereas:

$$h^*_\beta(\beta^*,\lambda^*) = \frac{1}{n} ln(\beta) + h(\beta,\lambda). \tag{33}$$

$$h^*_\lambda(\beta^*,\lambda^*) = \frac{1}{n} ln(\lambda) + h(\beta,\lambda). \tag{34}$$

Then the Bayes estimators for the power law process by the Laplace approximation are:

$$\hat{\beta}_{B.LC} = \left[\frac{|\Sigma^*|}{|\Sigma|}\right]^{1/2} exp\left\{n\left(h^*_\beta(\beta^*,\lambda^*) - h(\beta,\lambda)\right)\right\}. \tag{35}$$

$$\hat{\lambda}_{B.LC} = \left[\frac{|\Sigma^*|}{|\Sigma|}\right]^{1/2} exp\left\{n\left(h^*_\lambda(\beta^*,\lambda^*) - h(\beta,\lambda)\right)\right\}. \tag{36}$$

For a comparison and to determine the best method, the mean absolute percentage error (MAPE) is used according to the following formula [21]:

$$MAPE = \frac{1}{n}\sum_{t=1}^{n}\left|\frac{m(t)-\widehat{m}(t)}{m(t)}\right|, \tag{37}$$

where $m(t)$ represents the real value, $\widehat{m}(t)$ represents the estimated value.

4. Simulation

To compare the maximum likelihood and Bayes estimator of parameters λ and β, a simulation with 1000 repetitions is used. Multiple different cases are considered in the simulation study that depends on the different values of the sample size $n = 25, 50, 100$. For each case and $\beta = 0.5$,2.5 , $\lambda = 0.5$, 1.5 and specify default values for the prior distribution parameters as in the following table:

Table 1: Default values for the parameters of the prior distribution of the Bayes estimator

Case	a	b	c	d
I	0.5	0.5	0.5	0.5
II	1.5	0.5	1.5	0.5
III	0.5	1.5	0.5	1.5
IV	1.5	1.5	1.5	1.5

MAPE is used to measure the performance of the MLE and Bayes estimator.

Table 2: The simulated MAPE for the MLE and Bayes estimator for PLP.

n	parameter	MLE	Bay Ⅰ	Bay Ⅱ	Bay Ⅲ	Bay Ⅳ
25	β=0.5, λ=0.5	30.397	29.731	28.569*	28.674	29.879
	β=2.5, λ=1.5	30.298	22.888	22.563*	28.356	25.161
50	β=0.5, λ=0.5	19.914	19.687	19.321*	19.361	19.757
	β=2.5, λ=1.5	19.385	16.594	16.041*	18.097	17.676
100	β=0.5, λ=0.5	14.061	13.989	13.853*	13.875	14.005
	β=2.5, λ=1.5	11.844	11.097	11.093*	12.006	11.319

Numerical results in Table 2 show the MAPE of the PLP using the two methods MLE and Bayes method. From the comparison of the values of MAPE, it appears that the Bayesian model is the best method compared to other Bayesian models and the MLE method.

5. Application to a Real Data

In order to evaluate the applicability of the proposed methods, real data from the Mosul gas power plant is used. This data shows the intervals between successive failures in days for the Mosul gas power plant in Nineveh Governorate in Iraq during the period from 1/5/2019 to 30/6/2021.

Table 2: MAPE values for methods used to estimate the PLP parameters

Unite	Size	Method	$\hat{\beta}$	$\hat{\lambda}$	MAPE
M1	49	MLE	0.42268	11.63708	36.948
		Bay I	0.48394	8.52938	32.906
		Bay II	0.55998	6.02398	**28.303***
		Bay Ⅲ	0.56423	5.83326	28.306
		Bay IV	0.47995	8.81374	33.019
M2	50	MLE	0.4348	11.39495	39.985
		Bay I	0.49708	8.35728	37.576
		Bay II	0.57372	5.91491	**33.703***
		Bay Ⅲ	0.57829	5.72393	34.041
		Bay IV	0.49271	8.64254	37.308
M3	50	MLE	0.41016	12.39101	35.439
		Bay I	0.47025	9.08233	32.673
		Bay Ⅲ	0.54591	6.39036	**29.322***
		Bay II	0.54956	6.20069	29.62
		Bay IV	0.46682	9.3644	32.559

Table 2 shows the estimation of the PLP parameters using the proposed methods in the paper. From MAPE results, it is concluded that the Bay II model for estimation yields efficient estimators in representing the data.

The following Figures show the PLP estimated using estimation methods compared to the real cumulative values representing the intervals between successive failures of the stations.

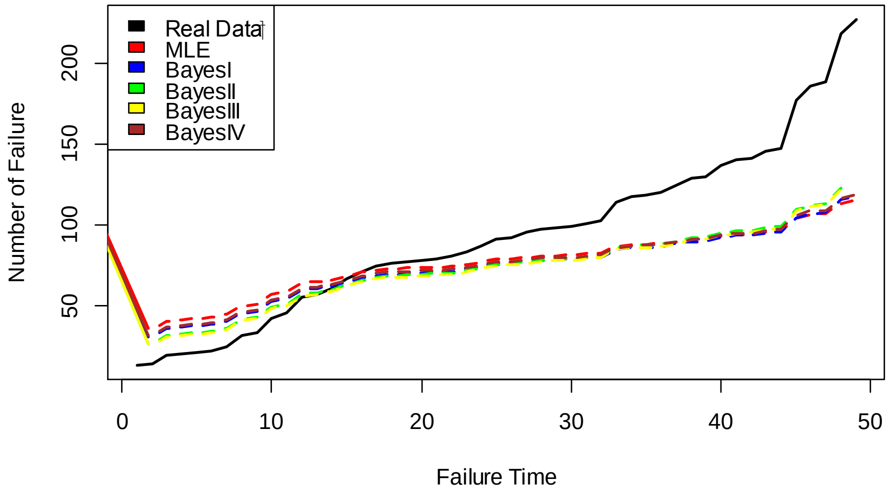

Figure 1: Estimated functions of the cumulative number of the intervals between successive failures for station M1 using different methods.

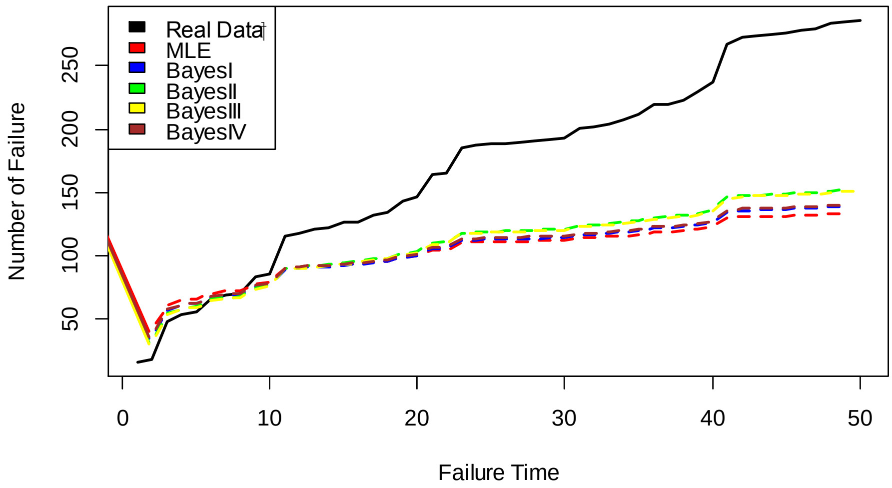

Figure 2: Estimated functions of the cumulative number of the intervals between successive failures for station M2 using different methods.

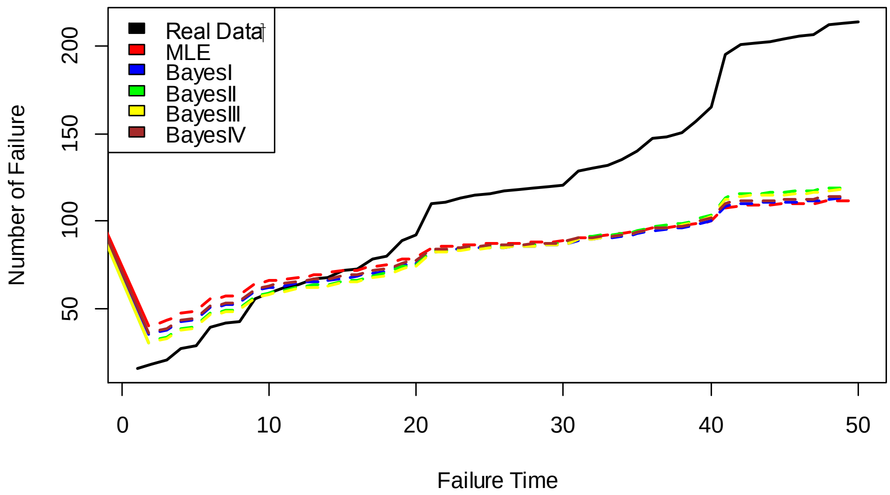

Figure 3: Estimated functions of the cumulative number of the intervals between successive failures for station M3 using different methods.

The previous Figures show the estimated functions of the cumulative number of intervals between the successive failures for the Mosul gas power plant using estimation methods. It is observed that the Bayesian model is the closest to the real data and this demonstrates the efficiency of this method of estimation compared with the MLE and other Bayes models.

6. Conclusion

In the applied aspect, it was concluded that the Bay II model is the best in estimating the power law process parameters of the gas power plant compared to the maximum likelihood estimation and the rest of the Bayes estimators. This is in agreement with the results of the simulation of the stochastic process and this indicates that the Bay II method gives efficient results in terms of estimating the parameters of the power law process. We recommend the beneficiary of the Mosul gas power plant to adopt the results reached in estimating the expected number of stops for the units under study.

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