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On Permutation Rho-Transitive in Permutation Rho-Algebras

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Abstract
Some of the novel algebraic topics explored in this paper, like Permutation ρ-algebra, permutation edge, permutation ρ-transitive, permutation ρ-ideal, permutation $\rho^{\#}$-ideal, and permutation ρ-subalgebra are discussed and looked into. Additionally, permutation ρ^{*}-algebra, permutation ρ-morphism, equivalence relation with its congruence classes and quotient permutation ρ-algebras were defined with specific results relating to our unique notions that have been developed and examined.

Keywords: Symmetric group, permutation sets, ρ-algebra, permutation ρ-algebra, permutation edge.

حول متعدية –روالتبديلي في جبر–رو التبديلي

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الخلاصة
في هذا البحث تم استكشاف بعض الموضوعات الجبرية الجديدة, مثل جبر –رو التبديلي و حافة التبديلي و متعدية–رو التبديلي و مثالية– رو التبديلي و مثالية– رو شارب التبديلي و الجبر الجزئي– رو التبديلي حيث تمت مناقشتها و التحقق منها , بالاضافة الى ذلك جبر–رو ستار التبديلي و مورفزم–رو التبديلي و علاقة التكافؤ مع صفوفها المرافقة و جبر –رو القسمة التبديلي تم تعريفها بنتائج محددة تتعلق بمفاهيمنا المقدمة التي تم تطويرها و دراستها.

1. Introduction

Imai and Is'eki proposed BCK-algebras and BCI-algebras, respectively [1, 2]. As in generally known, BCK-algebra is a proper subclass of BCI-algebra. Neggers and Kim [3] created the notion of d-algebras, which is a useful generalization of BCK-algebras, and then studied various links between d-algebras and BCK-algebras, as well as several other noteworthy interactions between d-algebras and oriented digraphs.

In 2017 [4] the notion of ρ -algebra is described and investigated some of the properties of a ρ-algebra that appear to be of interest. Various fields [5–12] investigate symmetric and

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alternating groups as well as their permutations. In recent years, some classes of algebra, group theory, and topology in non-classical sets, such as permutation sets [13], fuzzy sets [14-21], and soft sets [22,24], have examined and addressed some notions and features.

Let β be a permutation in symmetric group S_n, then β can be written as product of disjoint cycles in an essentially unique way. As the form $\beta=(b_1^1,b_2^1,...,b_{\alpha_1}^1)\,(b_1^2,b_2^2,...,b_{\alpha_2}^2)$ $...(b_1^{c(\beta)},b_2^{c(\beta)},...,b_{\alpha_{c(\beta)}}^{c(\beta)})$ where for each $i\neq j$ we have $\{b_1^i,b_2^i,...,b_{\alpha_i}^i\}\cap\{b_1^j,b_2^j,...,b_{\alpha_j}^j\}=\phi$ [25]. Therefore, $\beta=\lambda_1\lambda_2...\lambda_{c(\beta)}$, where λ_i disjoint cycles of length $|\lambda_i|=\alpha_i$ and $c(\beta)$ is the number of disjoint cycle factors including the 1-cycle of β. Also, $\alpha=\alpha(\beta)=(\alpha_1(\beta),\alpha_2(\beta),...,\alpha_{c(\beta)}(\beta))=(\alpha_1,\alpha_2,...,\alpha_{c(\beta)})$ is called the cycle type of β [26].

In this work, the concepts like, Permutation ρ-algebra, permutation edge, permutation ρ-transitive, permutation ρ-ideal, permutation $\rho^{\#}$-ideal, and permutation ρ-subalgebra are some of the innovative algebraic issues described and investigated in this article. Furthermore, permutation ρ^{*}-algebra, permutation ρ-morphism, equivalence relation, congruence class, and quotient permutation ρ-algebras were defined, with specific results relating to our unique conceptions established and investigated.

2. Preliminary

The definitions of ρ-algebra and permutation sets are reviewed in this section.

Definition 2.1:[4] Let $X\neq\emptyset$ and 0 be a constant with a binary operation $*$. We say that $(X,*,0)$ is a ρ *-algebra* if it satisfies the following conditions:

a) $x*x=0$,
b) $0*x=0$,
c) $x*y=0$ and $y*x=0$ implies $x=y$ $\forall x,y\in X$.
d) For all $x\neq y\in X-\{0\}$ imply that $x*y=y*x\neq 0$.

Definition 2.2:[13]

For any permutation $\beta=\prod_{i=1}^{c(\beta)}\lambda_i$ in a symmetric group S_n, where $\{\lambda_i\}_{i=1}^{c(\beta)}$ is a composite of pairwise disjoint cycles $\{\lambda_i\}_{i=1}^{c(\beta)}$ where $\lambda_i=(t_1^i,t_2^i,\dots,t_{\alpha_i}^i), 1\leq i\leq c(\beta)$, for some $1\leq \alpha_i, c(\beta)\leq n$. If $\lambda=(t_1,t_2,\dots,t_k)$ is $k-$cycle in S_n, we define $\beta-$set as $\lambda^\beta=\{t_1,t_2,\dots,t_k\}$ and it is called $\beta-$set of cycle λ. Also, the $\beta-$sets of $\{\lambda_i\}_{i=1}^{c(\beta)}$ are described as $\{\lambda_i^\beta=\{t_1^i,t_2^i,\dots,t_{\alpha_i}^i\}|1\leq i\leq c(\beta)\}$.

Definition 2.3: [13]

Suppose that λ_i^β and λ_j^β are $\beta-$sets in X, where $|\lambda_i|=\sigma$ and $|\lambda_j|=\upsilon$. Then $\lambda_i^\beta=\lambda_j^\beta$, if $\sum_{k=1}^{\sigma}t_k^i=\sum_{k=1}^{\upsilon}t_k^j$ and there exists $1\leq d\leq\sigma$, for each $1\leq r\leq\upsilon$ such that $t_d^i=t_r^j$. Also, we call λ_i^β and λ_j^β are disjoint $\beta-$sets in X, if and only if $\sum_{k=1}^{\sigma}t_k^i=\sum_{k=1}^{\upsilon}t_k^j$ and there exists $1\leq d\leq\sigma$, for each $1\leq r\leq\upsilon$ such that $t_d^i\neq t_r^j$.

3. On Permutation ρ-Algebra

Some new structures of ρ-algebra using β-sets are described and investigated in this section, like Permutation ρ-algebra, permutation edge, permutation ρ-transitive, permutation ρ-ideal, permutation $\rho^{\#}$-ideal, and permutation ρ-subalgebra.

Definition 3.1: Let $X = \left\{\lambda_i^{\beta}\right\}_{i=1}^{c(\beta)} = \left\{t_1^i, t_2^i, \ldots, t_{\alpha_i}^i\right\} | 1 \leq i \leq c(\beta)\}$ be a collection of β-sets, where β is a permutation in the symmetric group $G = S_n$. Let T= λ_k^{β} , for some $\lambda_k^{\beta} \in X$, where λ_k^{β} such that $\sum_{s=1}^{\alpha_k} t_s^k \leq \sum_{s=1}^{\alpha_i} t_s^i$ & $\left|\lambda_k\right| \leq \left|\lambda_i\right|, \forall \left(1 \leq i \leq c(\beta)\right) \ldots .(*)$. Moreover, if there are at least two disjoint β-sets say λ_h^{β} , λ_g^{β} such that $(*)$. Then, let T= λ_h^{β} if there exists $t_r^h \in \lambda_h^{\beta}$ with $t_r^h \leq t_s^g,\ \forall\ (1 \leq s \leq \alpha_g)$ or T= λ_g^{β} if there exists $t_r^g \in \lambda_g^{\beta}$ with $t_r^g \leq t_s^h,\ \forall\ (1 \leq s \leq \alpha_h)$. Also, let $\#: X \times X \longrightarrow X$ be a binary operation. We say $(X, \#, T)$ is a *permutation ρ-algebra* $(P - \rho - A)$, if $\#$ such that the conditions:

(1) $\lambda_i^{\beta} \# \lambda_i^{\beta} = T,$

(2) $T \# \lambda_i^{\beta} = T,$

(3) $\lambda_i^{\beta} \# \lambda_j^{\beta} = T \text{ and } \lambda_j^{\beta} \# \lambda_i^{\beta} = T \implies \lambda_i^{\beta} = \lambda_j^{\beta}, \forall \lambda_i^{\beta}, \lambda_j^{\beta} \in X.$

(4) $\lambda_i^{\beta} \# \lambda_j^{\beta} = \lambda_j^{\beta} \# \lambda_i^{\beta} \neq T,\ \forall \lambda_i^{\beta} \neq \lambda_j^{\beta} \in X - T$.

Example 3.2:

Let (S_{11}, o) be symmetric group and $\beta = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 \\ 5 & 8 & 4 & 1 & 3 & 7 & 6 & 9 & 2 & 11 & 10 \end{pmatrix}$ be a permutation in S_{11}. So, $= \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 \\ 5 & 8 & 4 & 1 & 3 & 7 & 6 & 9 & 2 & 11 & 10 \end{pmatrix} = (1\ 5\ 3\ 4)$ $(2\ 8\ 9)(6\ 7)(10\ 11)$. Therefore, we have $X = \left\{\lambda_i^{\beta}\right\}_{i=1}^{4} = \{\{1,5,3,4\}, \{2,8,9\}, \{10,11\}, \{6.7\}\}$ and $T = \{6,7\}$. Define $\#: X \times X \longrightarrow X$ by;

$$\lambda_i^{\beta} \# \lambda_j^{\beta} = \begin{cases} T, if \left(\lambda_i^{\beta} = T\right) or \left(\lambda_i^{\beta} = \lambda_j^{\beta}\right), \\ \lambda_i^{\beta}, if \left(\sum_{s=1}^{\alpha_i} t_s^i > \sum_{s=1}^{\alpha_j} t_s^j\right)\ or\ \left(\sum_{s=1}^{\alpha_i} t_s^i = \sum_{s=1}^{\alpha_j} t_s^j\ and\ |\lambda_i| > |\lambda_j|\right)) \\ \lambda_j^{\beta}, if \left(\sum_{s=1}^{\alpha_i} t_s^i < \sum_{s=1}^{\alpha_j} t_s^j\right) or \left(\sum_{s=1}^{\alpha_i} t_s^i = \sum_{s=1}^{\alpha_j} t_s^j\ and\ \& |\lambda_i| < |\lambda_j|\right), \end{cases},$$

where λ_i and λ_j are cycles for λ_i^{β} and λ_j^{β}, respectively. Here we consider $(X, \#, T)$ is a $(P - \rho - A)$. See table (1).

Table 1: $(X, \#, T)$ is a $(P - \rho - A)$.

#	{6,7}	{1,5,3,4}	{2,8,9}	{10,11}
{6,7}	{6,7}	{6,7}	{6,7}	{6,7}
{1,5,3,4}	{1,5,3,4}	{6,7}	{2,8,9}	{10,11}
{2,8,9}	{2,8,9}	{2,8,9}	{6,7}	{10,11}
{10,11}	{10,11}	{10,11}	{10,11}	{6,7}

Definition3.3: Let $(X, \#, T)$ be a $(P - \rho - A)$. We say $(X, \#, T)$ is a regular permutation ρ-algebra $(RP - \rho - A)$ if and only if $\lambda_i^{\beta} \# T = \lambda_i^{\beta}, \forall \lambda_i^{\beta} \in X$.

Example 3.4: Let $(X, \#, T)$ be a $(P-\rho-A)$in Example (3.2). Then, we consider that $(X, \#, T)$ is a $(RP-\rho-A)$.

Remark 3.5: If $(X, \#, T)$ is a $(P-\rho-A)$ and $\emptyset \neq S \subseteq X$ is closed under $\#$, then $T \in S$ and $(S, \#, T)$ is a $(P-\rho-A)$. Also, if $(X, \#, T)$is a $(RP-\rho-A)$. Then $(S, \#, T)$ is a $(RP-\rho-A)$.

Lemma 3.6: If $(X, \#, T)$ is a $(RP-\rho-A)$ and X has at least two β −sets. Then $(X, \#, T)$ is "non-associative".

Proof: Let $(X, \#, T)$ be a $(RP-\rho-A)$ and X has at least two β −sets, say λ_i^β is a β −set in X with $\lambda_i^\beta \neq T$. If it is associative, then $\left(\lambda_i^\beta \# \lambda_i^\beta\right) \# \lambda_i^\beta = \lambda_i^\beta \# \left(\lambda_i^\beta \# \lambda_i^\beta\right) = \lambda_i^\beta \# T = \lambda_i^\beta$ this is because X is a $(RP-\rho-A)$. By using 2 of Definition 3.1, we have $\left(\lambda_i^\beta \# \lambda_i^\beta\right) \# \lambda_i^\beta = T \# \lambda_i^\beta = T$. Then $\lambda_i^\beta = T$, but this a contradiction with our hypotheses $\lambda_i^\beta \neq T$. Hence $(X, \#, T)$ is "non-associative". That means regular permutation ρ -algebras are the "most non-associative".

Definition 3.7: Assume that $(X, \#, T)$ is a $(P-\rho-A)$ and $\lambda_i^\beta \in X$. Define $\lambda_i^\beta \# X = \{\lambda_i^\beta \# \lambda_m^\beta \mid \lambda_m^\beta \in X\}$. We say X is a *permutation edge ρ-Algebra* $(PE-\rho-A)$ if for any $\lambda_i^\beta \in X$, $\lambda_i^\beta \# X = \left\{\lambda_i^\beta, T\right\}$.

Example 3.8:
Let $\beta = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 \\ 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 1 & 2 \end{pmatrix}$ be a permutation in S_{12}. Since $\beta = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 \\ 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 1 & 2 \end{pmatrix} = (1\ 3\ 5\ 7\ 9\ 11)(2\ 4\ 6\ 8\ 10\ 12)$. So we have $X = \left\{\lambda_i^\beta\right\}_{i=1}^{2} = \{\{1, 3, 5, 7, 9, 11\}, \{2, 4, 6, 8, 10, 12\}\}$, and $T = \{1, 3, 5, 7, 9, 11\}$. Define $\#: X \times X \longrightarrow X$ by;

$$\lambda_i^\beta \# \lambda_j^\beta = \begin{cases} T, if \left(\lambda_i^\beta = T\right) or \left(\lambda_i^\beta = \lambda_j^\beta\right), \\ \lambda_j^\beta, if\ otherwies. \end{cases}$$

Here we get $(X, \#, T)$ is a $(PE-\rho-A)$.

Lemma 3.9: Assume that $(X, \#, T)$ is a $(PE-\rho-A)$. Then $(X, \#, T)$ is a $(RP-\rho-A)$.

Proof: We need to prove that $\lambda_i^\beta \# T = \lambda_i^\beta$, for any β −set $\lambda_i^\beta \in X$. If $\lambda_i^\beta = T$, then it is hold from 2 of Definition 3.1. Also, let $\lambda_i^\beta \neq T$, since $(X, \#, T)$ is $(PE-\rho-A)$, then either $\lambda_i^\beta \# T = \lambda_i^\beta$ or $\lambda_i^\beta \# T = T$. If $\lambda_i^\beta \# T = T$, then by 2 and 3 of Definition 3.1, we get $\lambda_i^\beta = T$, a contradiction. Hence $\lambda_i^\beta \# T = \lambda_i^\beta, \forall\ \lambda_i^\beta \in X$. Therefore, $(X, \#, T)$ is a $(RP-\rho-A)$.

Proposition 3.10: If $(X, \#, T)$ is a $(PE-\rho-A)$, then $\left(\lambda_i^\beta \# \left(\lambda_i^\beta \# \lambda_j^\beta\right)\right) \# \lambda_j^\beta = T$, $\forall\ \lambda_i^\beta, \lambda_j^\beta \in X$.

Proof: If $\lambda_i^\beta = T$, then $\left(\lambda_i^\beta \Cap \left(\lambda_i^\beta \Cap \lambda_j^\beta\right)\right) \Cap \lambda_j^\beta = T$ by 2 of Definition 3.1. Let $\lambda_i^\beta \neq T$. Assume $\left(\lambda_i^\beta \Cap \left(\lambda_i^\beta \Cap \lambda_j^\beta\right)\right) \Cap \lambda_j^\beta \neq T$, for some $\lambda_j^\beta \in X$. Let $\lambda_\alpha^\beta = \lambda_i^\beta \Cap \left(\lambda_i^\beta \Cap \lambda_j^\beta\right)$. Then $\lambda_\alpha^\beta \Cap \lambda_j^\beta \neq T$ and $\lambda_\alpha^\beta \neq T$. This means that $\lambda_i^\beta \neq \lambda_i^\beta \Cap \lambda_j^\beta \in \lambda_i^\beta \Cap X = \left\{\lambda_i^\beta, T\right\}$ and hence $\lambda_i^\beta \Cap \lambda_j^\beta = T$. It follows from Lemma 3.9 that

$$\left(\lambda_i^\beta \Cap \left(\lambda_i^\beta \Cap \lambda_j^\beta\right)\right) \Cap \lambda_j^\beta = \left(\lambda_i^\beta \Cap T\right) \Cap \lambda_j^\beta = \lambda_i^\beta \Cap \lambda_j^\beta = T.$$

A contradiction. Hence $\left(\lambda_i^\beta \Cap \left(\lambda_i^\beta \Cap \lambda_j^\beta\right)\right) \Cap \lambda_j^\beta = T, \forall \lambda_i^\beta, \lambda_j^\beta \in X$.

Definition 3.11: Assume that $(X, \Cap, T)$ is a $(P-\rho-A)$. We say it is a *permutation* ρ *-transitive algebra* $(P-\rho-TA)$ if $\lambda_i^\beta \Cap \lambda_k^\beta = T$ and $\lambda_k^\beta \Cap \lambda_j^\beta = T$ imply $\lambda_i^\beta \Cap \lambda_j^\beta = T$

Example 3.12: Let $(X, \#, T)$ be a $(P-\rho-A)$ in Example 3.8, thus we get $(X, \#, T)$ is a $(P-\rho-TA)$, since $T\#T = T$, $T\#\{2,4,6,8,10,12\} = T$, and $\{2,4,6,8,10,12\}\#\{2,4,6,8,10,12\} = T$ the all cases their compositions are equal T, also

(1) $T\#T = T$ & $T\#\{2,4,6,8,10,12\} = T \rightarrow T\#\{2,4,6,8,10,12\} = T$,

(2) $T\#\{2,4,6,8,10,12\} = T\&\{2,4,6,8,10,12\}\#,\{2,4,6,8,10,12\} = T \rightarrow$
$T\#\{2,4,6,8,10,12\} = T$

Proposition 3.13: Assume that $(X, \Cap, T)$ is a $(P-\rho-TA)$ and $(PE-\rho-A)$. Then

$$\left(\left(\lambda_i^\beta \Cap \lambda_j^\beta\right) \Cap \left(\lambda_i^\beta \Cap \lambda_k^\beta\right)\right) \Cap \left(\lambda_k^\beta \Cap \lambda_j^\beta\right) = T, \forall \lambda_i^\beta, \lambda_j^\beta, \lambda_k^\beta \in X.$$

Proof: Since $(X, \Cap, T)$ is a $(PE-\rho-A)$, then by Proposition 3.10, we obtain $\left(\lambda_i^\beta \Cap \left(\lambda_i^\beta \Cap \lambda_j^\beta\right)\right) \Cap \lambda_j^\beta = T$, $\forall \lambda_i^\beta, \lambda_j^\beta, \lambda_k^\beta \in X$. Assume that $\left(\left(\lambda_i^\beta \Cap \lambda_j^\beta\right) \Cap \left(\lambda_i^\beta \Cap \lambda_k^\beta\right)\right) \Cap \left(\lambda_k^\beta \Cap \lambda_j^\beta\right) \neq T$, for some $\lambda_i^\beta, \lambda_j^\beta, \lambda_k^\beta \in X$. That means $\left(\lambda_i^\beta \Cap \lambda_j^\beta\right) \Cap \left(\lambda_i^\beta \Cap \lambda_k^\beta\right) \neq T$, by 2 of Definition 3.1. However

$$\left(\lambda_i^\beta \Cap \lambda_j^\beta\right) \Cap \left(\lambda_i^\beta \Cap \lambda_k^\beta\right) \in \left(\lambda_i^\beta \Cap \lambda_j^\beta\right) \Cap X = \left\{\lambda_i^\beta \Cap \lambda_j^\beta, T\right\},$$

$$\left(\lambda_i^\beta \Cap \lambda_j^\beta\right) \Cap \left(\lambda_i^\beta \Cap \lambda_k^\beta\right) = \lambda_i^\beta \Cap \lambda_j^\beta \qquad \ldots\ldots\ldots(1)$$

Also, $\lambda_i^\beta \Cap \lambda_j^\beta \in \lambda_i^\beta \# X = \left\{\lambda_i^\beta, T\right\}$.

If $\lambda_i^\beta \Cap \lambda_j^\beta = T$, then

$$\begin{aligned} T &\neq \left(\left(\lambda_i^\beta \Cap \lambda_j^\beta\right) \Cap \left(\lambda_i^\beta \Cap \lambda_k^\beta\right)\right) \Cap \left(\lambda_k^\beta \Cap \lambda_j^\beta\right) \\ &= \left(T \Cap \left(\lambda_i^\beta \Cap \lambda_k^\beta\right)\right) \Cap \left(\lambda_k^\beta \Cap \lambda_j^\beta\right) \\ &= T \Cap \left(\lambda_k^\beta \Cap \lambda_j^\beta\right) = T \end{aligned}$$

Which is a contradiction. It follows that

$$\lambda_i^\beta \Cap \lambda_j^\beta = \lambda_i^\beta \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2)$$

Hence, $\lambda_i^\beta = \lambda_i^\beta \Cap \lambda_j^\beta (From\ (2))$

$$= \left(\lambda_i^\beta \Cap \lambda_j^\beta\right) \Cap \left(\lambda_i^\beta \Cap \lambda_k^\beta\right) \text{(From (1))}$$

$= \lambda_i^\beta \# \left(\lambda_i^\beta \# \lambda_k^\beta \right).$ (From (2))

That is, $\lambda_i^\beta = \lambda_i^\beta \# \left(\lambda_i^\beta \# \lambda_k^\beta \right)$ ……………………………(3)

If $\lambda_i^\beta \# \lambda_k^\beta \neq T$, then $\lambda_i^\beta \# \lambda_k^\beta = \lambda_i^\beta$, since X is a $(PE-\rho-A)$. From (3), we have that

$$\lambda_i^\beta = \lambda_i^\beta \# \left(\lambda_i^\beta \# \lambda_k^\beta \right) = \lambda_i^\beta \# \lambda_i^\beta = T.$$

This means that

$$T \neq \left(\left(\lambda_i^\beta \# \lambda_j^\beta \right) \# \left(\lambda_i^\beta \# \lambda_k^\beta \right) \right) \# \left(\lambda_k^\beta \# \lambda_j^\beta \right)$$
$$= \left(\lambda_i^\beta \# \lambda_i^\beta \right) \# \left(\lambda_k^\beta \# \lambda_j^\beta \right) \quad \left(from\ (2)\ and\ \lambda_i^\beta \# \lambda_k^\beta = \lambda_i^\beta \right)$$
$$= T \# \left(\lambda_k^\beta \# \lambda_j^\beta \right) = T, \text{ and this is a contradiction.}$$

Thus we conclude that

$$\lambda_i^\beta \# \lambda_k^\beta = T \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (4)$$

We claim that $\lambda_i^\beta \# \lambda_k^\beta = T$. If $\lambda_k^\beta \# \lambda_j^\beta = \lambda_k^\beta$, then

$$T \neq \left(\left(\lambda_i^\beta \# \lambda_j^\beta \right) \# \left(\lambda_i^\beta \# \lambda_k^\beta \right) \right) \# \left(\lambda_k^\beta \# \lambda_j^\beta \right)$$
$$= \left(\left(\lambda_i^\beta \# \lambda_j^\beta \right) \# T \right) \# \lambda_k^\beta \quad \left(\text{from (4) and } \lambda_k^\beta \# \lambda_j^\beta = \lambda_k^\beta \right)$$
$$= \left(\lambda_i^\beta \# \lambda_j^\beta \right) \# \lambda_k^\beta \quad \text{(From Lemma 3.9)}$$
$$= \lambda_i^\beta \# \lambda_k^\beta \quad \text{(From (2))}$$
$$= T. \quad \text{(From (4))}$$

Which is a contradiction. Thus, we have that $\lambda_i^\beta \# \lambda_k^\beta = T$ and $\lambda_k^\beta \# \lambda_j^\beta = T$. Since X is a $(P-\rho-TA)$, $\lambda_i^\beta \# \lambda_j^\beta = T$, and hence
$T \neq \left(\left(\lambda_i^\beta \# \lambda_j^\beta \right) \# \left(\lambda_i^\beta \# \lambda_k^\beta \right) \right) \# \left(\lambda_k^\beta \# \lambda_j^\beta \right) = \left(T \# \left(\lambda_i^\beta \# \lambda_k^\beta \right) \right) \# \left(\lambda_k^\beta \# \lambda_j^\beta \right) = T \# \left(\lambda_k^\beta \# \lambda_j^\beta \right) = T$, a contradiction. Hence $\left(\left(\lambda_i^\beta \# \lambda_j^\beta \right) \# \left(\lambda_i^\beta \# \lambda_k^\beta \right) \right) \# \left(\lambda_k^\beta \# \lambda_j^\beta \right) = T$.

Definition 3.14: Assume that $(X, \#, T)$ is a $(P-\rho-A)$ and $\emptyset \neq I \subseteq X$. I is called a *permutation ρ-subalgebra* $(P-\rho-SA)$ of X if $\lambda_i^\beta \# \lambda_j^\beta \in I$ whenever $\lambda_i^\beta \in I$ and $\lambda_j^\beta \in I$. Also, any $\emptyset \neq I \subseteq X$ is called a permutation ρ-ideal $(P-\rho-I)$ of X if it satisfies
1) $\lambda_i^\beta \# \lambda_j^\beta \in I$ and $\lambda_j^\beta \in I$ implies that $\lambda_i^\beta \in I$.
2) $\lambda_i^\beta \in I$ and $\lambda_j^\beta \in I$ implies that $\lambda_i^\beta \# \lambda_j^\beta \in I$.

Example 3.15:
Let (S_{12}, o) be a symmetric group and $\beta = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 \\ 6 & 3 & 5 & 2 & 4 & 7 & 1 & 9 & 8 & 12 & 10 & 11 \end{pmatrix}$ be a permutation in S_{12}. Since $\beta = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 \\ 6 & 3 & 5 & 2 & 4 & 7 & 1 & 9 & 8 & 12 & 10 & 11 \end{pmatrix} = (1\ 6\ 7)(2\ 3\ 5\ 4)(8\ 9)(10\ 12\ 11)$.
Therefore, we have $X = \left\{ \lambda_i^\beta \right\}_{i=1}^{4} = \{\{10,12,11\}, \{2,3,5,4\}, \{8,9\}, \{1,6,7\}\}$, and $T = \{1,6,7\}$.
Define $\#: X \times X \longrightarrow X$ by

$$\lambda_i^\beta \# \lambda_j^\beta = \begin{cases} T, if\left(\lambda_i^\beta = T\right) or \left(\lambda_i^\beta = \lambda_j^\beta\right), \\ \lambda_i^\beta, if(\sum_{s=1}^{\alpha_i} t_s^i > \sum_{s=1}^{\alpha_j} t_s^j) \ or \ (\sum_{s=1}^{\alpha_i} t_s^i = \sum_{s=1}^{\alpha_j} t_s^j \ and \ |\lambda_i| > |\lambda_j|), \\ \lambda_j^\beta, if(\sum_{s=1}^{\alpha_i} t_s^i < \sum_{s=1}^{\alpha_j} t_s^j) or \ (\sum_{s=1}^{\alpha_i} t_s^i = \sum_{s=1}^{\alpha_j} t_s^j \ and \ \&|\lambda_i| < |\lambda_j|), \end{cases}$$

where λ_i and λ_j are cycles for λ_i^β and λ_j^β, respectively. Here we consider $(X, \#, T)$ is a $(P - \rho - A)$. See table (2).

Table 2: $(X, \#, T)$ is a $(P - \rho - A)$.

#	{1,6,7}	{2,3,5,4}	{8,9}	{10,12,11}
{1,6,7}	{1,6,7}	{1,6,7}	{1,6,7}	{1,6,7}
{2,3,5,4}	{2,3,5,4}	{1,6,7}	{8,9}	{10,12,11}
{8,9}	{8,9}	{8,9}	{1,6,7}	{10,12,11}
{10,12,11}	{10,12,11}	{10,12,11}	{10,12,11}	{1,6,7}

Let $F = \{\{1,6,7\}, \{8,9\}\}$, and $K = \{\{1,6,7\}, \{2,3,5,4\}\}$ be subsets of X, then each of F and K is $(P - \rho - SA)$. Also, K is $(P - \rho - I)$, but F is not $(P - \rho - I)$ since {2,3,5,4} #{8,9} ∈ K and {8,9} ∈ K, but {2,3,5,4} ∉ K.

Proposition 3.16: Assume that $(X, \#, T)$ is a $(P - \rho - A)$. If $\lambda_i^\beta \neq \lambda_j^\beta$ and $\lambda_i^\beta \# \lambda_j^\beta = T$, then $\lambda_j^\beta \# \lambda_i^\beta \neq T$.

Proof: It follows from 3 of Definition 3.1.

Lemma 3.17: Assume that $(X, \#, T)$ is a $(P - \rho - A)$. If I is a $(P - \rho - I)$ of X, then $T \in I$.

Proof: Since $I \neq \emptyset$, there exists λ_i^β in I and hence $T = \lambda_i^\beta \# \lambda_i^\beta \in I$ by Definition (3.14).

Proposition 3.18: Assume that I is a $(P - \rho - I)$ of $(X, \#, T)$. If $\lambda_i^\beta \in I$ and $\lambda_j^\beta \# \lambda_i^\beta = T$, then $\lambda_j^\beta \in I$.

Proof: Assume that $\lambda_i^\beta \in I$ and $\lambda_j^\beta \# \lambda_i^\beta = T$. By Lemma 3.15 and Definition 3.14, we have that $\lambda_j^\beta \in I$.

Definition 3.19: Let $(X, \#, T)$ be a $(P - \rho - A)$. A $(P - \rho - I)$ I of X is called a *permutation* $\rho^\#$*-ideal* $(P - \rho^\# - I)$ of X if it satisfies the identity;
$\lambda_i^\beta \# \lambda_k^\beta \in I$ whenever $\lambda_i^\beta \# \lambda_j^\beta \in I$ and $\lambda_j^\beta \# \lambda_k^\beta \in I, \forall \lambda_i^\beta, \lambda_j^\beta, \lambda_k^\beta \in X$ ... ..$(\rho - i)$.

Example 3.20: Let $\beta = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ 4 & 3 & 2 & 1 & 6 & 7 & 9 & 5 & 10 & 8 \end{pmatrix}$ be a permutation in S_{10}. Since $\beta = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ 4 & 3 & 2 & 1 & 6 & 7 & 9 & 5 & 10 & 8 \end{pmatrix} = (1\ 4)(2\ 3)(5\ 6\ 7\ 9\ 10\ 8)$. Therefore, we have $X = \left\{\lambda_i^\beta\right\}_{i=1}^3 = \{\{1,4\}, \{2,3\}, \{5,6,7,9,10,8\}$, and $T = \{1,4\}$. Define $\#: X \times X \longrightarrow X$ by;

$$\lambda_i^\beta \# \lambda_j^\beta = \begin{cases} T, if \left(\lambda_i^\beta = T\right) or \left(\lambda_i^\beta = \lambda_j^\beta\right), \\ \lambda_i^\beta, if \left(\lambda_j^\beta = T\right) or \ (\lambda_i^\beta \text{and } \lambda_j^\beta \text{ are not disjoint with } |\lambda_i| > |\lambda_j|), \\ \lambda_j^\beta, if \ (\lambda_i^\beta \text{and } \lambda_j^\beta \text{ are not disjoint with } |\lambda_i| < |\lambda_j|), \end{cases}$$

where λ_i and λ_j are cycles for λ_i^β and λ_j^β, respectively. Here we consider $(X, \#, T)$ is a $(P-\rho-A)$. See table (3).

Table 3: $(X, \#, T)$ is a $(P-\rho-A)$.

#	{1,4}	{2,3}	{5,6,7,9,10,8}
{1,4}	{1,4}	{1,4}	{1,4}
{2,3}	{2,3}	{1,4}	{5,6,7,9,10,8}
{5,6,7,9,10,8}	{5,6,7,9,10,8}	{5,6,7,9,10,8}	{1,4}

Let $H = \{1,4\}$. Then H is a $(P-\rho-I)$ of X. Moreover, H is a $(P-\rho^{\#}-I)$ of X.

Remark 3.21: From Definition (3.19) every $(P-\rho^{\#}-I)$ is a $(P-\rho-I)$. However, it is not in general any $(P-\rho-I)$ is a $(P-\rho^{\#}-I)$. See K in Example 3.15, it is $(P-\rho-I)$, but not $(P-\rho^{\#}-I)$ since $\{2,3,5,4\}\#\{1,6,7\} \in K$ and $\{1,6,7\}\#\{8,9\} \in K$, but $\{2,3,5,4\}\#\{8,9\} \notin K$.

Definition 3.22: Let $(X, \#, T)$ be an arbitrary $(P-\rho-A)$ and not a $(PE-\rho-A)$. Define a binary operation $\oplus: X \times X \longrightarrow X$ by

$$\lambda_i^\beta \oplus \lambda_j^\beta = \begin{cases} \lambda_i^\beta, & if \ \ \lambda_i^\beta \# \lambda_j^\beta \neq T \\ T, & O.W. \end{cases}$$

Then we can easily see that $(X, \oplus, T)$ such that (1) ,(2) and (3) in Definition 3.1. Also, such that the condition of edge for any $\lambda_i^\beta \in X$ (i.e, $\lambda_i^\beta \oplus X = \left\{\lambda_i^\beta, T\right\}, \forall \lambda_i^\beta \in X$), but it is not $(P-\rho-A)$. So, we say $(X, \oplus, T)$ is *permutation edge weakly ρ-algebra* $(PE-W\rho-A)$ of $(P-\rho-A)$ $(X, \#, T)$. Suppose now that $\lambda_i^\beta \oplus X = T$. Then $\lambda_i^\beta \# \lambda_j^\beta = T$ for all $\lambda_j^\beta \in X$. In particular, $\lambda_i^\beta \# T = T = T \# \lambda_i^\beta$, so that $\lambda_i^\beta = T$. Hence, if $\lambda_i^\beta \neq T$, then $\lambda_i^\beta \oplus X = \left\{\lambda_i^\beta, T\right\}$. Moreover, if $(X, \oplus, T)$ is $(SPE-W\rho-A)$, then $(X, \oplus, T)$ is called *permutation edge weakly ρ-transitive algebra* $(SPE-W\rho-TA)$ if $\lambda_i^\beta \oplus \lambda_k^\beta = T$ and $\lambda_k^\beta \oplus \lambda_j^\beta = T$ imply $\lambda_i^\beta \oplus \lambda_j^\beta = T, \forall \lambda_i^\beta, \lambda_j^\beta, \lambda_k^\beta \in M$.

Proposition 3.23: Let $(X, \#, T)$ be $(P-\rho-A)$. Then $(X, \#, T)$ is a $(P-\rho-TA)$ if and only $(X, \oplus, T)$ is $(PE-W\rho-TA)$

Proof: If $(X, \#, T)$ is a $(P-\rho-TA)$, then $\lambda_i^\beta \oplus \lambda_k^\beta = T$ and $\lambda_k^\beta \oplus \lambda_j^\beta = T$ implies that $\lambda_i^\beta \# \lambda_k^\beta = T = \lambda_k^\beta \# \lambda_j^\beta$, so that $\lambda_i^\beta \# \lambda_j^\beta = T$ and $\lambda_i^\beta \oplus \lambda_j^\beta = T$ as well.

Conversely, if $(X, \oplus, T)$ is a $(PE-W\rho-TA)$, then $\lambda_i^\beta \# \lambda_k^\beta = T$ and $\lambda_k^\beta \# \lambda_j^\beta = T$ imply $\lambda_i^\beta \oplus \lambda_k^\beta = T = \lambda_k^\beta \oplus \lambda_i^\beta$, so that $\lambda_i^\beta \oplus \lambda_j^\beta = T$ and $\lambda_i^\beta \# \lambda_j^\beta = T$ as well.

Definition 3.24: Let $(X, \#, T)$ be a $(P-\rho-A)$. We say $(X, \#, T)$ is a *permutation ρ*-algebra* $(P-\rho^*-A)$ if it satisfies the identity $\left(\lambda_i^\beta \# \lambda_j^\beta\right) \# \lambda_i^\beta = T, \ \forall \lambda_i^\beta, \lambda_j^\beta \in X$.

Example 3.25: Let $(X,\#,T)$ be a $(P-\rho-A)$ in Example 3.8, thus we get $(X,\#,T)$ is a $(P-\rho-TA)$, Also, we have the following:
(1) $(T\#T)\#T=T$,
(2) $(\{2,4,6,8,10,12\}\#\{2,4,6,8,10,12\})\#\{2,4,6,8,10,12\}=T$,
(3) $(T\#\{2,4,6,8,10,12\})\#T=T$,
(4) $(\{2,4,6,8,10,12\}\#T)\#\{2,4,6,8,10,12\}=T$.
Hence $(X,\#,T)$ is a $(P-\rho^*-A)$.

Proposition 3.26: Let $(X,\#,T)$ be a $(P-\rho^*-A)$. Then $\emptyset\neq I\subseteq X$ is a $(P-\rho-I)$ if;
(1)-$T\in I$.
(2)- $\lambda_i^\beta\in I$ and $\lambda_j^\beta\#\lambda_i^\beta\in I$ implies $\lambda_j^\beta\in I$, for all $\lambda_i^\beta,\lambda_j^\beta\in X$.

Proof: To prove that I is a $(P-\rho-I)$, we need only to show that $\lambda_i^\beta\#\lambda_j^\beta\in I$, for every $\lambda_i^\beta\in I$, $\lambda_j^\beta\in I$. From Definition (3.24), we have that $\left(\lambda_i^\beta\#\lambda_j^\beta\right)\#\lambda_i^\beta=T$, for all $\lambda_i^\beta,\lambda_j^\beta\in X$, but $T\in I$ and hence it follows from Proposition 3.18 that $\lambda_i^\beta\#\lambda_j^\beta\in I$. Hence I is a $(P-\rho-I)$ of X.

Definition 3.27: Let each of $(X,\#,T_X)$ and $(Y,\#,T_Y)$ be $(P-\rho-A)$. A mapping $f:X\longrightarrow Y$ is called a *permutation ρ-morphism* if $f\left(\lambda_i^\beta\#\lambda_j^\beta\right)=f\left(\lambda_i^\beta\right)\#f\left(\lambda_j^\beta\right)$, $\forall\,\lambda_i^\beta,\lambda_j^\beta\in X$. Also, $f(T_X)=T_Y$.

Definition 3.28: Let $(X,\#,T)$ be $(P-\rho-A)$ and I be a $(P-\rho^\#-I)$ of X. For any $\lambda_i^\beta,\lambda_j^\beta\in X$, we define $\lambda_i^\beta\sim\lambda_j^\beta$ if and only if $\lambda_i^\beta\#\lambda_j^\beta\in I$ and $\lambda_j^\beta\#\lambda_i^\beta\in I$. We claim that $\sim$ is an *equivalence relation* on X. Since $T\in I$, we have that $\lambda_i^\beta\#\lambda_i^\beta=T\in I$, $i.e.$ $\lambda_i^\beta\sim\lambda_i^\beta$, for any $\lambda_i^\beta\in X$. If $\lambda_i^\beta\sim\lambda_j^\beta$ and $\lambda_j^\beta\sim\lambda_k^\beta$, then $\lambda_i^\beta\#\lambda_j^\beta$, $\lambda_j^\beta\#\lambda_i^\beta\in I$ and $\lambda_j^\beta\#\lambda_k^\beta$, $\lambda_k^\beta\#\lambda_j^\beta\in I$.
From the fact of the Definition 3.19, we have $\lambda_i^\beta\#\lambda_k^\beta\in I$ whenever $\lambda_i^\beta\#\lambda_j^\beta\in I$ and $\lambda_j^\beta\#\lambda_k^\beta\in I$. We have that $\lambda_i^\beta\#\lambda_k^\beta$, $\lambda_k^\beta\#\lambda_i^\beta\in I$ and hence $\lambda_i^\beta\sim\lambda_k^\beta$. Thus $\sim$ is transitive. The symmetry of $\sim$ is trivial. By condition $(\rho-ii)$, we can easily see that $\sim$ is a congruence relation on X.

Condition $(\rho-ii)$: Let $(X,\#,T)$ be $(P-\rho-A)$ and I be a $(P-\rho^\#-I)$ of X. If it satisfies $\lambda_i^\beta\#\lambda_j^\beta\in I$ and $\lambda_j^\beta\#\lambda_i^\beta\in I$ imply that $\left(\lambda_i^\beta\#\lambda_k^\beta\right)\#\left(\lambda_j^\beta\#\lambda_k^\beta\right)\in I$ and $\left(\lambda_k^\beta\#\lambda_i^\beta\right)\#\left(\lambda_k^\beta\#\lambda_j^\beta\right)\in I$, for all $\lambda_i^\beta,\lambda_j^\beta,\lambda_k^\beta\in X$. Then we say that I is a permutation ρ^*-ideal $(P-\rho^*-I)$ of X.

Proposition 3.29: Let $f:X\longrightarrow Y$ be a permutation ρ-morphism from a $(P-\rho-A)$ X into a $(P-\rho-TA)$ Y. Then $Kerf$ is a $(P-\rho^*-I)$ of X.

Proof: We only need to prove $(\rho-i)$ in Definition 3.19 and $(\rho-ii)$.
If $\lambda_i^\beta\#\lambda_j^\beta$, $\lambda_j^\beta\#\lambda_k^\beta\in Kerf$, then
$$f\left(\lambda_i^\beta\right)\#f\left(\lambda_j^\beta\right)=T_Y=f\left(\lambda_j^\beta\right)\#f\left(\lambda_k^\beta\right).$$
Since Y is a $(P-\rho-TA)$, we obtain $f\left(\lambda_i^\beta\right)\#f\left(\lambda_k^\beta\right)=T$ and hence

$\lambda_i^\beta \# \lambda_k^\beta \in Kerf$, which shows $(\rho - i)$ in Definition 3.19.
Let $\lambda_i^\beta \# \lambda_j^\beta$, $\lambda_j^\beta \# \lambda_i^\beta \in Kerf$. Then

$$f(\lambda_i^\beta) \# f(\lambda_j^\beta) = T_Y = f(\lambda_j^\beta) \# f(\lambda_i^\beta).$$

By 3 of Definition 3.1, we have that $f(\lambda_i^\beta) = f(\lambda_j^\beta)$. It follows that

$$\begin{aligned} f\left(\left(\lambda_i^\beta \# \lambda_k^\beta\right) \# \left(\lambda_j^\beta \# \lambda_k^\beta\right)\right) &= f\left(\lambda_i^\beta \# \lambda_k^\beta\right) \# f\left(\lambda_j^\beta \# \lambda_k^\beta\right) \\ &= \left(f(\lambda_i^\beta) \# f(\lambda_k^\beta)\right) \# \left(f(\lambda_i^\beta) \# f(\lambda_k^\beta)\right) \\ &= T_Y. \end{aligned}$$

And hence $\left(\lambda_i^\beta \# \lambda_k^\beta\right) \# \left(\lambda_j^\beta \# \lambda_k^\beta\right) \in Kerf.$
Similarly, $\left(\lambda_k^\beta \# \lambda_i^\beta\right) \# \left(\lambda_k^\beta \# \lambda_j^\beta\right) \in Kerf.$
Which proves $(\rho - ii)$.

Definition 3.30: We denote the congruence class containing λ_i^β by $\left[\lambda_i^\beta\right]_I$, i.e. $\left[\lambda_i^\beta\right]_I = \left\{\lambda_j^\beta \in X \mid \lambda_i^\beta \sim \lambda_j^\beta\right\}$. We say that $\lambda_i^\beta \sim \lambda_j^\beta$ if and only if $\left[\lambda_i^\beta\right]_I = \left[\lambda_j^\beta\right]_I$. Denote the set of all equivalence classes of X by X/I, i.e. $X/I = \left\{\left[\lambda_i^\beta\right]_I \mid \lambda_i^\beta \in X\right\}$.

Lemma 3.31: Let$(X, \#, T)$ be $(P - \rho - A)$ and I be a $(P-\rho^* - I)$ of X. Then $I = [T]_I$.

Proof: If $\lambda_i^\beta \in I$, then

$$\lambda_i^\beta \# T \in I \# X \subseteq I$$

And hence $\lambda_i^\beta \in [T]_I$, i.e. $I \subseteq [T]_I$. Since

$$\begin{aligned} [T]_I &= \left\{\lambda_i^\beta \in X \mid \lambda_i^\beta \sim T\right\} \\ &= \left\{\lambda_i^\beta \in X \mid \lambda_i^\beta \# T, T \# \lambda_i^\beta \in I\right\} \\ &= \left\{\lambda_i^\beta \in X \mid \lambda_i^\beta \# T \in I\right\} \quad (T \in I) \\ &\subseteq I \qquad \text{(From Definition 3.30)} \end{aligned}$$

It then follows that $I = [T]_I$.

Proposition 3.32: Let$(X, \#, T)$ be $(P - \rho - A)$ and I be a $(P-\rho^* - I)$ of X. If we define

$$\left[\lambda_i^\beta\right]_I \# \left[\lambda_j^\beta\right]_I = \left[\lambda_i^\beta \# \lambda_j^\beta\right]_I \quad \left(\lambda_i^\beta, \lambda_j^\beta \in X\right),$$

Then $(X/I, \#, T)$ is a $(P - \rho - A)$.

Proof: Since $\sim$ is a congruence relation on X, $\lambda_i^\beta \# \lambda_j^\beta \sim \lambda_i^{\beta\prime} \# \lambda_j^{\beta\prime}$ for any $\lambda_i^\beta \sim \lambda_i^{\beta\prime}$, $\lambda_j^\beta \sim \lambda_j^{\beta\prime}$. This means that

$$\left[\lambda_i^\beta\right]_I \# \left[\lambda_j^\beta\right]_I = \left[\lambda_i^\beta \# \lambda_j^\beta\right]_I$$

is well defined.
Let $\left[\lambda_i^\beta\right]_I, \left[\lambda_j^\beta\right]_I \in X/I$ with $\left[\lambda_i^\beta\right]_I \# \left[\lambda_j^\beta\right]_I = [T]_I = \left[\lambda_j^\beta\right]_I \# \left[\lambda_j^\beta\right]_I$.
Then $\left[\lambda_i^\beta \# \lambda_j^\beta\right]_I = [T]_I = \left[\lambda_j^\beta \# \lambda_i^\beta\right]_I$ and $\lambda_i^\beta \# \lambda_j^\beta$, $\lambda_j^\beta \# \lambda_i^\beta \in I$. Thus $\lambda_i^\beta \sim \lambda_j^\beta$ and

$$\left[\lambda_i^\beta\right]_I = \left[\lambda_j^\beta\right]_I.$$

Not that $(X/I, \#, T)$ is called the quotient permutation ρ -algebra $(QP-\rho-A)$.

Proposition 3.33: Let$(X, \#, T)$ be $(P-\rho-A)$ and I be a $(P-\rho^*-I)$ of X. Then the mapping $\pi: X \longrightarrow X/I$ defined by

$$\pi\left(\lambda_i^\beta\right) = \left[\lambda_i^\beta\right]_I$$

Is a permutation ρ -morphism of X onto the $(QP-\rho-A)$ X/I and the kernel of π is just the set I.

Proof: Since $\left[\lambda_i^\beta \# \lambda_j^\beta\right]_I = \left[\lambda_i^\beta\right]_I \# \left[\lambda_j^\beta\right]_I$, π is a permutation ρ-morphism. From Lemma (3.31), we know that

$$Ker\pi = \left\{\lambda_i^\beta \in X | \pi\left(\lambda_i^\beta\right) = [T]_I\right\}$$
$$= \left\{\lambda_i^\beta \in X | \left[\lambda_i^\beta\right]_I = [T]_I\right\}$$
$$= \left\{\lambda_i^\beta \in X | \lambda_i^\beta \sim T\right\} = [T]_I = I.$$

Proposition 3.34: If $f: X \longrightarrow Y$ is a permutation ρ -morphism from a $(P-\rho-A)$ $(X, \#, T)$ onto a $(P-\rho-TA)$ $(Y, \#, T)$, then $X/Kerf \cong Y$.

Proof: Assume that $\mu: X/Kerf \longrightarrow Y$ such that $\mu\left(\left[\lambda_i^\beta\right]_{Kerf}\right) = f\left(\lambda_i^\beta\right)$.

If $\left[\lambda_i^\beta\right]_{Kerf} = \left[\lambda_j^\beta\right]_{Kerf}$, then $\lambda_i^\beta \# \lambda_j^\beta$, $\lambda_j^\beta \# \lambda_i^\beta \in Kerf$, and so

$$f\left(\lambda_i^\beta\right) \# f\left(\lambda_j^\beta\right) = T = f\left(\lambda_j^\beta\right) \# f\left(\lambda_i^\beta\right).$$

By (3) of Definition (3.1), we have that $f\left(\lambda_i^\beta\right) = f\left(\lambda_j^\beta\right)$, i.e. $\mu\left(\left[\lambda_i^\beta\right]_{Kerf}\right) = \mu\left(\left[\lambda_j^\beta\right]_{Kerf}\right)$. This means that μ is well-defined. For any $\lambda_j^\beta \in Y$ there is an $\lambda_i^\beta \in X$ such that $f\left(\lambda_i^\beta\right) = \lambda_j^\beta$ since f is onto. Hence $\mu\left(\left[\lambda_i^\beta\right]_{Kerf}\right) = f\left(\lambda_i^\beta\right) = \lambda_j^\beta$, which means that μ is onto. If $\mu\left(\left[\lambda_i^\beta\right]_{Kerf}\right) = f\left(\lambda_i^\beta\right) = \lambda_j^\beta$, which means that μ is onto. If $\mu\left(\left[\lambda_i^\beta\right]_{Kerf}\right) \neq \mu\left(\left[\lambda_j^\beta\right]_{Kerf}\right)$ then either $\lambda_i^\beta \# \lambda_j^\beta \notin Kerf$ or $\lambda_j^\beta \# \lambda_i^\beta \notin Kerf$. Without loss of generality, we may assume $\lambda_i^\beta \# \lambda_j^\beta \notin Kerf$. It follows that $f\left(\lambda_i^\beta\right) \# f\left(\lambda_j^\beta\right) = f\left(\lambda_i^\beta \# \lambda_j^\beta\right) \neq T$ and hence $f\left(\lambda_i^\beta\right) \neq f\left(\lambda_j^\beta\right)$. This means that μ is one-to-one and onto. Since

$$\mu\left(\left[\lambda_i^\beta\right]_{Kerf} \# \left[\lambda_j^\beta\right]_{Kerf}\right) = \mu\left(\left[\lambda_i^\beta \# \lambda_j^\beta\right]_{Kerf}\right)$$
$$= f\left(\lambda_i^\beta \# \lambda_j^\beta\right)$$
$$= f\left(\lambda_i^\beta\right) \# f\left(\lambda_j^\beta\right)$$
$$= \mu\left(\left[\lambda_i^\beta\right]_{Kerf}\right) \# \mu\left(\left[\lambda_j^\beta\right]_{Kerf}\right)$$

μ is a permutation ρ -morphism. Thus we have that

$$X/Kerf \cong Y.$$

Completing the proof.

Proposition 3.35: Let $f: X \longrightarrow Y$ be an onto permutation ρ-morphism from the $(PE-\rho-A)$ $(X, \#, T_X)$ to the $(P-\rho-A)(Y, \#, T_Y)$. Then $(Y, \#, T_Y)$ is also a $(PE-\rho-A)$.

Proof: Consider $f\left(\lambda_i^\beta\right)=\lambda_j^\beta, f\left(\lambda_n^\beta\right)=\lambda_m^\beta$. Then

$$\lambda_j^\beta \# \lambda_m^\beta = f\left(\lambda_i^\beta\right) \# f\left(\lambda_n^\beta\right) = f\left(\lambda_i^\beta \# \lambda_n^\beta\right) \in \left\{f\left(\lambda_i^\beta\right), f\left(\lambda_n^\beta\right)\right\} = \left\{\lambda_j^\beta, T\right\}.$$

4. Conclusions

This work introduces some new extensions of ρ -algebras and investigates their properties using permutation sets, which are non-classical sets. Furthermore, non-classical sets such as nano sets [27] and neutrosophic sets [28-31] a result, instead of applying permutation sets in a future study, we will extend our conceptions and conclusions in this paper using nano and neutrosophic sets.

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