DOI: 10.24996/ijs.2019.60.6.22

ISSN: 0067-2904

H-essential Submodules and Homessential Modules

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Abstract

The main goal of this paper is introducing and studying a new concept, which is named H-essential submodules, and we use it to construct another concept called Homessential modules. Several fundamental properties of these concepts are investigated, and other characterizations for each one of them is given. Moreover, many relationships of Homessential modules with other related concepts are studied such as Quasi-Dedekind, Uniform, Prime and Extending modules.

Keywords: Essential submodules, H-essential submodules, Homessential modules.

المقاسات الجزئية الجوهرية من النمط H– و مقاسات هوم إسينشيل

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الخلاصة

إن الهدف الرئيس من هذا البحث هو تقديم ودراسة مفهوم جديد أطلقنا عليه اسم المقاس الجزئي الجوهري من النمط–H والذي استخدمناه في تعريف المفهوم الاخر والذي أطلقنا عليه اسم مقاس هوم إسينشيل (Homessential module). تم أعطاء العديد من النتائج المهمة حول هذين المفهومين، وتشخيص آخر لكل من هذين المفهومين. إضافة الى ذلك فقد تم دراسة علاقة مقاس هوم اسينشيل ببعض المقاسات الأخرى ذات العلاقة مثل مقاس كواسي ديدكند، المقاس المنتظم، المقاس الأولي والمقاس التوسعي.

1. Introduction

Throughout this paper, all rings are commutative with identity and all modules are unitary left R-modules. "A submodule V of a module U is called essential (simply $V \leq_e U$), if the intersection of V with any non-zero submodule of U is not equal to zero" [1]. In this paper, we introduce a new concept, named Homessential module. This concept needs to define a certain type of submodules named H-essential submodules, where a proper submodule V of U is called H-essential, if for each non-zero homomorphism $f \in \mathrm{Hom}_R(\frac{U}{V}, U)$; $f(\frac{U}{V})$ is an essential submodule of U. A module U is called Homessential if every proper submodule of U is H-essential.

In section two; we investigate the main properties of H-essential submodules; we determine the trace of the quotient module over any H-essential submodule, see Proposition (2.8). Also, we show that the intersection of any two H-essential submodules is also H-essential, see Proposition (2.11). Furthermore, another characterization of H-essential submodule is given, see Theorem (2.12), and we show that every rational submodule is H-essential (see 2.13). Moreover, we prove that in the class of polyform module, every essential submodule is H-essential, Corollary (2.15).

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Section three of this paper is devoted to introduce and study the concept of Homessential module, we give another characterization of Homessential module, see Proposition (3.6). Also, the relationships of Homessential module with some other related concepts such as quasi-Dedekind, uniform and prime and extending modules are studied, see the results (3.9), (3.10), (3.12), (3.14) (3.15), (3.16) and (3.17).

2. H-essential Submodules

This section is devoted to studying the main properties of H-essential submodules.

Definition (2.1): Let U be an R-module. A proper submodule V of U is called H-essential, if for each non-zero homomorphism $f \in \mathrm{Hom}_R(\frac{U}{V}, U)$; $f(\frac{U}{V})$ is essential submodule of U.

Examples (2.2):

1. Consider the $\mathbb{Z}$-module $\mathbb{Z}_4$, the submodule $(\bar{2})$ of Z_4 is H-essential, since the non-zero homomorphism in $\mathrm{Hom}_{\mathbb{Z}}(\frac{\mathbb{Z}_4}{(\bar{2})}, \mathbb{Z}_4)$ is only the inclusion homomorphism. In fact $f(\frac{\mathbb{Z}_4}{(\bar{2})})=(\bar{2})$, and $(\bar{2})$ is essential submodule of $\mathbb{Z}_4$.

2. In the $\mathbb{Z}$-module $\mathbb{Z}_6$, the submodule $(\bar{3})$ is not H-essential. In fact; there exists a homomorphism f: $\frac{\mathbb{Z}_6}{(\bar{3})} \to \mathbb{Z}_6$ with f $(\frac{\mathbb{Z}_6}{(\bar{3})})$ is not essential submodule of $\mathbb{Z}_6$.

Proposition (2.3): If V is H-essential submodule of U, then $rU \leq_e U$ for each $(0\neq)\ r \in \mathrm{ann}_R(V)$.

Proof: Let $(0\neq)\ r \in \mathrm{ann}_R(V)$. Define f: $\frac{U}{V} \to U$ by $f(u + V) = ru$ for each $u + V \in \frac{U}{V}$. It is clear that f is well defined and homomorphism. Since V is H-essential, then $f(\frac{U}{V}) = rU$ is essential submodule of U. □

As a consequence of Proposition (2.3) we have the following.

Corollary (2.4): Every non-zero H-essential submodule cannot be faithful.

We need to give the following lemmas.

Lemma (2.5): [1] If A and B are submodules of an R-module U such that $A \leq_e B \leq_e C$. Then $A \leq_e U$ if and only if $A \leq_e B$ and $B \leq_e U$.

The following Lemma can be easily shown.

Lemma (2.6): Let $\{V_i\}_{i\in I}$ be a family of essential submodules of an R-module U. Then $\sum_{i\in I} V_i \leq_e U$.

Proof. It's obvious from Lemma (2.5).

"Recall that a ring R is called Noetherian if every ideal of R is finitely generated" [2, P.55].

Proposition (2.7): Let R be a Noetherian ring, and U be an R-module. If V is H-essential submodule of U, then $\mathrm{ann}_R(V)U \leq_e U$.

Proof: Since R is a Noetherian ring, then $\mathrm{ann}_R(V)$ is finitely generated ideal of R. That is $\mathrm{ann}_R(V) = (r_1, r_2, \ldots, r_n)$ for some $r_i \in \mathrm{ann}_R(V)$. This implies that $\mathrm{ann}_R(V)U = \sum_{i=1}^{n} r_i U$. By Proposition (2.3), $r_iU \leq_e U$ for each i=1,2, . . ., n, and by Lemma (2.6), $(\mathrm{ann}_R(V))U \leq_e U$. □

"For R-modules U and V, the trace of an R-module U is defined by $T_V(U) = \sum_{\varphi} \varphi(U)$ where $\varphi \in \mathrm{Hom}_R(U, V)$ ([2], p.27)". If V=R, then the trace of U on R is denoted by T(U).

Proposition (2.8): Let U be an R-module and T(U) is the trace of U on R. If V is an H-essential submodule, then either $T(\frac{U}{V})$ is zero or $T(\frac{U}{V})U$ is essential submodule of U.

Proof: Suppose that $T(\frac{U}{V}) \neq 0$ and $(0\neq)\ \varphi \in \mathrm{Hom}_R(\frac{U}{V}, R)$. For each $u \in U$, one can define f_u: $R \to U$ by $f_u(r) = ru$; for all $r \in R$. It is clear that f_u is well defined and homomorphism. So $(f_u o \varphi) \in \mathrm{Hom}_R(\frac{U}{V}, U)$. Since V is H-essential submodule of U, then $(f_u o\varphi)\frac{U}{V} \leq_e U$ for each $u \in U$. This implies that $\varphi(\frac{U}{V})u \leq_e U$ for each $u \in U$. Since $T(\frac{U}{V})U = (\sum_{\varphi} \varphi(\frac{U}{V}))U$, so we have $\varphi(\frac{U}{V})U \leq (\sum_{\varphi} \varphi(\frac{U}{V}))U = T(\frac{U}{V})U \leq U$. By Lemma (2.6), we get $T(\frac{U}{V})U \leq_e U$. □

Proposition (2.9): Let U be an R-module. If V is an H-essential submodule of U with $\mathrm{ann}_R(V) \neq 0$, then $\mathrm{ann}_R(U) \subsetneq \mathrm{ann}_R(V)$.

Proof: It is clear that $\mathrm{ann}_R(U) \leq \mathrm{ann}_R(V)$. Now let $(0\neq)\ r \in \mathrm{ann}_R(V)$, by proposition (2.3), $rU \leq_e U$, hence $rU \neq 0$, that is $r \notin \mathrm{ann}_R(U)$. Therefore $\mathrm{ann}_R(U) \subsetneq \mathrm{ann}_R(V)$. □

Examples (2.10):

1. From Example (2.2)(1) , the submodule $(\bar{2})$ is H-essential of $\mathbb{Z}_4$. So by Proposition (2.9), $ann_{\mathbb{Z}}(\mathbb{Z}_4) \subsetneq ann_{\mathbb{Z}}(\bar{2})$. In fact, $ann_{\mathbb{Z}}(\bar{2}) = 2\mathbb{Z}$, and $ann_{\mathbb{Z}}(\mathbb{Z}_4) = 4\mathbb{Z}$.

2. Consider the $\mathbb{Z}$ -module $\mathbb{Q}$. $\mathbb{Z}$ is H-essential submodule of $\mathbb{Q}$ (as we will see in Example 2.14), but $ann_{\mathbb{Z}}$ ($\mathbb{Q}$)= $ann_{\mathbb{Z}}$ ($\mathbb{Z}$) since $ann_{\mathbb{Z}}$ ($\mathbb{Z}$) = 0.

Proposition (2.11): Let A and B be H-essential submodules of an R-module U, then A ∩ B is an H-essential submodule.

Proof: If A∩B = 0, then we are done. Assume that A∩B ≠ 0 and $0 \neq f \in$ Hom$_R$($\frac{U}{A\cap B}$, U). It is clear that Hom$_R$ ($\frac{U}{A\cap B}$, U) ≤ Hom$_R$($\frac{U}{A}$; U) + Hom$_R$($\frac{U}{B}$, U) [3]. So for all $f \in$ Hom$_R$ ($\frac{U}{A\cap B}$, U), $f = h + k$ where h ∈ Hom$_R$($\frac{U}{A}$, U) and $k \in$ Hom$_R$($\frac{U}{B}$, U) with $h(\frac{U}{A}) \leq_e$ U and $k(\frac{U}{B}) \leq_e$ U implies $h(\frac{U}{A}) + k(\frac{U}{B}) \leq_e$ U. Hence $f(\frac{U}{A\cap B}) \leq_e$ U, hence we are done. □

Theorem (2.12): Let U be an R-module, and 0 ≠ V ≤ U. Then the following statements are equivalent:

1. V is H-essential submodule.

2. For each $f \in$ End$_R$(U) such that V≤ ker$f$; f(U) $\leq_e$ U.

Proof: (1) ⇒ (2) Let $f \in$ End$_R$(U) and V ≤ kerf. Define g : $\frac{U}{V} \rightarrow$ U by g(u + V) = f(u). It is clear that g is well defined and homomorphism. By assumption, $g(\frac{U}{V}) \leq_e$ U. But by definition of g, $g(\frac{U}{V})$ = f(U), thus f (U) $\leq_e$ U.

(2) ⇒ (1) Let $f \in$ Hom$_R$($\frac{U}{V}$, U), and consider the natural epimorphism π: U → $\frac{U}{V}$, then fo $\pi \in$ End$_R$(U). But V ≤ ker (fo π), so by assumption foπ (U) $\leq_e$ U. This implies that $f(\frac{U}{V}) \leq_e$ U. That is V is H-essential submodule of U. □

"Recall that a submodule V of an R-module U is called rational if Hom$_R$($\frac{U}{V}$, E(U)) = 0, where E(U) is the injective hull of U" [2, P.12].

Proposition (2.13): Every non-zero rational submodule is H-essential.

Proof: Let V be a non-zero submodule of an R-module U. It is clear that Hom$_R$($\frac{U}{V}$, U) ≤ Hom$_R$($\frac{U}{V}$, E(U)), and since V is rational submodule, then Hom$_R$($\frac{U}{V}$,E(U)) = 0. This implies that Hom$_R$($\frac{U}{V}$ U) = 0, that is V is H- essential. In fact there is no non-zero homomorphism $f \in$Hom$_R$($\frac{U}{V}$, U) with f ($\frac{U}{V}$) $\leq_e$ U. □

Example (2.14): Consider the $\mathbb{Z}$-module $\mathbb{Q}$, where $\mathbb{Q}$ is the set of all rational numbers. Since $\mathbb{Z}$ is rational submodule of $\mathbb{Q}$ hence by Proposition (2.13), $\mathbb{Z}$ is H-essential submodule of $\mathbb{Q}$.

"Recall that an R-module U is called polyform if every essential submodule is rational" [4]. By proposition (2.13), we conclude the following.

Corollary (2.15): If U is polyform module, then every essential submodule of U is H-essential.

3. Homessential Module

In this section, we introduce the class of Homessential module.

Definition (3.1): An R-module U is called Homessential module, if every proper submodule of U is H-essential. A ring R is called Homessential, if R is Homessential R-module.

Examples (3.2):

1. The zero R-module is a Homessential module, since (0) has no non-zero H-essential submodule V of (0) such that f(V) is not essential submodule of (0).

2. $\mathbb{Z}_4$ is a Homessential module, since the only proper submodule of $\mathbb{Z}_4$ is ($\bar{2}$), and ($\bar{2}$) is H-essential submodule see (example (2.2)(1)).

3. The $\mathbb{Z}$-module $\mathbb{Z}_{p_\infty}$ is Homessential module. In fact, it's uniform module [5], then the result follows Remark (3.3)(4).

Remarks (3.3):

1. Every simple module is a Homessential module.

2. A semisimple module is not Homessential module.

3. Every integral domain R is Homessential R-module.

4. Every uniform module is Homessential module, where an R-module U is called uniform if every non-zero submodule is essential in U [2].

5. Every monoform module is Homessential, where a module U is called monoform if every non-zero submodule is rational [5], in fact this follows directly from Proposition (2.13).

Proof: 3. Let R be an integral domain, and I be a non-zero ideal of R. We can easily show that in an integral domain every non-zero ideal is an essential ideal. This implies that $\frac{R}{I}$ is singular R-module, hence, it is torsion. On the other hand, R is torsion free R-module, therefore $Hom_R(\frac{R}{I}, R) = 0$, that is R is Homessential module. In particular, Z is Homessential.

4. It is obvious. □

Proposition (3.4): A direct summand of a Homessential module is Homessential.

Proof: Let $U=U_1\oplus U_2$ be a Homessential module, where U_1 and U_2 be R-submodules of U, and V be a submodule of U_1. Let $0 \neq f \in End_R(U_1)$ such that $V \leq \ker f$. Consider the following sequence of homomorphism:

$$U_1\oplus U_2 \xrightarrow{\rho} U_1 \xrightarrow{f} U_1 \xrightarrow{i} U_1\oplus U_2$$

Note that $i \circ f \circ \rho \in End_R(U)$, and $V \leq \ker(i \circ f \circ \rho)$. Since U is Homessential, then by Theorem (2.12), $(i \circ f \circ \rho)(U_1\oplus U_2) \leq_e U_1\oplus U_2 = U$. This implies that $f(U_1) \leq_e U$ and so that $f(U_1) \leq_e U_1$ [1]. □

Proposition (3.5): Let U be an R-module. Then U is Homessential R-module if and only if U is Homessential $\frac{R}{I}$-module for each ideal $I \leq ann_R U$.

Proof: Assume that U Homessential R-module. Since $I \leq ann_R U$, then $Hom_R(\frac{U}{V}, U) = Hom_{\frac{R}{I}}(\frac{U}{V}, U)$ for each submodule V of U [6, Ex.3, P.51]. By assumption $f(\frac{U}{V}) \leq_e U$ for each $f \in Hom_R(\frac{U}{V}, U)$, hence $f(\frac{U}{V}) \leq_e U$ for each $f \in Hom_{\frac{R}{I}}(\frac{U}{V}, U)$. The convers is in similar way. □

The following theorem gives another characterization of Homessential module.

Proposition (3.6): Let U be an R-module, and V be a non-zero submodule of U. Then the following statements are equivalent:

1. U is a Homessential module.

2. For each $f \in End_R(U)$ with $\ker f \neq 0$; $f(U) \leq_e U$.

Proof: (1) ⇒ (2) Let U be a Homessential module, and $f \in End_R(U)$, with $\ker f \neq 0$. Put $V = \ker f$, by assumption V is H-essential submodule of U. By Theorem (2.12), $f(U) \leq_e U$.

(2) ⇒ (1) Let V be a non-zero proper submodule of U, and $f \in Hom_R(\frac{U}{V}, U)$. Note that $V \leq \ker(f \circ \pi)$, where $\pi: U \to \frac{U}{V}$ is the natural epimorphism. Since $V \neq 0$, then $\ker(f \circ \pi) \neq 0$. By Theorem (2.12), V is an H-essential submodule, hence U is Homessential module □

"Recall that a submodule V of a module U is said to be closed if V has no proper essential submodule in U" [1, p.18]

Proposition (3.7): Let U be a Homessential module. If U is a semisimple module, then for every $f \in Hom_R(\frac{U}{V}, U)$, f is an epimorphism.

Proof: Let V is a proper submodule of U and $f \in Hom_R(\frac{U}{V}, U)$. By the definition of Homessential module, $f(\frac{U}{V}) \leq_e U$. But U is semisimple, so that $f(\frac{U}{V})$ is closed submodule of U [7]. This implies that $f(\frac{U}{V}) = U$, hence f is an epimorphism. □

"Recall that an R-module U is said to be quasi-Dedekind, if for every non-zero homomorphism $f \in End(U)$, $\ker f = 0$" [8]

Proposition (3.8): Let U be a quasi-Dedekind module with $ann_R(U)$ is a maximal ideal of R, then U is Homessential.

Proof: Put $A= \frac{R}{ann_R(U)}$, since $ann_R(U)$ is a maximal ideal, then A is field, and clearly U is quasi-Dedekind A-module, therefore U is free A-module. This implies that $U \cong \bigoplus_{i\in I} A_i$ and $\forall i \in I$, $A_A \cong A_i$ [7, Lem (4.4.1), P.88]. Since U is indecomposable [8, Rem (1.3), p.24], then $U \cong A$ as A-module. Thus U is simple, and by Remarks (3.3)(1), U is Homessential. □

Corollary (3.9): If U is a finitely generated quasi-Dedekind module, then U is Homessential.

Proof: Since U is a finitely generated quasi-Dedekind module, then U is uniform [8], and by Remark (3.3)(4), U is Homessential. □

"A ring R is called regular (in the sense of Von Newmann) if for each a∈R there is exists b∈R such that a=aba" [2, P.4]. Under certain condition Homessential module can be quasi-Dedekind as the following proposition shows.

Proposition (3.10): Let U be a Homessential R- module. If $End_R(U)$ is a regular ring, then U is quasi-Dedekind.

Proof: Let $0\neq f\in End_R(U)$. We have to show that ker$f$=0, otherwise; since U is a Homessential module, so by Proposition (3.6), f(U) $\leq_e$ U. But $End_R(U)$, then U=f(U)⊕ $kerf$ [7, Exc. 17(a), P.272]. This implies that $f(U)\cap kerf=0$. Since ker$f\neq$0, so $f(U)$ is not essential in U, which is a contradiction. □

Remark (3.11): The condition "$End_R(U)$. is regular" cannot be dropped from Proposition (3.10), for example; Z_4 is Homessential Z-module but not quasi Dedekind, since $End_R(Z_4)$. = Z_4 which is not regular.

It is known that if U is a semisimple module, then $End_R(U)$ is regular [9, Cor.(2.22, P.52)]. So by this fact and Proposition (3.10) we have the following.

Corollary (3.12): If U is semisimple and Homessential module, then U quasi-Dedekind.

Remarks (3.13):

1. The condition "semisimple" in Corollary (3.12) cannot be dropped. For example, the $\mathbb{Z}$-module $\mathbb{Z}_{p_\infty}$is uniform module, so it is Homessential (see Remark (3.3)(4)) but it is not quasi-Dedekind module, since it is not semisimple module.

2. In general, Homessential module need not be prime module; for example: the $\mathbb{Z}$ -module $\mathbb{Z}_4$ is Homessential but not prime module.

The following Corollary gives a necessary condition under which Homessential module can be prime.

Corollary (3.14): Let U be a Homessential module. If U is semisimple, then:

1. U is a prime module.

2. ann_R(U) is a prime ideal of R.

Proof: 1. The result follows by Corollary (3.12), and [8, Prop.(1.7), p.26].

2. The result follows by Corollary (3.12), and [8, Cor.1.8, p.26]. □

"Recall that an R-module P is said to be projective if for every epimorphism f: B → C and for every homomorphism g: P → C there is a homomorphism h: P → B with g = hf" [7, Def. (5.3.1) (b), p. 116].

Corollary (3.15): Let R be a regular ring and U is a finitely generated projective module. If U is Homessential, then U is a quasi-Dedekind module.

Proof: Assume that U is a Homessential module. Since U is a finitely generated projective module over regular ring, then End_R(U) is regular [7,Exc.17(c), P.272], and by Proposition (3.10), we are done. □

"Recall that an R-module U is called Z-regular, if every cyclic submodule of U is direct summand and projective" [10].

Corollary (3.16): Let U be a Z-regular module. If U is a Homessential module, then U is quasi-Dedekind.

Proof: Since U is a Z-regular module, then End_R(U) is regular [10]. But U is Homessential, then by proposition (3.10), U is a quasi-Dedekind. □

"Recall that an R-module U is called extending if every submodule of U is essential in a direct summand of U" [2, p.118], and U is indecomposable, if the only decomposition U=A⊕B are those in which either A=(0) or B=(0) [7, p.285].

Proposition (3.17): Every non-zero extending and indecomposable module is Homessential.

Proof: Since every non-zero extending and indecomposable module is uniform, then the result follows by Remark (3.3)(4). □

Acknowledgements: The authors express their gratitude to referees for them valuable suggestions and helpful comments.

References

1. Goodearl K. **1976**. *Ring theory, Non-singular ring and modules*, Marcel Dekker, New York.
2. Tercan, A. and Yucel C. **2016**. *Module theory, extending modules and generalizations,* Springer International Publishing Switzerland.
3. Hadi I.M. and Ghwai Th. Y. **2011**. *Essentially quasi-Invertible submodules and essentially quasi-Dedekind modules, Ibn AL-Haitham J. Pure and Applied Sci*., **24**(3): 217-228.
4. Ahmed, M. A. **2018**. *St-Polyform modules and related concepts*. *Baghdad Sci. J*., **15**(3): 335-343.
5. Hadi I.M. and Marhun H. K. **2014.** *Small monoform modules, Ibn AL-Haitham J. Pure and Applied Sci*., **27**(2): 229-240.
6. Larsen M. D. and McCarthy P. J. **1971**. *Multiplicative theory of ideals,* New York and London: Academic Press.
7. Kasch, F. **1982**. *Modules and rings*, Academic Press, London.
8. Mijbass A.S. **1997.** *Quasi-Dedekind* modules, Ph. D. Thesis, *University of Baghdad*, Iraq.
9. Muller S.M.B. and Mohamed S. **1990.** *Continuous and discrete modules, London Math. Soc. Lecture Note Sen*, **147**: 67-71.
10. Zelmanowitz, J. **1972.** *Regular modules*, Transactions of the American Mathematical Society, **163**.