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Compare Some Shrinkage Bayesian Estimation Method for Gumbel-Max Distribution with Simulation

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Abstract

The Gumbel max distribution is one of the most important statistical distributions because it has many applications, especially in the field of water research and the prediction of earthquakes and volcanoes. This work includes a comparison of several Shrinkage Bayesian methods, namely the Shrinkage Bayesian Square Loss Function, Shrinkage Bayesian Lenix Function, and Shrinkage Bayesian Unix Function to estimate the distribution parameter. The research also includes several simulated experiments according to the sample size change and the real value of the distribution parameter. The number of experiments is 15 simulated experiments based on the difference sample size and true distribution parameter values. The results of the simulation experiments are compared depending on each of the absolute least difference criteria $(ALDC)$ and the mean squared error (Mse). The comparison results show that the estimation method is affected by the sample size, and the real value of the distribution parameter. The best estimation method is the Shrinkage Bayesian Lenix Function. The Bayesian methods can be applied to other statistical distributions such that the Lindley Weibull distribution and logistics distribution.

Key words: Gumbel-Max Distribution, Shrinkage Bayesian Square Loss Function (BSLF), Shrinkage Bayesian Lenix Function (BLENIXF), Shrinkage Bayesian Unix Function (BUNIXF)

مقارنه بين بعض طرق تقدير البايزي المقلصه لـ توزيع Gumbel−Max مع المحاكاه

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الخلاصة :

يعد توزيع Gumbel max احد اهم التوزيعات الاحصائية لانه تحتوي على العديد من التطبيقات , خاصة في مجال ابحاث المياه والتنبؤ بالزلازل والبراكين. نتيجة لذلك جاء هذا البحث الذي تضمن مقارنة بين عدد من طرق بايزي االمقلصه في ظل دوال الخساره (التربيعيه ,لينكس و يونيكس) من اجل تقدير معامل التوزيع . اشتمل البحث على عدة تجارب محاكاة حسب تغيير حجم العينة والقيمة الحقيقية لمعامل التوزيع, وكان عدد التجارب (15) تجربة محاكاة بناء على حجوم عينات مختلفه و قيم معلمات توزيع مختلفه تمت مقارنة نتائج تجارب المحاكاة اعتمادا على كل من (معايير الفروق الصغرى المطلقة ومتوسط الخطا التربيعي (Mse), واظهرت نتائج المقارنة ان طريقة التقدير تأثرت ب (حجم العينة, القيمة الحقيقة لمعامل التوزيع),

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افضل طريقة تقدير هي (BLENLX). يمكن تطبيق طرق بايزي على التوزيعات الاحصائية الخرى مثل توزيع ليندلي وبيل والتقزيع اللوجستي).

1-Introduction

The study of estimation methods for the statistical distribution parameter is still one of the important topics in scientific research due to its importance in applications that depend on these distributions. For this purpose, many studies have been carried out. The study introduced by Tatsuya (1998) includes estimated variance by using a modified shrinkage estimation method. The experimental results show that the estimation method gives minimum mean square error values [1]. The study introduced by Mahdi S., and Cenac M. in (2005) that compares the maximum likelihood, weighted moments and moments estimation methods for Gumbel distribution by different simulation experiments according to the difference in sample size and the true value of the parameter of the distribution. The simulation results show that the weighted moment method is the best [2]. The study introduced by Sara van Erp and others in 2018 includes regression estimates by using the shrinkage Bayesian method. This study also compares the Ridge, Lasso, and Hyperlasso with other methods. The experimental results show that the Lasso gives the best estimation method [3]. The study introduced by Gholami G. et al in 2020 estimates the parameter of the Gamma Gumbel distribution by the maximum likelihood method and Bayesian estimation method. The estimation results of real data with a sample size of 20 observations show that the Bayesian method gives the best estimator for the parameters of the Gamma Gumbel distribution [4].

The study introduced by Abdulali BAA, et al. in 2022 estimates the parameter of the generalized extreme value distribution that applies to air pollution, estimation procedure of the parameter of distribution includes maximum likelihood and moment methods and compares estimators by the mean square error and root mean square error criteria by deferent simulation experiments, the estimator's result show that maximum likelihood estimation method is the best [5].

In this study, a number of simulation experiments with different methods, namely the shrinking Bayesian estimation, sample size, and the true value of the distribution parameter are compared based on the least absolute difference and mean square error.

The lack of an estimator close to the distribution parameter can lead to the inability to apply the different distribution functions (reliability, survival, failure and entropy) and the occurrence of problems for the distance from the real values.

This research aims to find the best estimator for the distribution parameter depending on the estimation method, sample size and the real value of the distribution parameter.
Finding an estimator that is close to the parameter of the statistical distribution leads to the possibility of applying all statistical functions in the application, including reliability, survival, failure and entropy.

2- Gumbel-Max Distribution

The Gumbel-Max distribution was first proposed by Emil Julius Gumbel (1891–1966). The distribution is used in many weather experiments and to predict rain, earthquakes and floods. The distribution has the following characteristics
[1,6,7,8,9]: -
The probability density function is given as follows

$$f(x,b,c) = b^{-1}\, e^{-\left(\frac{x-c}{b}\right)} e^{-e^{-\left(\frac{x-c}{b}\right)}}, \tag{2.1}$$

and the Cumulative distribution function is

$$F(x,b,c) = e^{-e^{-\left(\frac{x-c}{b}\right)}} \quad . \qquad \dots (2.2)$$

The three centroid moments will be

$$the\ mean = c + b\gamma \ , \qquad \dots (2.3)$$

where γ represents the Euler–Mascheroni constant.

$$\text{The median} = c - bLn\big(Ln(2)\big) \qquad \dots (2.4)$$

$$\text{and the mode} = c \qquad \dots (2.5)$$

The three variations moments will be [10,11] as follows:

$$\text{The variance} = \frac{\pi^2}{6}\, b^2 \qquad \dots (2.6)$$

$$\text{The skewness} = 1.14 \qquad \dots (2.7)$$

$$\text{The kurtosis} = \frac{12}{5} \qquad \dots (2.8)$$

3-Shrinkage Bayesian Estimation methods

These methods were introduced by Xu in 2003 and all methods depend on the maximum likelihood estimation method [10, 12, 13].

If we have a random sample $(X_1, X_2, \dots, X_n)$ with size (n) each of them with Gumbel-Max, then the logarithm of likelihood function $L(b,c)$ will be as follows:-

$$L(b,c) = Ln\left(\prod_{i=1}^{n} f(x_i,b,c)\right)$$

$$L(b,c) = -\sum_{i=1}^{n}\left[\frac{x_i - c}{b}\right] - n\,Ln(b) - \sum_{i=1}^{n} e^{-\left[\frac{x_i-c}{b}\right]}$$

The partial derivatives for each parameter will be

$$\frac{\partial L(b,c)}{\partial c} = \frac{1}{b}\left[n - \sum_{i=1}^{n} e^{-\left[\frac{x_i-c}{b}\right]}\right] \qquad \dots (3.1)$$

$$\frac{\partial L(b,c)}{\partial b} = \sum_{i=1}^{n}\left[\frac{x_i - c}{b^2}\right] - \frac{n}{b} - \sum_{i=1}^{n}\left[\frac{x_i - c}{b^2}\right] e^{-\left[\frac{x_i-c}{b}\right]} \qquad \dots (3.2)$$

The solving of the system $\frac{\partial L(b,c)}{\partial b} = \frac{\partial L(b,c)}{\partial c} = 0$ yields the maximum Likelihood (ML) estimates of (b,c) and as numerical solutions of the following equations [14, 15].

$$c = b\left[Ln(n) - Ln\left(\sum_{i=1}^{n} e^{-\left[\frac{x_i}{b}\right]}\right)\right] \qquad \dots (3.3)$$

$$\bar{x} = b + \left[\frac{\sum_{i=1}^{n} x_i e^{-\left[\frac{x_i}{b}\right]}}{\sum_{i=1}^{n} e^{-\left[\frac{x_i}{b}\right]}}\right] \qquad \dots (3.4)$$

The estimate of b is explicitly obtained from the first equation and the estimate of c is then obtained from the second equation after the substitution of the estimated value.

(3-1) The Bayesian Shrinkage estimator under squared error loss functions

This shrinking Bayesian estimator involves finding a new estimator according to the square loss function so that the posterior distribution of the parameter b is the gamma distribution G (b, t) with the following pdf [2,6,14,16]: -

$F(b,t) = \frac{t^n\, b^{n-1}\, e^{-tb}}{\gamma(n)}$ with $t = Ln\left[\sum_{i=1}^{n} e^{-\frac{x_i}{b}}\right]$ such that t represents the limit that depends on the random variable. The squared error loss function is defined as follows [17,18,19,20]:

$L1\left(b,\hat{b}s\right)=\left[b-\hat{b}s\right]^2.$

The posterior risk function of b can be calculated as:

$\rho_1\left(b,\hat{b}_{s1}\right)=E_b\left[\hat{b}_{s1}-b\right]^2$

$=\hat{b}_{s1}{}^2-2\hat{b}_{s1}\frac{n}{t}+\frac{n(n+1)}{t^2}$

$with\ b\sim gamma$ (n,t) the partial derivative with respect to $\hat{b}_{s1}$. We obtain:

$\frac{\partial\rho_1}{\partial\hat{b}_{s1}}\left(b,\hat{b}_{s1}\right)=2\hat{b}_{s1}-2\frac{n}{t}$

$\frac{\partial\rho_1}{\partial\hat{b}_{s1}}\left(b,\hat{b}_{s1}\right)=0$

$\hat{b}_{s1}=\frac{n}{t}$

The shrinkage estimator is:

$\hat{b}_{sh1}=k\left(\hat{b}_{s1}-b_0\right)+b_0$ Which $\hat{b}_{s1}$ is the generalized Bayesian estimator, the risk function of $\hat{b}_{sh1}$ is defined as follows:

$\rho_1\left(b,\hat{b}_{sh1}\right)=E\left(\hat{b}_{sh1}-b\right)^2$

$=$ Eb[k $(\hat{b}_{s1}-b_0)+(b_0-b)]^2$

Taking the partial derivative

$$\frac{\partial\rho_1\left(b,\hat{b}_{sh1}\right)}{\partial k}=2k\hat{b}_{s1}{}^2+(2-4k)b_0\hat{b}_{s1}-2(1-k)b_0{}^2-\frac{2n\left(\hat{b}_{s1}+b_0\right)}{t}$$

$$\hat{b}_{sh1}=\left[\frac{t\left(b_0{}^2-b_0\right)\left(\frac{n}{t}\right)+n\left[\left(\frac{n}{t}\right)+60\right]}{t\left[\left(\frac{n}{t}\right)-b_0\right]^2}\right]\left(\left[\frac{n}{t}\right]-60\right)+b_0 \tag{3.5}$$

(3-2) *Bayesian Shrinkage Estimator under Linux Loss Function*

This shrinking Bayesian estimator involves finding a new estimator according to the Linux loss function

The loss function will be [15]

$L2=e^{a\Delta}-a\Delta-1$ with $\Delta=\frac{b_{s2}}{b}$.

The Posterior risk function of the parameter b under the Linex will be as follows:

$\rho_2\left(b,\hat{b}_{s2}\right)=E_b[\exp(c)\,(\frac{\hat{b}_{s2}}{b}-1)-C(\frac{\hat{b}_{s2}}{b}-1)-1]$

$=\exp(c)\,E_b\,[\exp(c\frac{\hat{b}_{s2}}{b})]-C\hat{b}_{s2}E_b\left(\frac{1}{b}\right)+C-1$

b$\sim Gamma$ (n,t) and $\frac{1}{b}\sim inverse\ gamma(\frac{n}{t})$ with the following pdf:

$f(\frac{1}{b})=\frac{t^n}{\Gamma(n)}b^{-(n+1)}e^{-\frac{t}{b}}$ such that $\Gamma(n)=(n-1)!$.

$E\left(\frac{1}{b}\right)=\frac{t}{n-1}$.

$\rho_2\left(b,\hat{b}_{s2}\right)=e^a(\frac{t}{t-a\hat{b}_{s2}})^n-a\hat{b}_{s2}\frac{t}{n-1}+a-1$

$\frac{\partial\rho_2(b,\hat{b}_{s2})}{\partial\hat{b}_{s2}}=ant^ne^{-a}(t-a\hat{b}_{s2})^{-(n-1)}-\left(\frac{at}{n-1}\right)$

$\hat{b}_{s2}=\frac{1}{a}[t-(nt^{n-1}e^{-a})^{\left(\frac{1}{n+1}\right)}]$

$=\exp(-a)\,[\frac{t}{t-a\hat{b}_{sh2}}]^n-a\hat{b}_{sh2}\left(\frac{t}{n-1}\right)+a-1$

$\hat{b}_{sh2}=k(\hat{b}_{s2}-b_0)+b_0$ with

$$\rho_2(b,\hat{b}_{sh2}) = \exp(-c)\ (\frac{t}{t-a\hat{b}_{sh2}})^n - a\hat{b}_{sh2}\frac{t}{n-1} + a - 1$$

$$\frac{\partial\rho_2(b,\hat{b}_{s2})}{\partial k} =$$

$$an(\hat{b}_{s2}-b_0)\exp(-a)\, t^n\left[t - ak(\hat{b}_{s2}-b_0) - ab_0\right]^{-(n+1)} - at\frac{(\hat{b}_{s2}-b_0)}{(n-1)}$$

$$\hat{b}_{sh2} = [\frac{(t-ab_0)-(n(n-1)t^{n-1}\exp(-a))^{-\left(\frac{1}{n+1}\right)}}{a\left[\left(\frac{1}{a}\right)[t-nt^{n-1}\exp(-a)]^{\left(\frac{1}{n+1}\right)}\right]} - 60\left[\frac{1}{a}\left(t-\left(nt^{n-1}\exp(-a)^{\frac{1}{n+1}}\right)\right) - 60\right] + 60 \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (3.6)$$

(3-3) Bayesian Shrinkage Estimator under Unix Loss Function

This shrinking Bayesian estimator involves finding a new estimator according to the Unix loss function so that the loss function will be as follows [2]

$L_3(b,\hat{b}_{s3}) = w\frac{\sum_{i=1}^{n}(x_i-\hat{b}_{s3})^2}{b^2} + (1-w)(\frac{\hat{b}_{s3}}{b}-1)^2$ such that $0 < w < 1$.

The posterior risk function will be :

$$\rho_3(b,\hat{b}_{s3}) = \frac{w}{n}\sum_{i=1}^{n}(x_i-\hat{b}_{s3})Eb(\frac{1}{b^2}) + (1-w)E_b(\frac{\hat{b}_{s3}}{b}-1)^2$$

by taking the partial derivation for $\rho_3(b,\hat{b}_{s3})$ *with respect to* $\hat{b}_{s3}$

$\frac{\partial\rho_3(b,\hat{b}_{s3})}{\partial\hat{b}_{s3}}$ and make it equal to zero we get

$\hat{b}_{s3} = w\bar{x} + (1-w)E\frac{b^{-1}}{b^{-2}}$ such that $\bar{x} = \frac{\sum_{i=1}^{n}x_i}{n}$, $b^{-1} \sim inverse\ gamma\ (n,t)$

$E(\frac{1}{b}) = \frac{t}{(n-1)}$ *then* $\hat{b}_{s3} = w\bar{x}$+(1-w)$\frac{n-2}{t}$.

Now we have two special cases if w=0 then $\hat{b}_{s3} = \frac{n-2}{t}$ and if w=1 then $\hat{b}_{s3} = \bar{x}$. The previous estimators can be proposed as a special case of the Bayesian estimators that means w=1, the posterior risk function of the Bayesian Shrinkage Estimator will be as follows:

$$\rho_3(b,\hat{b}_{sh3}) = \frac{w}{n}\left[\sum_{i=1}^{n}(x_i-k)\left((\hat{b}_{s3}-b_0)-b_0\right)^2\right]E_b b^{-2}$$

$$+ (1-w)E_b\left[\frac{k(\hat{b}_{s3}-b_0)+b_0}{b}-1\right]^2$$

By taking the partial derivative of $\rho_3(b,\hat{b}_{sh3})$ with respect to k. Assuming $\frac{\partial\rho_3(b,\hat{b}_{sh3})}{\partial k} = 0$, then k=$\frac{w\bar{x}-b_0}{\left[w\bar{x}+\frac{(1-w)(n-2)}{t}\right]} - 60 + \frac{(1-w)(n-2)}{t\left[w\bar{x}+\frac{(1-w)(n-2)}{t}-b_0\right]}$

Hence, the shrinkage Bayesian Estimator will be:

$\hat{b}_{sh} = k(\hat{b}_{s3}-b_0) + b_0$

$\hat{b}_{sh} = \left[\frac{w\bar{x}-b_0}{\left[w\bar{x}+\frac{(1-w)(n-2)}{t}-b_0\right]} + \frac{(1-w)(n-2)}{t[w\bar{x}+\frac{(1-w)(n-2)}{t}-b_0]}\right]$ *[w$\bar{x} + \frac{(1-w)(n-2)}{t} - b_0] + b_0$

Now in case w=0 then

$$\hat{b}_{sh31} = \left[\frac{-b_0}{\left(\frac{n-2}{t}\right)-b_0} + \frac{(n-2)}{t[(\frac{n-2}{t}-b_0]}\right] * \left(\frac{n-2}{t}-b_0\right) + b_0 \qquad (3.7)$$

And in case of w=1 then $\hat{b}_{sh32} = \bar{x} + b_0$. (3.8)

4-The simulation processes

Simulation processes of the proposed distribution depend on the cumulative distribution function such that [11,21,22]

$F(x,b,c) = e^{-e^{-\left(\frac{x-c}{b}\right)}}$

Tacking the randomize function $R = F(x,b,c)$ then $R = e^{-e^{-\left(\frac{x-c}{b}\right)}}$

By solving the equation, we get

$$x = b\left[Ln\big(Ln(R)\big)\right] + c \quad (4.1)$$

Such that ($x \sim$Gumbel − Max distribution $with$ (b,c))

The simulation experiment will be $n = 15,25,50,75,100$ and $b = 0.25,0.50,0.75$

5-Experimental Results

The experimental results for the simulation experiments will be as follows [4,23].

Table (1) gives the estimators for each simulation method such that the best estimator is the absolute least difference criteria ($ALDC$)

$$ALDC = \min\left(|\hat{\alpha} - \alpha|\right) \quad (4.2)$$

Table (2) gives the mean square error and it is the best estimation method

$$Mse = \frac{\sum_{i=1}^{it}[\hat{\alpha}_i - \alpha]^2}{it} \quad (4.3)$$

where it represents the number of iterations ($it = 1000$).

After carrying out various simulation experiments, the following tables and figures for the simulation results were as follows

Table 1: Show the estimation methods

$\hat{\alpha}$	n	BSLF	BLINEX	BUNIX	$C.C$	$the\ best$
0.25	15	0.227327	0.280901	0.268891	0.018891	3
0.25	25	0.289401	0.245681	0.286445	0.004319	2
0.25	50	0.249193	0.249865	0.250126	0.000126	3
0.25	75	0.249968	0.25005	0.250032	3.18E-05	1
0.25	100	0.250018	0.250001	0.250025	5.98E-07	2
0.5	15	0.503174	0.496907	0.509935	0.003093	2
0.5	25	0.470059	0.500279	0.471194	0.000279	2
0.5	50	0.599538	0.499979	0.599148	2.14E-05	2
0.5	75	0.55508	0.499976	0.555036	2.38E-05	2
0.5	100	0.453222	0.457244	0.43057	0.042756	2
0.75	15	0.749995	0.750005	0.749998	2.37E-06	3
0.75	25	0.719991	0.786767	0.785987	0.030009	1
0.75	50	0.742365	0.748702	0.743945	0.001298	2
0.75	75	0.749821	0.750012	0.749832	1.18E-05	2
0.75	100	0.748885	0.743	0.748896	0.001104	3

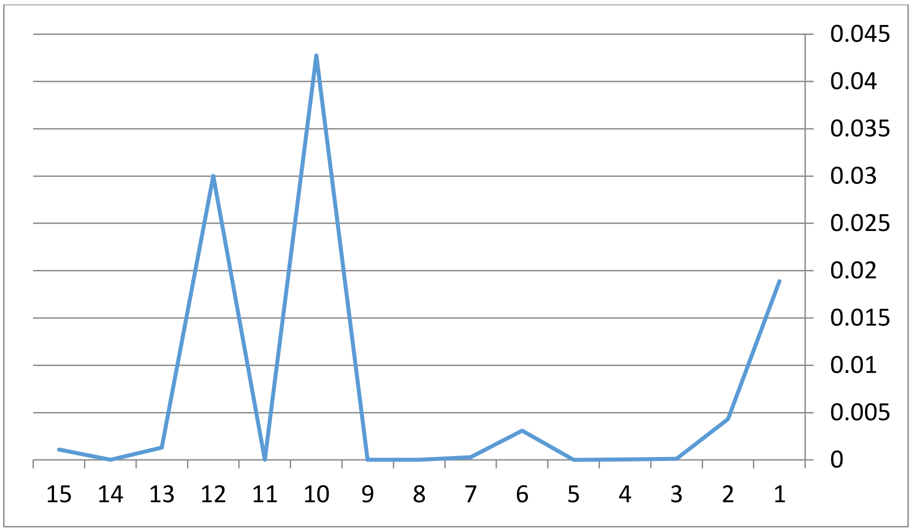

Figure 1: represent the $(C.C)$ values for each simulation experiment

Table1 and Figure (1) show that the estimators affected by both the sample size and the real value of the distribution parameter according to $(ALDC)$

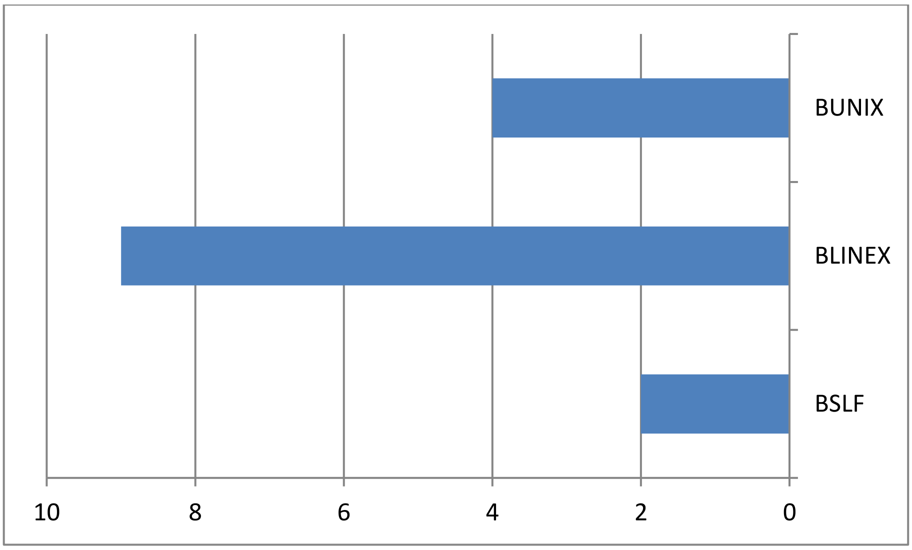

Figure 2: the number of best (ALD) 0for each simulation experiment

Figure (2) show that the best estimation method is (BLINEX), the number of simulation experiments in which the (BLINEX) method is the best according to the criterion $(ALDC)$, (9) times , and percentage(60%) .

Table 2: show the Mean Square Error and the best estimation method

$\hat{\alpha}$	n	BSLF	BLINEX	BUNIX	$Min(mse)$	$the\ best$
0.25	15	7.57E-04	1.18E-03	3.98E-04	3.98E-04	3
0.25	25	2.80E-03	1.87E-05	2.79E-03	1.87E-05	2
0.25	50	6.69E-07	2.56E-08	5.85E-08	2.56E-08	2
0.25	75	1.13E-09	2.82E-09	1.14E-09	1.13E-09	1
0.25	100	5.79E-10	3.21E-11	1.10E-09	3.21E-11	2
0.5	15	1.18E-05	1.13E-04	1.12E-03	1.18E-05	1
0.5	25	1.43E-03	2.29E-06	1.38E-03	2.29E-06	2
0.5	50	2.63E-02	1.45E-07	2.62E-02	1.45E-07	2
0.5	75	6.87E-03	6.88E-10	6.86E-03	6.88E-10	2
0.5	100	3.16E-03	2.05E-03	6.23E-03	2.05E-03	2
0.75	15	2.07E-09	3.24E-11	1.44E-09	3.24E-11	2
0.75	25	9.12E-04	1.78E-03	1.78E-03	9.12E-04	1
0.75	50	8.50E-05	3.09E-06	5.46E-05	3.09E-06	2
0.75	75	9.26E-08	3.63E-10	9.76E-08	3.63E-10	2
0.75	100	2.52E-06	9.91E-12	2.48E-06	9.91E-12	2

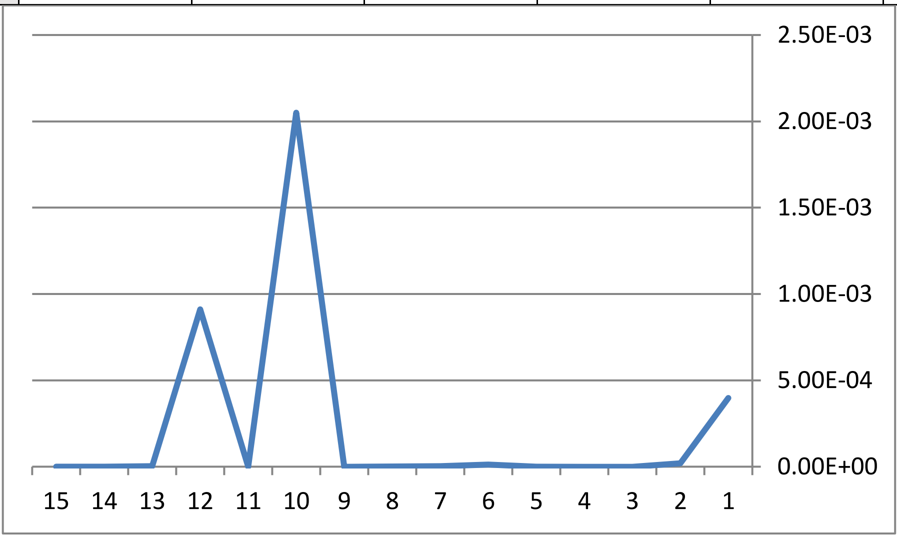

Figure 3: the ($Min(MSE)$) values for each simulation experiment

The previous table and figure showed the effect of the resulting estimator on both the sample size and the real value of the distribution parameter according to ($Min(mse)$)

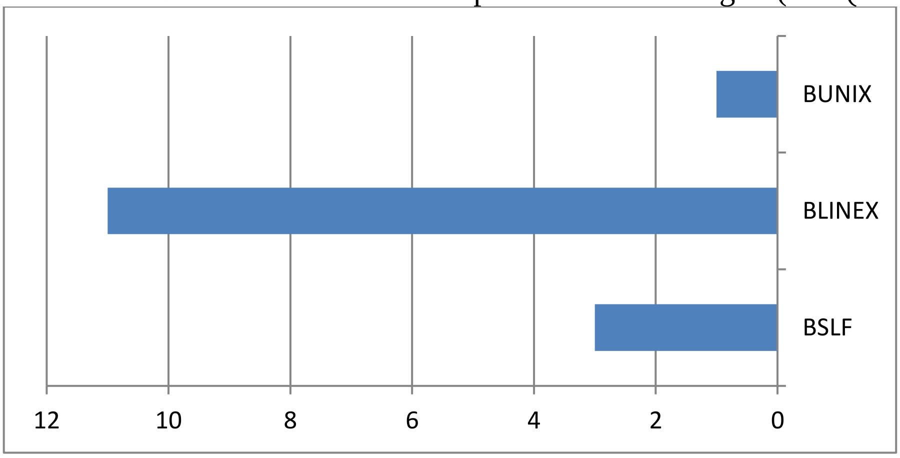

Figure 4: the number of best ($Min(mse)$) for each simulation experiment

The previous Figure shows that the best estimation method is (BLINEX), the number of simulation experiments in which the (BLINEX) method is the best according to the criterion $(Min(mse)\&(ALDC)$, (11) and percentage(73%)

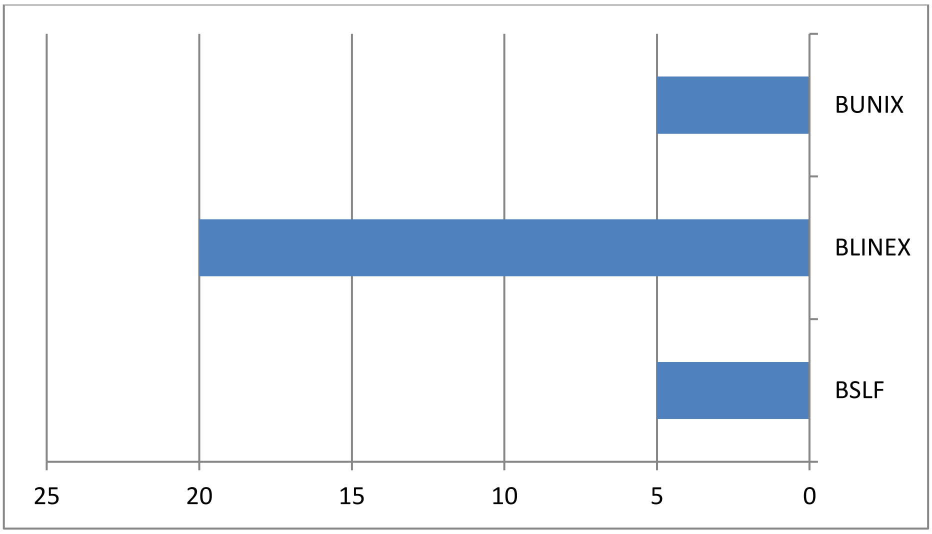

Figure 5: the number of best $(ALDC\&Min(mse))$ for each simulation experiment

The previous Figure shows that the best estimation method is (BLINEX), the number of simulation experiments in which the (BLINEX) method is the best according to the criterion $(Min(mse)\&(ALDC)$), (20) and percentage(67%)

6- Conclusions and Suggestions

1- The Shrinkage Bayesian estimation method is affected by the sample size.
2- The estimation method is affected by the simulation value of the distribution parameter.
3- The best shrinkage Bayesian estimation method is (BLINEX).
4- The shrinking Bayesian methods can be compared with both (the moment's method and noise method) to note the results.
5- Other parameters of the distributions can be estimated (Weibull and exponential distribution) by shrinking Bayesian methods to observe the results.

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