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Pure-Hollow Modules and Pure-Lifting Modules

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Abstract

Let R be a commutative ring with identity, and W be a unitary left R-module. In this paper we, introduce and study a new class of modules called pure hollow (Pr-hollow) and pure-lifting (Pr-lifting). We give a fundamental, properties of these concept. also, we, introduce some conditions under which the quotient and direct sum of Pr-lifting modules is Pr-lifting.

Keywords: small submodule, hollow module, lifting module, Pure hollow, pure lifting

المقاسات المجوفة من النمط Pr −hollow ومقاسات الرفع من النمط Pr−lifting

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الخلاصة:

لتكن R حلقة إبداليه ذات عنصر محايد وليكن W مقاسا أيسرا متحايدا. في هذا البحث سنقدم و ندرس صنف جديد من أصناف المقاسات تدعى المقاسات المجوفة من النمط − (Pr−hollow) ومقاسات رفع من النمط (Pr−lifting) سنعطي الخواص الجوهرية لهذه المفاهيم وكذلك ليكون قسمة ومجموع ومقاسات مجوفة من النمط Pr−hollow ومقاس رفع من النمط Pr −lifting

Introduction

Throughout this paper R is a commutative ring with unity and all modules are unitary R-modules. in [1],[2] also in [3],[4],[5] researchers introduced the concepts of large-hollow module and large-lifting module with some conditions. Let W be an R-module, a submodule K of a module W is called small in W, denoted by $K \ll W$, if every submodule H of W, we have $K + H = W$, then $H = W$ in [6],[7],[8] also recall that in [9],[10]. In [11] A proper submodule K of W is called pure small (Pr-small) submodule of W denoted by $K \ll_{Pr} W$,if $K + H = W$, then H is a pure submodule of W, In [12] H is named pure submodule denoted ($H \leq_P W$) if $H \cap IW = IH$ for every ideal I of R. An R-module W is called hollow R-module if whole proper submodule K of R-module W is small submodule in W [13],[14]. Bethinks that, every small be Pr-small there for every hollow R-module is Pr-hollow, where an R-module W is called Pr-hollow R-module if every proper submodule K of R-module W is Pr-small submodule in W. in [15] An R-module W is called lifting if for every submodule K of W there is a decomposition

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$W = H \oplus L$ and $K \cap L \ll L$, where L is a submodule of W , equivalently W is called lifting module in [16],[17] if and only if for every submodule A of W there exists a submodule K of A, ($K \leq A$) such that $W = K \oplus X$ with $A \cap X \ll W$, in [18], [19]. In [20] Let W be an R-module W , is called large-lifting (L- lifting), if for every submodule A of W there exists a submodule K of A , such that $W = K \oplus X$ and $A \cap X \ll_L W$ where X is a submodule of W. in [21],[22], with [23], and [14], M be an R-module, a proper submodule N of M, is called large-small (L-small) submodule of M denoted by ($N \ll_L M$), if $N + K = M$ where $K \leq M$, then K is essential (Large) submodule of M ($K \leq_e M$). Every lifting module is Large lifting module. This paper consists two sections, in section one we introduce the concept of pure hollow (Pr-hollow) modules, and we give some properties that we need it in this paper. In section two we give the concept of pure lifting (Pr-lifting) modules and some of its properties, such that an R-module W is said to be Pr-lifting, if for each submodule K of W there, exists a submodule L of K such that W= $L \oplus H$ and $K \cap H \ll_{Pr} W$ where H submodule of W.

2. Pure Hollow Modules.

In this, we introduce pure-hollow modules as generalization of hollow modules, also We give some rules properties of these modules.

Definition 2.1:
An R-module W is called pure-hollow (Pr-hollow) module, if all submodule of W is Pr-small submodule of W.
Remarks and Examples 2.2:
1. Since every small is, Pr-small in [11], then every hollow module is Pr-hollow module. Thus Z_4 as Z-module is hollow and hence, is Pr-hollow module.
2.The converse of (1) is not true in general, for example in Z_6 as Z-module is not hollow modules but $Z_6$is Pr-hollow.
3. Every simple module is hollow, so is Pr-hollow module.
4. If W be semi simple module, then W is Pr-hollow module.
This clear since recall that in [11] every semisimple is Pr-small, then is Pr-hollow.
5. If W is Pr-hollow module, then a submodule of W need not Pr-hollow for example Z_{P^∞} as Z-module is Pr-hollow since it is hollow in [15], but Z a submodule of Z_{P^∞} is not Pr-hollow.
6. If W is Pr-hollow R-module, then it is decomposable.
Proof: Suppose K, H are submodule of W such that $W = K + H$ either H or K is pure which means W decomposable. for example, Z_6=2Z_6+3 Z_6.
Proposition 2.3: If W is direct summand, then every submodule module of W is Pr-hollow.
Proof: Suppose H, K are a proper, submodules of W, such that $K \leq H$ and H is direct summand of W. Since W is Pr-hollow, then $H \ll_{Pr} W$ and since H is direct summand of W by [7], if $K \ll_{Pr} W$ then $K \ll_{Pr} H$ implies H is, Pr-hollow.

Proposition 2.4: If $W_1$and W_2 are R-modules such that $\varphi: W_1 \to W_2$ is an epimorphism if W_2 is Pr-hollow then $W_1$is Pr-hollow.
Proof: Let K be a proper submodule of W_1, thus $\varphi(K)$ is a proper submodule of W_2 , if not, then φ (K) =W_2, so K=W_1 and this contradiction. Since W_2 is Pr-hollow, then $\varphi(K) \ll_{Pr} W_2$, and hence by [11], $\varphi^{-1}(\varphi (K)) \ll_{Pr} W_1$, so $K \ll_{Pr} W_1$ and hence W_1 is Pr-hollow.
Corollary 2.5: If W is an R-module and H is a submodule of W if $\frac{W}{H}$ is Pr-hollow, then W is Pr-hollow.
Proof: Let $\frac{k}{H}$ be a submodule of $\frac{W}{H}$ such that $H < K \leq W$ since $\frac{W}{H}$ is Pr-hollow, then $\frac{k}{H} \ll_{Pr} \frac{W}{H}$, by [11], we get $K \ll_{Pr} W$ we get W is Pr-hollow.

Corollary 2.6: If W is Pr-hollow then $\frac{W}{K}$ is Pr-hollow for $K \leq W$.

Theorem 2.7: If W_1 and W_2 are R-modules and if W= $W_1 \oplus W_2$ such that W is duo modules, then W is Pr-hollow if and only if, W_1 and W_2 are Pr-hollow with $H \cap W_i \neq W_i$ for i=1,2 and $H \leq W$.

Proof: →) Clearly by [11].

←) Suppose H is a proper submodule of W and W_1 , W_2 are Pr-hollow. Since W is duo module, then $H = (H \cap W_1) \oplus (H \cap W_2)$, hence $H \cap W_1$ and $H \cap W_2$ are proper submodule of W_1 and W_2, also since W_1 and W_2 are Pr-hollow, then $H \cap W_1 \ll_{Pr} W_1$ and $H \cap W_2 \ll_{Pr} W_2$ thus by [11] we get $(H \cap W_1) \oplus (H \cap W_2) \ll_{Pr} W_1 \oplus W_2$ and cause $H \ll_{Pr} W$.

Proposition 2.8: Let $W = W_1 \oplus W_2$ be an R- modules with W_1 and W_2 are submodules of W and Wis distributive, then W is Pr-hollow if and only if, W_1 and W_2 are Pr-hollow, such that $K \cap W_i \neq W_i$, i=1,2 and $K \leq W$.

Proof: →) Clearly by Theorem 2.7.

←)Since W is distributive, then $K = (K \cap W_1)+(K \cap W_2)$, where K be a proper submodule of W . So, by proof of Theorem 2.7, we get W is Pr-hollow.

3. Pr-Lifting Modules.

We introduce Pr-lifting module as generalization of lifting module.

Recall that an R-module W is called lifting module if for every submodule K of W, then exists a submodule L of K such that $W = L \oplus H$ and $K \cap H \ll H$ in [18] and [15].

Definition 3.1: An R-module W is called Pure-lifting (Pr-lifting), If for each submodule K of W there exists a submodule L of K such that W= $L \oplus H$ and $K \cap H \ll_{Pr} W$, then H is submodule of W.

Remarks 3. 2: 1. All lifting module is Pr-lifting.

Proof: Suppose W lifting R-module clearly every submodule K of W there exists submodule L such that W= $L \oplus H$ and $K \cap H \ll W$ by [15], by [11] $K \cap H \ll_{Pr} W$. The convers of remark is not true since $\cap H \ll_{Pr} W$, need not true in general $K \cap H \ll W$ by [11].

2. Z_6 is Pr-lifting module since the Pr-small submodule of Z_6 is $3Z_6$ and $2Z_6$, $Z_6 = 2Z_6 \oplus 3Z_6$ thus $K=2Z_6$ then L= {0}, Z_6= {0} +Z_6, $Z_6 \cap 2Z_6 = 2Z_6 \ll_{Pr} Z_6$, also $K=3Z_6$, L={0} , H=Z_6, Z_6= {0} +Z_6, $Z_6 \cap 3Z_6 = 3Z_6$, $3Z_6 \ll_{Pr} Z_6$ but not lifting module since $2Z_6$ not small in Z_6 and $3Z_6$ not small of Z_6.

3. Every semisimple R-module is Pr-lifting, For, example Z_6 is semisimple implies Z_6 is Pr-lifting.

4. The convers is not true in general, for example: If W=Z_8 as Z-module the submodule $2Z_8 = \{\bar{0}, \bar{2}, \bar{4}, \bar{6}\}$ of Z_8, $4Z_8$ is a submodule of $2Z_8$ such that, $Z_8 = 4Z_8 \oplus Z_8$ and $Z_8 \cap 2Z_8 = 2Z_8 \ll_{Pr} Z_8$ also $4Z_8$ has only $\{\bar{0}\}$ a submodule of Z_8 such that $Z_8 = \{\bar{0}\} \oplus Z_8$, $Z_8 \cap 4Z_8 = 4Z_8 \ll_{Pr} Z_8$. Thus Z_8 is Pr-lifting but not semi-simple

Proposition 3. 3: Every Pr-hollow is Pr-lifting module.

Proof: Let W be Pr-hollow module and H be a submodule of W and suppose W={ $\bar{0}$}$\oplus$ W and $H \cap W = H \ll_{Pr} W$ since W is Pr-hollow, then W is Pr-lifting module by Definition 3.1.

Example for this, Z_6 as Z-module is Pr-hollow since all submodule of Z_6 is Pr-small implies Pr-lifting.

Remark 3.4: The converse of Proposition 3.3 is not true in general for example Z_{12} is Pr-lifting but not Pr-hollow.

Remark 3.5: Every local module is, Pr-lifting.

Proof: Since every local is hollow hence it is Pr-hollow by remarks and examples (2.2) (1), there, fore it is Pr-lifting by Proposition 3.3.

Proposition 3. 6: If W is an indecomposable, then W is Pr-hollow if and only if is, Pr-lifting.

Proof: →) By Proposition 3.3.
←)Let W be Pr-lifting and H be a proper submodule of W and suppose $K \leq H$ such that $W = K \oplus L$ where $L \leq W$ and L∩ $H \ll_{Pr} W$, since W is indecomposable, then either K=0 or, $K = W$. if $K = W$, then $H = W$ and this contradiction, so K=0 and hence $W = L$, so H= $H \cap W = H \cap L \ll_{Pr} W$ hence $H \ll_{Pr} W$ and so W is Pr-hollow.
Recall that in [17], if H is a direct summand of W, then H is pure in W.
Non, we give characterization of Pr-lifting module

Theorem 3.7: If W is an R-module, then the following are equivalent:
1.W is Pr-lifting module.
2.All submodule H of W can be written as $H = K \oplus M$ where K direct summand of W and $M \ll_{Pr} W$.
3. All submodule H of W there exists a direct summand K of W such that $K \leq H$ and $\frac{H}{K} \ll_{Pr} \frac{W}{K}$.
Proof: 1→2) suppose W is Pr-lifting, let H be a submodule of W, then there exists K a submodule of H such that $W = K \oplus L$ and $L \cap H \ll_{Pr} W$ where L be submodule of W. Now $H = H \cap W$, H=H∩ $(K \oplus L) = K \oplus$(H∩ L) by Modular Law. Suppose $V = K$ and U =H∩ L so $H = V \oplus U$, then V direct summand of W and $U \ll_{Pr} W$.
2→3) Assume $H \leq W$, and $H = V \oplus U$ such that V is direct summand of W and $U \ll_{Pr} W$, impose $\frac{K}{V} \leq \frac{W}{V}$ with $\frac{H}{V} + \frac{K}{V} = \frac{W}{V}$ then $\frac{V \oplus U}{V} \oplus \frac{K}{V} = \frac{W}{V}$ implies $W = V + U + K = U + K$ since $U \ll_{Pr} W so$ $V \leq_P W$, since V direct summand of W, then V is pure in W, hence $\frac{K}{V} \leq_P \frac{W}{V}$ by [11], so $\frac{H}{V} \ll_{Pr} \frac{W}{V}$.
3→1) Suppose H be a submodule of W, there exist a submodule K of H such that $W = K \oplus L$ and $\frac{H}{K} \ll_{Pr} \frac{W}{k}$ by (3) hence $H \ll_{Pr} W$ by [7] since H∩ $L \leq H \leq W$, we get $H \cap L \ll_{Pr} W$ by [11]
Proposition 3.8: If W is Pr-lifting module and K, L are submodules of W such that $W = K + L$, then there exist H is direct summand of W such that $(H + K) \leq_P W$.
Proof: Suppose W be Pr-lifting, then by theorem (3.6) $L = H + C$, where H is a direct summand of W and $C \ll_{Pr} W$, since $W = K + L$, so $W = H + C + K$ but $C \ll_{Pr} W$, hence $H + K \leq_P W$.
Proposition 3.9: Any direct summand of Pr-lifting module is Pr-lifting module.
Proof: Suppose W is Pr-lifting and W=$W_1 \oplus W_2$ let H be a submodule of W_1,so $H \leq W$ since W is Pr-lifting by Theorem 3.7 implies $H = V \oplus L$ where V direct summand of W and $L \ll_{Pr} W$. By [7] we get $L \ll_{Pr} W_1$ since $L \leq W_1 \leq W$ since W_1direct summand of W and $L \ll_{Pr} W$, then $L \ll_{Pr} W_1$ to prove V is direct summand of W_1. Since $W_{1=}W_1 \cap W = W_1 \cap (V \oplus U)$ = $V \oplus (W_1 \cap U)$ by modular law clearly V direct summand of W_1 there for W_1 is Pr-lifting.
Theorem 3.10: If W is an R-module, then the following statements, are equivalent:
1.W is Pr-lifting module.
2. For each submodule H of W, there exists $\varphi \in End(W)$ such that φ^2= φ, φ $(W) \leq H$ and (1-$\varphi)(H) \ll_{Pr} W$.
Proof: 1→2) let H be a submodule of W, then there exists a submodule K of H such that W=$K \oplus L$ and $L \cap H \ll_{Pr} W$ where L be a submodule of W. Let $\varphi: W \to K$ be a projection map, clearly φ^2=φ, and W= $K \oplus L$ =$\varphi(W)$ ⊕ (1-$\varphi)(W)$, $\varphi(W) \leq H$. Now (1-$\varphi)(H)$ =H ∩(1-$\varphi)(W)$=$H \cap L \ll_{Pr} W$, so (1-$\varphi)(H) \ll_{Pr} W$.
2 → 1) Let H be a submodule of W, then there exists $\varphi \in End(W)$ such that φ^2=φ, $\varphi(W) \leq H$ and (1-$\varphi)(H) \ll_{Pr} W$. clear that $W = \varphi(W) \oplus (1 - \varphi)(W)$. now, let K=$\varphi(W)$ and L= (1-$\varphi)(W)$, hence $H \cap L = H \cap$ (1-$\varphi)(W)$, to show that, H ∩(1-$\varphi)(W)$ =(1-$\varphi)(H)$, put x=(1-$\varphi)(m) \in H$ ∩(1-$\varphi)(W)$, since$(1 - \varphi)^2$=(1-φ) so x=$(1 - \varphi)^2(m)$= (1-$\varphi)(m) \in (1 - \varphi)(H)$.

Now let x = (1-φ)(m) ∈(1-φ)(H); $m \in H$, then $x \in$ (1-φ)(W), x= (1-φ)(m) ∈ H, hence $x \in H \cap$ (1-φ)(W) so $H \cap L = H \cap (1-\varphi)(W) = (1-\varphi)(H) \ll_{Pr} W$, hence $H \cap L \ll_{Pr} W$, so W is Pr-lifting.

Proposition 3. 11: If W is indecomposable module, then W is not Pr-lifting for every non trivial submodule K of W.

Proof: Let W be Pr-lifting, for every non trivial submodule K of W, we have $K = H + L$ by Theorem 3.7, where $L \ll_{Pr} W$ and H direct summand of W, since W is indecomposable this, contradiction, hence W is not Pr-lifting.

Proposition 3.12: If W is Pr-lifting module and K is a submodule of W, then $\frac{W}{K}$ need not be Pr-lifting module for example Z_4 as Z-module is Pr-lifting and $2Z_4$ is submodule of Z_4 but $\frac{Z_4}{2Z_4} \cong Z_2$ since Z_2 is indecomposable this not Pr-lifting by Proposition 2.11.

Proposition 3. 13: If W is Pr-lifting module and K is a submodule of W such that for each direct summand H of W, $\frac{H+K}{K}$ direct summand of $\frac{W}{K}$, then $\frac{W}{K}$ is Pr-lifting.

Proof: Impose $\frac{L}{K} \leq \frac{W}{K}$ since W is Pr-lifting there exist $H \leq L$ such that W=H$\oplus B, B \leq W, \frac{L}{H} \ll_{Pr} \frac{W}{H}$assist hypothesis $\frac{H+K}{K}$ direct summand of $\frac{W}{K}$ implies $H+K$ direct summand of W, implies, $\frac{L}{H+K} \ll_{Pr} \frac{W}{H+K}$ then $H+K \leq L$ and then $\frac{H+K}{K} \leq \frac{L}{K}$, since $\frac{H+K}{K}$direct summand of$\frac{W}{K}$ now, $\frac{\frac{L}{K}}{\frac{H+K}{K}} \ll_{Pr} \frac{\frac{W}{K}}{\frac{H+K}{K}}$ implies $\frac{W}{K}$ is Pr-lifting assist Theorem 3.6.

Corollary 3. 14: If W is Pr-lifting and distributive module and H is a submodule of W then $\frac{W}{H}$ is Pr-lifting.

Proof: Impose K is a direct summand of W, such that $W = K \oplus L$ for some submodule of W, hence $\frac{W}{H} = \frac{K\oplus L}{H} = \frac{K+H}{H} + \frac{L+H}{H}$ and since W is distribution module, then $(K+H) \cap (L+H) = ((K+H) \cap L)$ +((K+H)∩ $H) = (K \cap L) + (H \cap L) + (K \cap H) + H = H$ hence $\frac{W}{H} = \frac{K+H}{H} \oplus \frac{L+H}{H}$ and assist Proposition 3.12, $\frac{W}{H}$ is Pr-lifting .

Recall that if W=$W_1 \oplus W_2$ be an R-module, then $\frac{W}{A} = \frac{A+W_1}{A} \oplus \frac{A+W_2}{A}$ for every fully invariant, submodules A of W [8].

Corollary 3.15: If W is Pr-lifting module if K is fully invariant submodule of W then $\frac{W}{K}$ is Pr-lifting.

Proof: Impose $W = W_1 \oplus W_2$ since W is fully invariant then $\frac{W}{A} = \frac{W_1+A}{A} \oplus \frac{W_2+A}{A}$ for all submodule A of W is full invariant this assist Proposition 3.13 we have $\frac{W}{A}$ is Pr-lifting.

Proposition 3. 16: If W =W_1+W_2 is an R-module, such that $= Ann_R(W_1) + Ann_R(W_2)$, if $W_1$and $W_2$are Pr-lifting then W is Pr-lifting.

Proof: Impose H=$H_1 \oplus H_2$ is a submodule of W since $R = Ann_R(W_1) + Ann_R(W_2)$ for some $H_1 \leq W_1$, $H_2 \leq W_2$ assist theorem (3.6) we have H_1=K_1+L_1 and H_2=K_2+L_2 since $H_1 and H_2$ are Pr-lifting such that K_1direct summand of W_1 and K_2direct summand of W_2 , $L_1 \ll_{Pr} W_1$, $L_2 \ll_{Pr} W_2$now $K_1 \oplus K_2$is direct summand of $W_1 \oplus W_2$=W , nakedly $L_1 \ll_{Pr} W_1$, $L_2 \ll_{Pr} W_2$ implies $L_1 \oplus L_2 \ll_{Pr} W_1 \oplus W_2$=$W$ hence W is Pr-lifting assist definition of Pr-lifting.

Corollary 3. 17: If W=$W_1 \oplus W_2$is duo module if $W_1$and $W_2$are Pr-lifting then W is Pr-lifting.

Proof: Impose W is duo module and K is a submodule of W, then K is fully invariant, hence $K = K \cap W = K \cap (W_1 \oplus W_2) = (K \cap W_1) \oplus (K \cap W_2)$ and assist Proposition 3. 15 we have W is Pr-lifting.

Proposition 3. 18: Let W=$\oplus_{i\in I}W_i$ be a fully stable module, if W_i is Pr-lifting for each $i \in I$ then W is Pr-lifting.
Proof: Impose K is a submodule of W, K= $\oplus_{i\in I}(K \cap W_i)$ [7] since $K \cap W_i \leq W_i$and $W_i$is Pr-lifting then $K \cap W_i$= L_i+H_i where $L_i$direct summand of W_i and $H_i \ll_{Pr} W_i$ hence K=$\oplus_{i\in I}$(K∩ W_i)= $\oplus_{i\in I}(L_i \cap H_i)$= $\oplus_{i\in I}(L_i)$+ $\oplus_{i\in I}(H_i)$, we can readily that $\oplus_{i\in I}(L_i)$ is direct summand of $\oplus_{i\in I.}W_i$=W and since $H_i \ll_{Pr} W_i$ then $\oplus_{i\in I}(H_i) \ll_{Pr} \oplus_{i\in I.}W_i$=$W$ assist[21] hence assist Theorem 3.6), we have W is Pr-lifting.
Proposition 3.19: If W is faithful, finitely generated and multiplication R-module then W is Pr-lifting if and only if, R is Pr-lifting.
Proof:→) Impose W is Pr-lifting and , J is an ideal of R, assist, Theorem 3.6, there exists L direct summand of W and $K \ll_{Pr} W$ such that $H = JW = L + K$, $K \leq W$ and $H \leq W$, since W is multiplication, then there exists I and F are ideal of R such that $K = IW$ and $L = FW$, so $JW = IW + FW = (J + F)W$, and since W is faithful, finitely generated and multiplication module, then we get $J = IF$ assist [18], to show I direct summand of R, impose $W = L \oplus B$ and $B = AW$ for some$A$ is ideal of R so $RW = W = IW \oplus AW = (I + A)W$ and since W is cancelation [14] hance $R = I + A$, now to show $\mathrm{I} \cap \mathrm{A} = 0$, since W is faithful , finitely generated and multiplication , then 0=IW ∩ AW $= (\mathrm{I} \cap \mathrm{A})\mathrm{W}$ so $\mathrm{I} \cap \mathrm{A} = 0$, and hence I is direct summand of R and since $\mathrm{K} = \mathrm{Fw} \ll_{\mathrm{Pr}} \mathrm{W}$ then $\mathrm{F} \ll_{\mathrm{Pr}}\mathrm{R}$ assist Theorem 3.12, so R is Pr-lifting.
←)If R is Pr-lifting and H is a submodule of W where W R-module, since W is multiplication then there exists J is an ideal of R such that $\mathrm{H} = \mathrm{JW}$, so there exist I is direct summand of R and $\mathrm{K} \ll_{\mathrm{Pr}}$ R such that $\mathrm{J} = \mathrm{I} + \mathrm{K}$ hence $\mathrm{JW} = (\mathrm{I} + \mathrm{K}), \mathrm{W} = \mathrm{IW} + \mathrm{KW}$ so, $\mathrm{H} = \mathrm{IW} + \mathrm{KW}$, to show IW is direct summand of W, impose $\mathrm{R} = \mathrm{I} + \mathrm{U}$ for some U is ideal of R, hence $\mathrm{W} = \mathrm{RW} = (\mathrm{I} + \mathrm{U})\mathrm{W} = \mathrm{IW} + \mathrm{UW}$ and since W is faithful finitely generated and multiplication, then $\mathrm{IW} \cap \mathrm{UW} = (\mathrm{I} \cap \mathrm{U})\mathrm{W} = 0\mathrm{W}$, so IW is direct summand of W and since $\mathrm{K} \ll_{\mathrm{Pr}}\mathrm{R}$ then $\mathrm{U} \ll_{\mathrm{Pr}}\mathrm{W}$ assist Theorem 3.12, hence W is Pr-lifting.

4. Conclusions

We will try to generalize the twigged of Pr-hollow R-module to some other concepts in future works. In this lucubration, the twigged of Pr-hollow R-modules is studied as a generalization of hollow submodule and some properties of this concepts are investigated also, we lucubration Pr-lifting as generalization of lifting module with some properties such as:

1. Every hollow submodule of R-module Wis Pr-hollow.
2. Every simple module is Pr-hollow.
3. If W be semisimple module, then W is Pr-hollow module.
4. If W is Pr-hollow module, then not all submodule is Pr-hollow module.
5. Every Pr-hollow module is decomposable.
6. All lifting module is Pr-lifting module.
7. Every semisimple R-module is Pr-lifting module.
8. Every hollow module is Pr-lifting.
9. Every Pr-local is, Pr-lifting.
10. If W is an indecomposable, then W is Pr-hollow if and only if W is Pr-lifting.
11. Any direct summand of Pr-lifting module is Pr-lifting module.
12. If W is indecomposable module, then W is not Pr-lifting for every non -trivial submodule K of W.

References

[1] D. K And Tribak R., “On Hollow-lifting Module”, *Taiwanese J. Math*, vol.11, no. 2, pp.545-568, 2007.
[2] Muna Abbas Ahmed “Prime Hollow Modules” *Iraqi Journal of science*, vol.51, no. 4, pp.628-632, 2010

[3] Hadi I. M.A. and Aidi, S.H. “On e-Hollow modules” *International journal of Advanced Sc.And Technical Reserch*, vol. 3, no. 5, pp,2249-9954, 2015.

[4] Layla S. Mahmood, Bothaynah N. Shihab, Hatam Y. Khalaf,Semihollow Modules andSemilifting Modules”, *International Journal of Advanced Scientific and Technical Research*, vol. 3, no. 5, pp. 375- 382. 2015

[5] Amira A., Abduljaleel and Sahira M., Yaseen “ On Large-Small submodule and Large-hollow” *J. Phys.:Conf.Ser.*1818, 2021.

[6] Zhou D.X. and Zhang X.R.”Small-Essential Submodules and Morita Duality”, *Southeast Asian Bull.Math*., vol. 35, pp. 1051-1062, 2011.

[7] Mehdi S. Abbas and Mohammad F. Manhal, “d-Small Submodule and d-Small”, *International Journal of Algebra*, voI. 12, 2018.

[8] Ali Kabban, Wasan Khalid, “On Jacobson-Small Submodules”, *Iraqi Journal of Science*, vol. 60, no.7, pp.1584-1591, 2019.

[9] Wisam A. Ail, N. S. Al. Mothafar “On Quasi-Small Prime Submodules” *Iraqi Journal of Science*, vol, 63, no,4, pp. 1692-1699, 2022.

[10] Muna Abbas Ahmed, Iman Abdulhadi Dhari, Zainab Abed Atiya, “Purely small Submodules and Purely Hollow Modules”, *Iraqi Journal of science*, vol.63, no. 12, pp. 5487-5495, 2022.

[11] Khawla Ahmed, N. S. Al. Mothafar, “Pr-Small Submodules of Modules and Pr-Radicals”, *Journal of Interdisciplinary Mathematics*, 2023. To appear.

[12] R. AL-Shaiban, N. S. AL-Mothafar “Quasi J-Regular Modules”, *Iraqi Journal of Science*, vol.61, no.6, pp.1473-1478, 2020.

[13] Noor M. M. and Wasan K. h., “Generalized-Hollow Liftingg Modules”, *Iraqi Journal of Science*, vol. 59, no. 2B, pp. 917-921, 2018.

[14] Kasch F., *Modules and Rings, Academic Press, Inc-London*, 1982.

[15] Enas M. K. and Wasan Kh., “On μ-lifting Modules”, *Iraqi journal of science*, vol. 60, no. 2, pp. 371-380, 2019.

[16] N. H. Garib, "Some Generalizations of Lifting Modules", M. Sc. thesis, Mosul University, (1989).

[17] Al-Redeeni H.S. and Al-Bahrani B. H.,” On (G^*-) T-lifting Modules and T-H supplemented Modules, M.Sc. Thesis, University of Baghdad, Colledge of Sciences.

[18] W. Khalid and A. S. Wadi, “Generalized Radical g- Lifting Modules”, *Int. J. of Science and Research*, vol. 6, no. 7, pp. 2211-2214, 2015.

[19] Ali K. and Wasan Kh., “On Lifting Modules”, *Journal of Physics: Conference Series*, vol.1530, no. 1, 012025, 2020.

[20] D. Keskin & N. Orhan, “Generalization of Weak Lifting Modules”, *Soochow J. Math*, vol. 32, no. 1, pp. 71-76. 2006.

[21] Amira A. Abduljaleel, Sahira M. Yaseen, “On Large-Lifting and Large-Supplemented Module”, *Iraqi Journal of Science*, vol. 63, no. 4, pp. 1729-1735, 2022.

[22] Sarah Sh. and Bahar H., “Some Generalization on δ -Lifting modules”, *Iraqi journal of science*, vol.53, no. 3, pp. 633-643, 2012.

[23] Al-Redeeni H.S. and Al-Bahrani B. H.,” On (G^*-) T-lifting Modules and T-H supplemented Modules, M. Sc., Thesis/University of Baghdad.